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Towards Computing Suboptimal Controls in a Zero-Sum Linear-Quadratic Differential Game: Artificial Parameter Approach

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Abstract

We consider a zero-sum finite horizon linear-quadratic differential game. Suboptimal state-feedback controls of the players in this game are derived. This derivation is based on the approximate solution (with a novel error's estimate) of the corresponding Riccati matrix differential equation by the method of artificial parameter. The theoretical results are illustrated by the approximate solution of the problem of pursuit-evasion engagement between two flying vehicles.

1 Introduction

The solution of a finite horizon zero-sum linear-quadratic differential game with fixed initial and free terminal states is one of the fundamental results in the theory of differential games. This solution reduces the original game to qualitative analysis and solution of the terminal-value problem for the game-theoretic Riccati matrix differential equation (see, e.g., [3, 9] and references therein). More precisely, if the solution of this terminal-value problem exists in the entire time-interval of the game's duration, then the game has the saddle point in the state-feedback controls. The solution of the aforementioned terminal-value problem for the game-theoretic Riccati matrix differential equation determines the gains of the players' optimal state-feedback controls, as well as the game's value. However, to verify the existence of the solution to the terminal-value problem for the game-theoretic Riccati matrix differential equation in the aforementioned time-interval and to derive this solution (if it exists) is a rather complicated task. This task cannot be treated, in general, by an exact analytical method because of high dimension and nonlinearity of the Riccati equation. In the literature, there are known different conditions for the existence of the solution to the game-theoretic Riccati matrix differential equation in the entire time-interval of the game's duration (see, e.g., [2, 5, 7, 19, 20, 22] and references therein). However, these conditions are not based directly on the game's data. Moreover, the verification of their validity requires rather complicated analytical/numerical calculations.

In the recent work [10], the method of auxiliary (artificial) parameter is developed for the qualitative analysis and approximate analytical solution of the terminal-value problem for the game-theoretic Riccati matrix differential equation associated with the finite horizon linear-quadratic differential game. The easily verified sufficient condition for the existence of the solution to this terminal-value problem in the entire time-interval of the game's duration is obtained. Different versions of the artificial parameter method, as well as their applications to approximate analytical solution of various initial/boundary-value problems for ordinary and partial differential equations (mainly scalar ones), are studied extensively in several recent decades (see, e.g., [11, 16, 17, 13, 15, 24, 23, 25] and references therein). To the best of our knowledge, the application of the artificial parameter method to the approximate solution of a Riccati matrix differential equation with a terminal condition was proposed for the first time by [8]. In this short conference paper, the method of continuation in parameter was applied for the approximate solution of the terminal-value problem for the Riccati matrix differential equation associated with the finite horizon linear-quadratic optimal control problem.

In the present paper, based on the auxiliary (artificial) parameter method, proposed in [10], we obtain a novel sufficient condition for the existence of the solution to the terminal-value problem for the game-theoretic Riccati matrix differential equation in the entire time-interval of the game's duration. Like the condition of the work [10], the condition of the present paper is easily verified, and these conditions can complement each other in the qualitative analysis and approximate analytical solution of the aforementioned terminal-value problem. Using the condition for the existence of the solution to this problem, obtained in the present paper, we derive a novel estimate of the error in its approximate solution. Furthermore, using this approximate solution, we design the suboptimal state-feedback controls of the players in the original differential game and establish novel estimates of the closeness of the guaranteed results of these controls to the game value. The theoretical results of the paper are applied to qualitative analysis and approximate solution of a real-life pursuit-evasion game.

The paper is organized as follows. In the next section, the problem is rigorously formulated. In Section 3, the sufficient condition for the existence of the solution to the terminal-value problem for the game-theoretic Riccati matrix differential equation in the entire time-interval of the game's duration is established. The approximate analytical solution to this problem and the estimate of the approximation's error are derived. In Section 4, using the results of Sections 2 and 3, the suboptimal state-feedback controls of the players in the considered game are presented. The closeness of the guaranteed results of these controls to the value of the game is established. In Section 5, the results of Sections 2 – 4 are applied to qualitative analysis and approximate solution of a pursuit-evasion engagement problem between two flying vehicles. Section 6 is devoted to the conclusions.

2 Problem statement

Consider the following dynamic system controlled by two decision makers:

$$\frac{dx}{dt} = A(t)x + B(t)u + C(t)v, \quad t \in [0, t_f], \quad x(0) = x_0, \tag{1}$$

where $x \in \mathbb{R}^n$ is a state variable; $u \in \mathbb{R}^r$ and $v \in \mathbb{R}^s$ are controls of the decision makers (players); $A(t)$, $B(t)$ and $C(t)$ are given matrices of corresponding dimensions; $x_0 \in \mathbb{R}^n$ is a given vector; $t_f > 0$ is a given time instant; the matrix-valued functions $A(t)$, $B(t)$ and $C(t)$ are continuous in the interval $[0, t_f]$.

The cost functional, to be minimized by the control u (the minimizer) and maximized by the control v (the maximizer), is

$$J(u,v) = x^T(t_f)Fx(t_f) + \int_0^{t_f} \big[x^T(t)D(t)x(t) + u^T(t)G_u(t)u(t) - v^T(t)G_v(t)v(t)\big]dt, \tag{2}$$

where F is a given symmetric positive semi-definite matrix of corresponding dimension; $D(t)$, $G_u(t)$ and $G_v(t)$ are given matrix-valued functions of corresponding dimensions, continuous in the interval $[0,t_f]$; for any $t \in [0,t_f]$, $D(t)$ is symmetric positive semi-definite, while $G_u(t)$ and $G_v(t)$ are symmetric positive definite.

