Laterally complete and projectable hulls of semilinear residuated lattices

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Abstract

The existence of lateral completions of ℓ-groups is an old problem that was first solved, for conditionally complete vector lattices, by Nakano [5]. The existence and uniqueness of lateral completions of representable ℓ-groups was first obtained as a consequence of the orthocompletions of Bernau [1], and later the proofs were simplified by Conrad [3], who also proved the existence and uniqueness of lateral completions of ℓ-groups with zero radical. Finally, Bernau [2] solved the problem for ℓ-groups in general.

In this work, we address the problem of the existence and uniqueness of lateral, projectable, and strongly projectable completions of residuated lattices. In particular, we push the methods of Conrad through to the case of the representable GMV-algebras.

The leading idea is to construct, for any given semilinear residuated lattice, an orthocomplete extension such that the former is dense in the latter. This extension is obtained as the direct limit of a family of residuated lattices that are constructed using partitions of the algebra of polars of the original residuated lattice.

In order to complete the proof we still need another hypothesis, which is the relativized law of double negation with respect to arbitrary elements, an abstraction of the law of double negation in which commutativity and integrality have been dropped, and determines the wide class of Generalized MV-algebras (see [4]). This, together with the density, allows us to obtain the completions of the given residuated lattice.

Two negative elements of a residuated lattice are *disjoint* if and only if their join is 1, and a residuated lattice is called *laterally complete* if every set of disjoint elements has an infimum. Using this notion of disjointness, we define the binary relation $\perp$ of *orthogonality* between elements of a residuated lattice $\mathbf{L}$. The orthogonality relation induces a self-adjoint operator $(\ )^{\perp}$ on $\mathcal{P}(L)$, which, in turn, induces a closure operator $(\ )^{\perp\perp}$. The closed sets of this closure operator are called the *polars* of $\mathbf{L}$. The polars of a residuated lattice $\mathbf{L}$ are shown to be universes of convex subalgebras of $\mathbf{L}$, and form a complete Boolean algebra $\mathrm{P}(\mathbf{L})$.

Residuated lattices with the property that all polars are normal are of a special interest for this work. They can be characterized as those satisfying the quasi-identity:

$$x \perp y \Rightarrow \lambda_z(x) \perp \rho_w(y).$$

In particular, the semilinear residuated lattices, that is, the residuated lattices satisfying the equation $\lambda_z((x \vee y)\backslash x) \vee \rho_w((x \vee y)\backslash y) \approx 1$, also called representable residuated lattices, have the property that all polars are normal convex subalgebras.

A nonempty set of polars $\mathcal{C}$ is a *partition* if for every two different $C, D \in \mathcal{C}$, $C \cap D = \{1\} \notin \mathcal{C}$, and the join (in the Boolean algebra $\mathrm{P}(\mathbf{L})$) of $\mathcal{C}$ is L. If $\mathcal{C}$ and $\mathcal{A}$ are two partitions, then $\mathcal{A}$ is said to be a *refinement* of $\mathcal{C}$ if every element of $\mathcal{C}$ is a join of elements of $\mathcal{A}$, and we denote it by $\mathcal{C} \preccurlyeq \mathcal{A}$. It is shown that, with this relation, the set of partitions is a directed ordered set. For every partition $\mathcal{C}$, we define a residuated lattice $\mathbf{L}_{\mathcal{C}}$, and for every refinement $\mathcal{A}$ of $\mathcal{C}$, we define a homomorphism $\phi_{\mathcal{C}\mathcal{A}} : \mathbf{L}_{\mathcal{C}} \to \mathbf{L}_{\mathcal{A}}$, in such a way that $\{\phi_{\mathcal{C}\mathcal{A}} : \mathbf{L}_{\mathcal{C}} \to \mathbf{L}_{\mathcal{A}}\}_{\mathcal{C}\preccurlyeq\mathcal{A}}$ is a directed system in the category of semilinear residuated lattices and homormorphisms of residuated lattices. We call the direct limit of this system $\mathcal{O}(\mathbf{L})$.

As a technical tool, we study direct limits of systems of algebras in general, and characterize them as a quotient of a subalgebra of the product of the algebras of the system. Thus, using this representation, we can prove that:

Theorem. *The direct limit* $\mathbf{S}$ *of a system of algebras* $\{f_{ij} : \mathbf{A}_i \to \mathbf{A}_j\}_I$ *satisfies every quasi-equation* κ *which is satisfied by algebras of the system with arbitrary large index, that is, if for every* $i \in I$*, there exists* $j \in I$ *such that* $i \leqslant j$ *and* $\mathbf{A}_j \models \kappa$*, then* $\mathbf{S} \models \kappa$*.*

Note that from this result it is immediate to prove the already known fact that quasivarieties are closed under taking direct limits.

Always using this characterization of the direct limit, we prove that $\mathcal{O}(\mathbf{L})$ is a semilinear residuated lattice, and that it densely includes $\mathbf{L}$. We then prove one of the key results of this paper:

Theorem. *If* $\mathbf{L}$ *is a semilinear residuated lattice, then* $\mathcal{O}(\mathbf{L})$ *is laterally complete.*

Although $\mathbf{L}$ is densely included in $\mathcal{O}(\mathbf{L})$, which is laterally complete, this might not be the smallest laterally complete residuated lattice including $\mathbf{L}$. Being laterally complete is not preserved by taking arbitrary intersections, so taking the intersection of all the laterally complete sublattices of $\mathcal{O}(\mathbf{L})$ containing $\mathbf{L}$ could result in a non laterally complete residuated lattice. In order to prevent this from happening, we need to add another hypothesis, restricting our study to the class of Generalized MV-algebras. We can then prove our main result:

Theorem. *If* $\mathbf{L}$ *is a semilinear GMV-algebra, then it has a unique semilinear GMV lateral completion, up to isomorphisms.*

We also study the notions of *projectability* and *strong projectability* for residuated lattices. We prove that, if $\mathbf{L}$ is a semilinear residuated lattice, then $\mathcal{O}(\mathbf{L})$ is projectable. We also prove that $\mathcal{O}_{<\omega}(\mathbf{L})$, which is the direct limit of the subsystem given by the finite partitions, is projectable as well. For GMV-algebras we prove that lateral completeness and projectability imply strong projectability.

References

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