Let Ω_u be the set of all functions $\omega_u(t,x) : [0,t_f] \times \mathbb{R}^n \to \mathbb{R}^r$, which are measurable w.r.t. $t \in [0,t_f]$ for any fixed $x \in \mathbb{R}^n$ and satisfy the local Lipschitz condition w.r.t. $x \in \mathbb{R}^n$ uniformly in $t \in [0,t_f]$. Similarly, let Ω_v be the set of all functions $\omega_v(t,x) : [0,t_f] \times \mathbb{R}^n \to \mathbb{R}^s$, which are measurable w.r.t. $t \in [0,t_f]$ for any fixed $x \in \mathbb{R}^n$ and satisfy the local Lipschitz condition w.r.t. $x \in \mathbb{R}^n$ uniformly in $t \in [0,t_f]$. In what follows, we assume that $u(t,x) \in \Omega_u$, $v(t,x) \in \Omega_v$.

We assume that both players are aware of all the data, presenting in (1) – (2), and of the current $(time, state)$-position $\big(t,x\big)$ of the system (1). In what follows, we call the problem, consisting of the system (1), the cost functional (2), the aforementioned objectives of the players and the information pattern, the Linear-Quadratic Differential Game (LQDG). The sets Υ_u and Υ_v of the players' state-feedback admissible controls, the saddle point $(u^*(t,x),\, v^*(t,x))$ consisting of the optimal players' state-feedback strategies, the game value $J^*(x_0)$ are defined in a standard fashion (see, e.g., the book [9] and references therein).

Let

$$\widetilde{D}(t) \triangleq D(t) + FA(t) + A^T(t)F - FS(t)F, \quad t \in [0,t_f] \tag{3}$$

and the matrix-valued function $\Phi(t)$, $t \in [0,t_f]$ be the unique solution of the terminal-value problem

$$\frac{d\Phi(t)}{dt} = -\big(A(t) - S(t)F\big)^T\Phi(t), \qquad t \in [0,t_f], \quad \Phi(t_f) = I_n,$$

where $S(t) = B(t)G_u^{-1}(t)B^T(t) - C(t)G_v^{-1}(t)C^T(t)$, $t \in [0,t_f]$.

Matrix $\Phi(t)$ is invertible for all $t \in [0,t_f]$.

Define the matrices

$$\mathcal{S}(t) = \Phi^T(t)S(t)\Phi(t), \qquad \mathcal{D}(t) = \Phi^{-1}(t)\widetilde{D}(t)\big(\Phi^T(t)\big)^{-1}.$$

Proposition 1. *Assume that the terminal-value problem*

$$\frac{dK}{dt} = K\mathcal{S}(t)K - \mathcal{D}(t), \qquad t \in [0,t_f], \quad K(t_f) = 0, \tag{4}$$

has the solution $K = K(t)$ in the entire interval $[0,t_f]$, and

$$M(t) = \Phi(t)K(t)\Phi^T(t) + F, \qquad t \in [0,t_f]. \tag{5}$$

Then the functions

$$u^*(t,x) = -G_u^{-1}(t)B^T(t)M(t)x \in \Omega_u, \quad (t,x) \in [0,t_f] \times \mathbb{R}^n \tag{6}$$

and

$$v^*(t,x) = G_v^{-1}(t)C^T(t)M(t)x \in \Omega_v, \quad (t,x) \in [0,t_f] \times \mathbb{R}^n \tag{7}$$

are the optimal controls of the minimizer and maximizer, respectively, in the LQDG. Moreover, the pair $\big(u^*(t,x), v^*(t,x)\big)$ *is the saddle point of the LQDG, the value of this game has the form*

$$J^*(x_0) = J\big(u^*(t,x), v^*(t,x)\big) = x_0^T M(0)x_0 \tag{8}$$

and, for any $u(t,x) \in \Upsilon_u\big(v^*(t,x)\big)$ *and any* $v(t,x) \in \Upsilon_v\big(u^*(t,x)\big)$*, the following saddle-point inequality is valid:*

$$J\big(u^*(t,x), v(t,x)\big) \le J^*(x_0) \le J\big(u(t,x), v^*(t,x)\big).$$

Remark 1. *The matrix function* $M(t)$ *given by (5) satisfies the terminal value problem*

$$\frac{dM}{dt} = -MA(t) - A^T(t)M + MS(t)M - D(t), \quad M(t_f) = F. \tag{9}$$

The known LQDG solvability condition [6] assumes the existence of the solution of (9) in the entire interval $[0,t_f]$*. The condition, based on the Riccati equation in (4), having the zero terminal condition, is considerably simpler. The solution of (4), if exists, is unique.*

Following [10], along with the problem (4), let us consider the terminal-value problem

$$\frac{d\mathcal{K}}{dt} = \varepsilon\mathcal{K}\mathcal{S}(t)\mathcal{K} - \mathcal{D}(t), \quad t \in [0,t_f], \quad \mathcal{K}(t_f) = 0, \tag{10}$$

where ε is an artificial parameter varying in the interval $[0,1]$.

For $\varepsilon = 1$, the problem (10) becomes the problem (4), while, for $\varepsilon = 0$, the problem (10) becomes the simplest terminal-value problem

$$\frac{d\mathcal{K}_0}{dt} = -\mathcal{D}(t), \quad t \in [0,t_f], \quad \mathcal{K}_0(t_f) = 0,$$

which has the unique solution

$$\mathcal{K}_0 = \mathcal{K}_0(t) = \int_t^{t_f} \mathcal{D}(\xi)d\xi \tag{11}$$

in the entire interval $[0,t_f]$.

The objectives of the paper are:

(a) using the formal expansion of the solution to the problem (10) in the power series of ε, to derive a novel sufficient condition for the existence of the solution to the problem (4) in the entire interval $[0,t_f]$;

(b) using this condition, to construct the approximation of the solution to the problem (4) and to obtain a novel estimate for the approximation's error;

(c) using the approximation of the solution to the problem (4), to design the suboptimal players' controls in the LQDG and to obtain novel estimates for the closeness of the guaranteed results of these controls to the game value.

3 Solution of the problem (4)

Similarly to [10], we look for the solution to the terminal-value problem (10) in the form of the power series with respect to ε

$$\mathcal{K}(t,\varepsilon)=\sum_{k=0}^{+\infty}\mathcal{K}_k(t)\varepsilon^k, \tag{12}$$

where $\mathcal{K}_0(t)$ is given by (11).

Substituting (12) into the problem (10) and equating the coefficients for ε^k, $(k=1,2,...)$ on both sides of the resulting differential equation and terminal condition yield the terminal-value problems for the matrix-valued coefficients of the series (12):

$$\dot{\mathcal{K}}_k(t)=\sum_{j=0}^{k-1}\mathcal{K}_j(t)\mathcal{S}(t)\mathcal{K}_{k-j-1}(t),\quad \mathcal{K}_k(t_f)=0,\ k=1,2,... \tag{13}$$

Solving these problems, we obtain the expressions for the matrix-valued coefficients in the series (12)

$$\mathcal{K}_k(t)=\sum_{j=0}^{k-1}\int_{t_f}^{t}\mathcal{K}_j(\xi)\mathcal{S}(\xi)\mathcal{K}_{k-j-1}(\xi)d\xi,\ t\in[0,t_f],\ k=1,2,.... \tag{14}$$

Since the matrix-valued functions $\mathcal{S}(t)$ and $\mathcal{D}(t)$ are continuous in the interval $[0,t_f]$, then there exist the following finite values:

$$\alpha_{\mathcal{S}}\triangleq\max_{t\in[0,t_f]}\big\|\mathcal{S}(t)\big\|,\quad \alpha_0\triangleq\max_{t\in[0,t_f]}\big\|\mathcal{K}_0(t)\big\|. \tag{15}$$

Lemma 1. *The matrix-valued coefficients (11), (14) of the series (12) satisfy the following estimates:*

$$\big\|\mathcal{K}_k(t)\big\|\le\alpha_0^{k+1}\alpha_{\mathcal{S}}^k(t_f-t)^k,\ t\in[0,t_f],\ k=0,1,2,.... \tag{16}$$

The lemma is proved by the induction over k, based on the explicit solution (14) of the problems (13).

For given $\varepsilon\in[0,1]$ and $t\in[0,t_f]$, the series (12) is called to be strong convergent [18] if the series

$$\sum_{k=0}^{+\infty}\big\|\mathcal{K}_k(t)\big\|\varepsilon^k$$

is convergent. Based on the results of [18] and using Lemma 1, we directly have the following assertion.

Proposition 2. *Let, for given $\varepsilon\in[0,1]$ and $t\in[0,t_f]$, the series*

$$\sum_{k=0}^{+\infty}\alpha_0^{k+1}\alpha_{\mathcal{S}}^k(t_f-t)^k\varepsilon^k,\quad k=0,1,2,... \tag{17}$$

converge. Then, for these ε and t, the series (12) strongly converges.

Lemma 2. *Let, for given $\varepsilon\in[0,1]$ and $t\in[0,t_f]$, the following inequality be satisfied:*

$$\alpha_0\alpha_{\mathcal{S}}(t_f-t)\varepsilon<1.$$

Then, for these ε and t, the series (17) converges.

The lemma is proved by exploiting the d'Alembert (ratio) criterion (see, e.g., [14]).
Setting formally $\varepsilon = 1$ in (12), we obtain the series

$$\mathcal{K}(t,1) = \sum_{k=0}^{+\infty} \mathcal{K}_k(t). \tag{18}$$

Corollary 1. *Let the following inequality be satisfied:*

$$\beta \triangleq \alpha_0 \alpha_{\mathcal{S}} t_f < 1. \tag{19}$$

Then, the series (18) strongly converges uniformly with respect to $t \in [0, t_f]$. Moreover, the sum $\mathcal{K}(t,1)$ of this series is a differentiable matrix-valued function in the interval $[0, t_f]$.

The main result of this section is as follows.

Theorem 1. *Let the inequality (19) be satisfied. Then, the terminal-value problem (4) has the unique solution in the entire interval $[0, t_f]$ and this solution is $K = K(t) = \mathcal{K}(t, 1)$.*

By differentiating the series (18) and exploiting the Cauchy product (see, e.g., [14]), it is proved that the function $\mathcal{K}(t,1)$, given by (18), satisfies the Riccati equation (4). This, along with Corollary 1, proves the theorem.

Remark 2. *Note, that the condition (19) for the existence of the solution to the terminal-value problem (4) in the entire interval $[0, t_f]$ is the novel one and it considerably differs from such a condition obtained in the work [10].*

Consider the matrix-valued function

$$K_m(t) \triangleq \sum_{k=0}^{m-1} \mathcal{K}_k(t), \qquad t \in [0, t_f], \qquad m \geq 1. \tag{20}$$

The following statement on the approximation of the function $K(t)$ is a consequence of Theorem 1.

Theorem 2. *Let the inequality (19) be satisfied. Then, the following inequality is valid:*

$$\big\| K(t) - K_m(t) \big\| \leq \mu_m(t), \quad t \in [0, t_f], \quad m \geq 1, \tag{21}$$

where

$$\mu_m(t) = \alpha_0 \frac{\big(\alpha_0 \alpha_{\mathcal{S}} (t_f - t)\big)^m}{1 - \alpha_0 \alpha_{\mathcal{S}} (t_f - t)}, \tag{22}$$

and

$$\lim_{m \to +\infty} \mu_m(t) = 0 \tag{23}$$

uniformly with respect to $t \in [0, t_f]$.

Proof. Due to (18) and (20),

$$K(t) - K_m(t) = \mathcal{K}(t,1) - K_m(t) = \sum_{k=m}^{+\infty} \mathcal{K}_m(t),$$

yielding by (16) and (19) the estimate

$$\|K(t)-K_m(t)\| \le \sum_{k=m}^{+\infty} \alpha_0^{k+1}\alpha_{\mathcal{S}}^k (t_f-t)^k = \alpha_0 \frac{\big(\alpha_0\alpha_{\mathcal{S}}(t_f-t)\big)^m}{1-\alpha_0\alpha_{\mathcal{S}}(t_f-t)} = \mu_m(t).$$

The limit equality (23) is a direct consequence of the inequality (19). This completes the proof of the theorem. □

Remark 3. *Note, that the estimate (21)-(22) for the error in the approximation of the solution to the terminal-value problem (4) in the entire interval $[0,t_f]$ is the novel one and it considerably differs from such an estimate obtained in the work [10]*

4 Main results: suboptimal players' controls in the LQDG

Due to the equations (5), (20) and Theorem 2, the matrix function $M(t)$ is approximated by the matrices

$$M_m(t) \triangleq \Phi(t)K_m(t)\Phi^T(t)+F, \quad t\in[0,t_f], \quad m\ge 1. \tag{24}$$

Remark 4. *Due to Theorem 2 and the equations (5) and (24), the following estimate holds:*

$$\|\Delta M_m(t)\| \le \frac{c_\Phi^2\alpha_0\beta^m}{1-\beta}, \quad t\in[0,t_f], \quad m\ge 1, \tag{25}$$

where

$$\Delta M_m(t) \triangleq M(t)-M_m(t), \tag{26}$$

$c_\Phi \triangleq \max_{t\in[0,t_f]} \|\Phi(t)\|$.

4.1 Suboptimal maximizer's control of the LQDG

Using the equations (7) and (24), we construct (similarly to [10]) the maximizer's linear feedback control

$$v_m(t,x) = G_v^{-1}(t)C^T(t)M_m(t)x, \quad (t,x)\in[0,t_f]\times\mathbb{R}^n. \tag{27}$$

The control (27) is admissible in the LQDG.

Substituting $v=v_m(t,x)$ into the system (1) and the cost functional (2), we obtain after a routine algebra the following new system and cost functional:

$$\frac{dx}{dt} = A_{v,m}(t)x + B(t)u, \quad t\in[0,t_f], \quad x(0)=x_0, \tag{28}$$

$$J_{v,m}(u) = x^T(t_f)Fx(t_f) + \int_0^{t_f} \big[x^T(t)D_{v,m}(t)x(t)+u^T(t)G_u(t)u(t)\big]dt, \tag{29}$$

where

$$A_{v,m}(t) = A(t)+S_v(t)M_m(t),$$
$$D_{v,m}(t) = D(t)-M_m(t)S_v(t)M_m(t),$$
$$S_v(t) = C(t)G_v^{-1}(t)C^T(t). \tag{30}$$

In (28) and (29), $x \in \mathbb{R}^n$ is a state variable, while $u \in \mathbb{R}^r$ is a control.

Consider the optimal control problem of the minimization of the cost functional (29) with respect to u along trajectories of the system (28).

Let

$$S_u(t) = B(t)G_u^{-1}(t)B^T(t).$$

By virtue of the results of [12], we have the following proposition.

Proposition 3. *If the terminal-value problem for the Riccati matrix differential equation*

$$\frac{dL}{dt} = -LA_{v,m}(t) - A_{v,m}^T(t)L + LS_u(t)L - D_{v,m}(t),$$
$$t \in [0, t_f], \quad L(t_f) = F. \tag{31}$$

has the solution $L = L_m(t)$ in $[0, t_f]$, then the optimal control in the problem (28), (29) is

$$u = u_m^*(t,x) \triangleq -G_u^{-1}(t)B(t)^T L_m(t)x, \quad (t,x) \in [0, t_f] \times \mathbb{R}^n.$$

The optimal value of the cost functional in the optimal control problem (28), (29) is

$$J_{v,m}^*(x_0) \triangleq J_{v,m}\big(u_m^*(t,x)\big) = x_0^T L_m(0)x_0. \tag{32}$$

The value $J_{v,m}^*(x_0)$ is called the guaranteed result of the control $v_m(t,x)$ in the LQDG. If, for any $x_0 \in \mathbb{R}^n$, $\lim_{m\to+\infty} J_{v,m}^*(x_0) = J^*(x_0)$, then the control $v_m(t,x)$ is called a suboptimal maximizer's control in the LQDG.

Theorem 3. *Let the inequality (19) be satisfied. Then, there exist an integer $m_{v,0} \geq 1$ and a number $c_{v,0} > 0$ such that the guaranteed result of the control $v_m(t,x)$ satisfies the inequality*

$$\big|J^*(x_0) - J_{v,m}^*(x_0)\big| \leq c_{v,0}\|x_0\|^2\beta^{2m}, \qquad m \geq m_{v,0}, \tag{33}$$

meaning that $v_m(t,x)$ is the suboptimal maximizer's control in the LQDG.

Proof. Let us denote

$$\Delta L_m(t) \triangleq M(t) - L_m(t), \quad t \in [0, t_f], \quad m \geq 1.$$

Due to (3), (9), (30), (31), we obtain after a routine algebra the terminal-value problem for $\Delta L_m(t)$

$$\begin{aligned}\frac{d\Delta L_m(t)}{dt} &= -\Delta L_m(t)\big(A(t) - S(t)\mathcal{M}(t)\big)\\ &-\big(A(t) - S(t)\mathcal{M}(t)\big)^T\Delta L_m(t)\\ -\Delta L_m(t)S_v(t)\Delta M_m(t) &- \Delta M_m(t)S_v(t)\Delta L_m(t)\\ -\Delta L_m(t)S_u(t)\Delta L_m(t) &- \Delta M_m(t)S_v(t)\Delta M_m(t),\\ &\Delta L_m(t_f) = 0.\end{aligned} \tag{34}$$

By virtue of the results of [1], we can rewrite the problem (34) in the equivalent integral form

$$\begin{aligned}\Delta L_m(t) = \int_t^{t_f} \Psi^T(\sigma,t)\Big[&\Delta L_m(\sigma)S_v(\sigma)\Delta M_m(\sigma)\\ +\Delta M_m(\sigma)S_v(\sigma)\Delta L_m(\sigma) &+ \Delta L_m(\sigma)S_u(\sigma)\Delta L_m(\sigma)\\ +\Delta M_m(\sigma)S_v(\sigma)\Delta M_m(\sigma)\Big]&\Psi(\sigma,t)d\sigma, \qquad t \in [0, t_f],\end{aligned}$$

where, for any given $t \in [0, t_f]$, the $n \times n$-matrix-valued function $\Psi(\sigma, t)$ satisfies the problem

$$\frac{d\Psi(\sigma,t)}{d\sigma} = \big(A(\sigma) - S(\sigma)M(\sigma)\big)\Psi(\sigma,t), \quad \sigma \in [t, t_f],$$
$$\Psi(t,t) = I_n.$$

By applying the successive approximations method [4] with the zero initial guess, it is shown that there exist an integer $m_{v,0} \geq 1$ and a number $c_{v,0} > 0$ such that

$$\max_{t\in[0,t_f]} \big\|\Delta L_m(t)\big\| \leq c_{v,0}\beta^{2m}, \ m \geq m_{v,0}. \tag{35}$$

Due to (8) and (32), the inequality (35) directly yields the estimate (33). □

Remark 5. *Due to (25)–(26), the matrix-valued function $M_m(t)$ approximates the matrix-valued function $M(t)$ with the accuracy of the order of β^m. However, as it is seen from the inequality (33), the suboptimal maximizer's control $v_m(t,x)$, based on $M_m(t)$, yields its guaranteed result $J^*_{v,m}(x_0)$ in the LQDG which approximates the value $J^*(x_0)$ of this game with the accuracy of the order of β^{2m}. Moreover, due to Remark 2, this accuracy differs considerably from such an accuracy obtained in the work [10].*

4.2 Suboptimal minimizer's control of the LQDG

Using the equations (6) and (24), we construct (similarly to [10]) the minimizer's linear feedback control

$$u_m(t,x) = -G_u^{-1}(t)B^T(t)M_m(t)x, \quad (t,x) \in [0,t_f] \times \mathbb{R}^n. \tag{36}$$

The control (36) is admissible in the LQDG. Substituting $u = u_m(t,x)$ into the system (1) and the cost functional (2) yields the following new system and cost functional:

$$\frac{dx}{dt} = A_{u,m}(t)x + C(t)v, \quad t \in [0,t_f], \quad x(0) = x_0, \tag{37}$$

$$J_{u,m}(v) = x^T(t_f)Fx(t_f)$$
$$+\int_0^{t_f} \big[x^T(t)D_{u,m}(t)x(t) - v^T(t)G_v(t)v(t)\big]dt, \tag{38}$$

where

$$A_{u,m}(t) = A(t) - S_u(t)M_m(t),$$
$$D_{u,m}(t) = D(t) + M_m(t)S_u(t)M_m(t).$$

In (37) and (38), $x \in \mathbb{R}^n$ is a state variable, while $v \in \mathbb{R}^s$ is a control.

Consider the optimal control problem of the maximization of the cost functional (38) with respect to v along trajectories of the system (37). The following statement holds, similar to Proposition 3.

Proposition 4. *If the terminal-value problem for the Riccati matrix differential equation*

$$\frac{dN}{dt} = -NA_{u,m}(t) - A^T_{u,m}(t)N - NS_v(t)N - D_{u,m}(t),$$
$$t \in [0,t_f], \quad N(t_f) = F,$$

has the solution $N = N_m(t)$ in $[0, t_f]$, then the optimal control problem (37), (38) has the solution (the optimal control)

$$v = v_m^*(t, x) \stackrel{\triangle}{=} G_v^{-1}(t)C(t)^T N_m(t)x, \quad (t, x) \in [0, t_f] \times \mathbb{R}^n.$$

The optimal value of the cost functional in the optimal control problem (37), (38) is

$$J_{u,m}^*(x_0) \stackrel{\triangle}{=} J_{u,m}\big(v_m^*(t, x)\big) = x_0^T N_m(0)x_0.$$

The value $J_{u,m}^*(x_0)$ is called the guaranteed result of the control $u_m(t, x)$ in the LQDG. If, for any $x_0 \in \mathbb{R}^n$, $\lim_{m\to+\infty} J_{u,m}^*(x_0) = J^*(x_0)$, then the control $u_m(t, x)$ is called a suboptimal minimizer's control in the LQDG.

Theorem 4. *Let the inequality (19) be satisfied. Then, there exist an integer $m_{u,0} \geq 1$ and a number $c_{u,0} > 0$ such that the guaranteed result of the control $u_m(t, x)$ satisfies the inequality*

$$\big|J^*(x_0) - J_{u,m}^*(x_0)\big| \leq c_{u,0}\|x_0\|^2\beta^{2m}, \qquad m \geq m_{u,0}, \tag{39}$$

meaning that $u_m(t, x)$ is the suboptimal minimizer's control in the LQDG.

The theorem is proved in the same lines as Theorem 3.

Similarly to Remark 5, the following should be noted.

Remark 6. *Due to (25)–(26), the matrix-valued function $M_m(t)$ approximates the matrix-valued function $M(t)$ with the accuracy of the order of β^m. However, as it is seen from the inequality (39), the suboptimal minimizer's control $u_m(t, x)$, based on $M_m(t)$, yields its guaranteed result $J_{u,m}^*(x_0)$ in the LQDG which approximates the value $J^*(x_0)$ of this game with the accuracy of the order of β^{2m}. Moreover, due to Remark 2, this accuracy differs considerably from such an accuracy obtained in the work [10].*

5 Example: pursuit-evasion engagement of two flying vehicles

Consider the following system:

$$\begin{aligned} \frac{dx_1}{dt} &= x_2, \qquad t \in [0, t_f], \qquad x_1(0) = 0, \\ \frac{dx_2}{dt} &= -u + v, \qquad t \in [0, t_f], \qquad x_2(0) = x_{20}. \end{aligned} \tag{40}$$

The system (40) is a linearized kinematic model of a planar pursuit-evasion engagement between two flying vehicles called a pursuer and an evader. In this model, it is assumed that the pursuer and the evader are directly controlled by their lateral accelerations $u = u(t)$ and $v = v(t)$, respectively. The state coordinates $x_1 = x_1(t)$ and $x_2 = x_2(t)$ are the relative lateral separation and the relative lateral velocity of the vehicles. For more details of such an engagement, one can see, for instance, the work [21] and references therein.

Let the 2×2-matrix-valued function $\mathcal{P}(t)$ be the unique solution of the terminal-value problem

$$\frac{d\mathcal{P}(t)}{dt} = -\mathcal{P}(t)\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad t \in [0, t_f], \qquad \mathcal{P}(t_f) = I_2.$$

A scalar state variable, playing an important role in analysis and solution of linear pursuit-evasion problems, is the so-called zero-effort miss distance (ZEMD) defined for the system (40) as follows (see, e.g., [21]):

$$z(t) = [1, 0]\mathcal{P}(t)\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \qquad t \in [0, t_f],$$

yielding $z(t) = x_1(t) + (t_f - t)x_2(t)$, $t \in [0, t_f]$.

The state variable $z(t)$ satisfies the initial-value problem

$$\frac{dz}{dt} = -(t_f - t)u + (t_f - t)v, \quad t \in [0, t_f], \quad z(0) = z_0 \overset{\triangle}{=} t_f x_{20}. \tag{41}$$

Furthermore, $z(t)$ satisfies the equality

$$z(t_f) = x_1(t_f),$$

and it has the following physical interpretation. If players' controls $u(t) \equiv 0$ and $v(t) \equiv 0$ in the interval $[\bar{t}, t_f]$, $(\bar{t} \in [0, t_f))$, then the miss distance $|x_1(t_f)|$ equals to $|z(\bar{t})|$.

In what follows in this section, we consider the pursuit-evasion engagement described by the scalar system (41). The behaviour of the players in this engagement is evaluated by the cost functional

$$J(u, v) = \int_0^{t_f} \Big[g_{z,1} \exp\big(g_{z,2}t\big)z^2(t) + g_u u^2(t) - g_v v^2(t)\Big]dt, \tag{42}$$

where the scalar coefficient $g_{z,1} > 0$ is a given constant; g_u and g_v are given positive constants and $g_u \neq g_v$; the scalar coefficient $g_{z,2}$ is a given nonzero constant. Thus, in this example, $n = r = s = 1$, $A(t) \equiv 0$, $B(t) = -(t_f - t)$, $C(t) = t_f - t$, $F = 0$, $G_u = g_u$, $G_v = g_v$, and $D(t) = g_{z,1} \exp\big(g_{z,2}t\big)$.

Remark 7. *It should be noted that the equation of dynamics (41) in the differential game (41),(42) is similar to such an equation in the example of the work [10]. However, the cost functional (42) differs considerably from the cost functional in the example of the work [10] meaning that the game (41),(42) considerably differs from the differential game in the example of the work [10].*

Since $F = 0$, in this example, the functions $M(t) = K(t)$ satisfy the terminal value problem

$$\frac{dM}{dt} = S(t)M^2 - D(t), \quad M(t_f) = 0. \tag{43}$$

where

$$S(t) = g_{uv}(t_f - t)^2,$$

$$g_{uv} = 1/g_u - 1/g_v.$$

Due to [12], for $g_u < g_v$, the solution of (43) exists in the whole interval $[0, t_f]$ for any $t_f > 0$. If $g_u > g_v$, then for sufficiently large t_f, the interval $[0, t_f]$ can contain a conjugate point, meaning that the solution of (43) does not exist in the whole interval $[0, t_f]$. In general, a task of checking the game solvability is difficult. In this case, the sufficient condition (19) serves as a practical instrument of verifying the LQDG solvability.

In this example, we chose the following parameters:

$$t_f = 1.1, \quad g_{z1} = 0.5, \quad g_{z2} = 0.1, \quad g_u = 0.11, \quad g_v = 0.1.$$

For these parameters, by direct integration in (11),

$$\mathcal{K}_0(t) = 5\left(\exp(0.11) - \exp(0.1t)\right).$$

The estimates (15) are $\alpha_{\mathcal{S}} = 1.1$, $\alpha_0 = 0.5814$ (see Figure 1), yielding $\beta = 0.7035$.

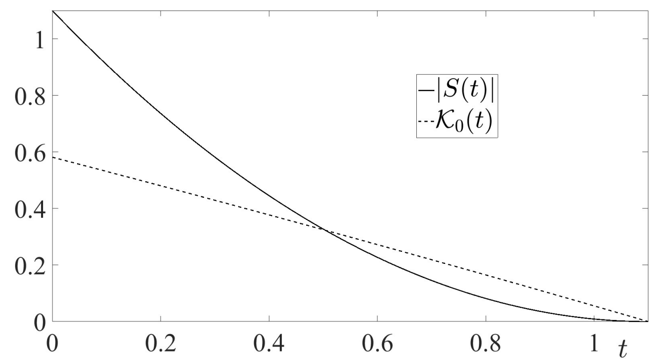

Figure 1: Functions $|S(t)|$ and $\mathcal{K}_0(t)$.

In this example, the approximating functions $M_m(t)$ are

$$M_m(t) = \sum_{k=0}^{m-1} \mathcal{K}_k(t), \quad m \geq 1.$$

In Figure 2, the absolute values of the approximation errors $\Delta M_m(t)$ are depicted for $m = 1, 2, 3$. It is seen that for $m = 3$ the approximation is very accurate.

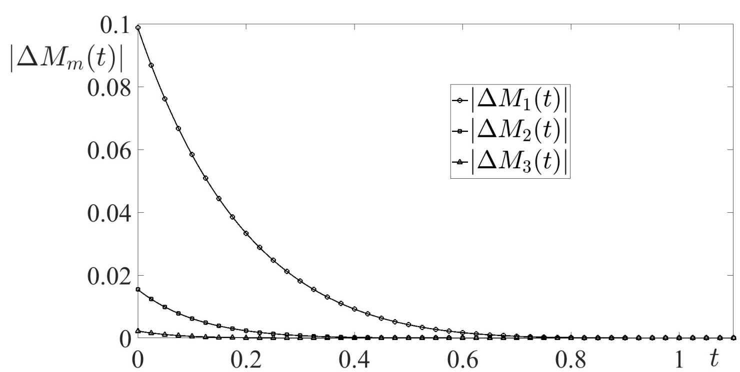

Figure 2: Approximation errors $|\Delta M_m(t)|$, $m = 1, 2, 3$.

The values

$$\delta M_m = \sup_{t \in [0, t_f]} \frac{|M(t) - M_m(t)|}{|M(t)|}$$

are presented for $m = 1, 2, 3$ in Table 1.

δM_1	δM_2	δM_3
0.1452	0.0227	0.0033

Table 1: Values of δM_m, $m = 1, 2, 3$.

The LQDG value along with the guaranteed results (32) for the suboptimal maximizer are presented in Table 2. The relative approximation errors

$$\delta J^*_{v,m} = \frac{|J^* - J^*_{v,m}|}{J^*}, \tag{44}$$

are shown in Table 3. The corresponding outcomes for the suboptimal minimizer are presented in Table 4. The relative approximation errors

$$\delta J^*_{u,m} = \frac{|J^* - J^*_{u,m}|}{J^*}$$

are shown in Table 5. These results illustrate Theorems 3 and 4 on the approximation of the game value by the guaranteed results of the suboptimal maximizer and minimizer, respectively.

Optimal	Suboptimal maximizer		
J^*	$J^*_{v,1}$	$J^*_{v,2}$	$J^*_{v,3}$
0.680378	0.671404	0.680239	0.680377

Table 2: Game outcomes for suboptimal maximizer.

$\delta J^*_{v,1}$	$\delta J^*_{v,2}$	$\delta J^*_{v,3}$
0.0132	$2.039 \cdot 10^{-4}$	$1.453 \cdot 10^{-6}$

Table 3: Comparison results for suboptimal maximizer.

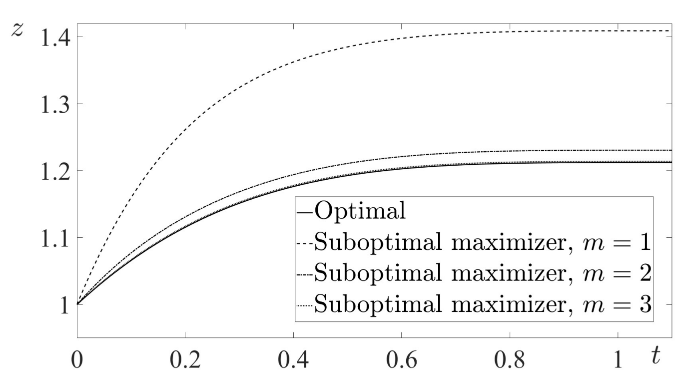

Figure 3: Trajectories: optimal and suboptimal maximizer, $m = 1, 2, 3$.

Optimal	Suboptimal minimizer		
J^*	$J^*_{u,1}$	$J^*_{u,2}$	$J^*_{u,3}$
0.680378	0.690811	0.680509	0.680379

Table 4: Game outcomes for suboptimal minimizer.

$\delta J^*_{u,1}$	$\delta J^*_{u,2}$	$\delta J^*_{u,3}$
0.0153	$1.922 \cdot 10^{-4}$	$1.362 \cdot 10^{-6}$

Table 5: Comparison results for suboptimal minimizer.

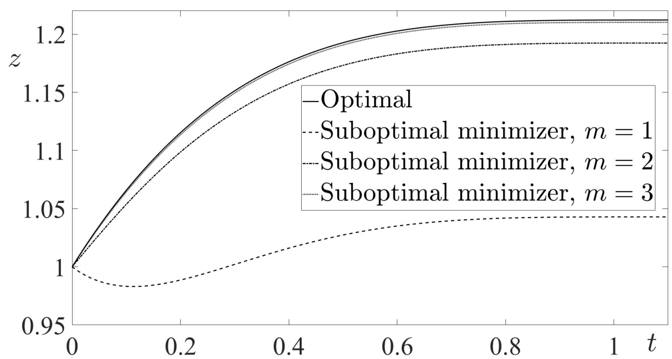

Figure 4: Trajectories: optimal and suboptimal minimizer, $m = 1, 2, 3$.

The optimal and suboptimal trajectories are depicted in Figures 3 and 4 for the suboptimal maximizer and minimizer, respectively. These figures justify a reasonable conjecture that the convergence $J^*_{v,m} \to J^*$ and $J^*_{u,m} \to J^*$ for $m \to \infty$, guaranteed by Theorems 3 and 4, leads to the C-norm convergence of the trajectories.

In Figures 5 and 6, the values of the solvability parameter β and the game value J^* are depicted as functions of t_f. It is seen that $\beta < 1$ for $t_f < 1.19$, and the LQDG is solvable in accordance with Theorem 1. For $t_f > t_f^* = 1.72$, the solution of the problem (4) does not exist in the whole interval $[0, t_f]$ (it contains a conjugate point), and the LQDG is not solvable. For $t_f \in [1.19, 1.71]$, the LQDG is solvable, although $\beta > 1$, demonstrating that the solvability condition (19) is sufficient and not necessary.

6 Conclusions

In this paper, the zero-sum finite horizon linear-quadratic differential game was considered. The existence of solution to this game and its approximation were studied. This study is based on the qualitative analysis and approximate solution of the terminal-value problem for the game-theoretic Riccati matrix differential equation associated with the considered game by the solvability conditions. Using the artificial (auxiliary) parameter method [10], the novel sufficient condition for the existence of solution to this terminal-value problem in the entire time-interval of the game's duration was obtained. The analytical form's approximation of this solution was derived. The novel estimate for the error of this approximation was established. Based on

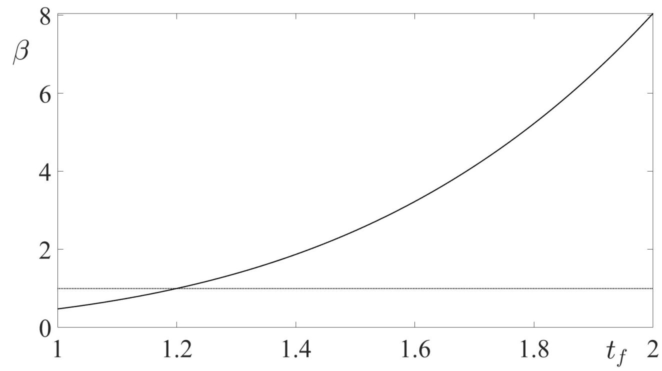

Figure 5: The value of β.

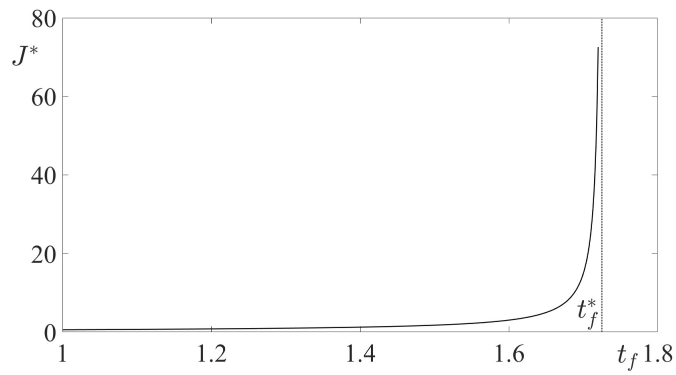

Figure 6: The game value J^*.

the approximate solution to the aforementioned terminal-value problem for the game-theoretic Riccati matrix differential equation, the players' suboptimal state-feedback controls in the considered differential game were designed. The guaranteed results of these controls were derived. The novel estimates for the closeness of these guaranteed results to the value of the game were established. It was shown that the error in the approximation of the game value by each of these guaranteed results is the square of the order of smallness in the approximation's error of the solution to the terminal-value problem for the game-theoretic Riccati matrix differential equation. Based on the theoretical results of the paper, the linear-quadratic pursuit-evasion game, modeling the planar engagement of two flying vehicles with zero-order (ideal) linear control dynamics, was considered. The existence of the game's solution was established and its analytical approximation was derived. The obtained numerical results show that three (and even two) iterations in the approximation of the solution to the corresponding terminal-value problem for the game-theoretic Riccati differential equation yield very accurate approximation of the game value by the guaranteed results of the corresponding pursuer's and evader's suboptimal state-feedback controls. In the future research, the presented results will be applied to a generalized zero-sum linear-quadratic differential game and to the Nash Equilibrium linear-quadratic differential game.

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