Idempotent generated algebras and Boolean powers of commutative rings

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1 Introduction

Boolean powers were introduced by Foster [5]. It was noticed by Jónsson in the review of [6], and further elaborated by Banaschewski and Nelson [1], that the Boolean power of an algebra A by a Boolean algebra B can be described as the algebra of continuous functions from the Stone space of B to A, where A has the discrete topology. It follows that a Boolean power of the group $\mathbb{Z}$ is an ℓ-group generated by its singular elements; that is, elements $g > 0$ satisfying $h \wedge (g - h) = 0$ for all h with $0 \leq h \leq g$. Conrad [4] called such ℓ-groups *Specker ℓ-groups* because they arise naturally in the study of the Baer-Specker group—the product of countably many copies of $\mathbb{Z}$. Similarly, a Boolean power of the ring $\mathbb{R}$ is an $\mathbb{R}$-algebra generated by its idempotents. In analogy with the ℓ-group case, these algebras were termed *Specker $\mathbb{R}$-algebras* in [3].

As a common generalization of these two cases, for a commutative ring R, we introduce the notion of a *Specker R-algebra* and show that Specker R-algebras are Boolean powers of R. For an indecomposable ring R, this yields an equivalence between the category of Specker R-algebras and the category of Boolean algebras. Together with Stone duality this produces a dual equivalence between the category of Specker R-algebras and the category of Stone spaces.

2 Specker R-algebras and Boolean Powers of R

Throughout R will be a commutative ring with 1, and we assume that all algebras are commutative and all algebra homomorphisms are unital (that is, preserve 1). We denote the Boolean algebra of idempotents of a ring S by $\mathrm{Id}(S)$.

We call an R-algebra S *idempotent generated* if S is generated as an R-algebra by a set of idempotents. If the idempotents belong to some Boolean subalgebra B of $\mathrm{Id}(S)$, we say that B *generates S*.

We call a nonzero idempotent e of S *faithful* if for each $a \in R$, whenever $ae = 0$, then $a = 0$. Let B be a Boolean subalgebra of $\mathrm{Id}(S)$ that generates S. We say that B is a *faithful generating algebra of idempotents of S* if each nonzero $e \in B$ is faithful.

Definition 2.1. We call an R-algebra S a *Specker R-algebra* if S is a commutative R-algebra that has a faithful generating algebra of idempotents.

To build Specker R-algebras from Boolean algebras we introduce a construction which has its roots in the work of Bergman [2] and Rota [7]. For a Boolean algebra B, let $R[B]$ be the quotient ring $R[\{x_e : e \in B\}]/I_B$ of the polynomial ring over R in variables indexed by the elements of B modulo the ideal I_B generated by the following elements, as e, f range over B:

$$x_{e \wedge f} - x_e x_f, \quad x_{e \vee f} - (x_e + x_f - x_e x_f), \quad x_{\neg e} - (1 - x_e), \quad x_0.$$

Let y_e be the image of x_e in $R[B]$. Then $R[B]$ is a Specker R-algebra with $\{y_e : e \in B\}$ a faithful generating algebra of idempotents.

Theorem 2.2. *Let S be a commutative R-algebra. The following are equivalent.*

1. *S is a Specker R-algebra.*
2. *S is isomorphic to $R[B]$ for some Boolean algebra B.*
3. *S is isomorphic to a Boolean power of R.*
4. *There is a Boolean subalgebra B of $\mathrm{Id}(S)$ such that S is generated by B and every Boolean homomorphism $B \to \mathbf{2}$ lifts to an R-algebra homomorphism $S \to R$.*

Here $\mathbf{2}$ denotes the two-element Boolean algebra. As we have noted, if $S = R[B]$ for some Boolean algebra B, then $\{y_e : e \in B\}$ is a faithful generating algebra of idempotents of S. While it is not the unique faithful generating algebra, it is unique up to isomorphism:

Theorem 2.3. *Let S be a Specker R-algebra. If B and C are both faithful generating algebras of idempotents of S, then B and C are isomorphic.*

In general, the algebra $\mathrm{Id}(R[B])$ is larger than $\{y_e : e \in B\}$, due to presence of nontrivial idempotents of R. In fact, $\mathrm{Id}(R[B])$ is isomorphic to the coproduct of $\mathrm{Id}(R)$ and B. The situation simplifies when R is *indecomposable*; that is, when $\mathrm{Id}(R) = \{0, 1\}$.

3 Specker algebras over an indecomposable ring

Lemma 3.1. *If R is indecomposable, then for each Boolean algebra B, we have $\mathrm{Id}(R[B]) = \{y_b : b \in B\}$ and $\mathrm{Id}(R[B])$ is isomorphic to B.*

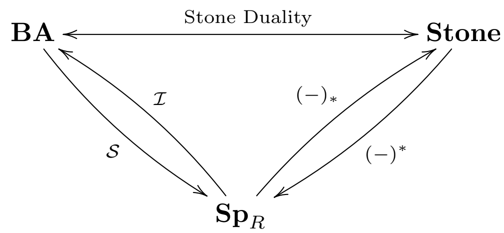
Theorem 3.2. *If R is indecomposable, then an idempotent generated commutative R-algebra S is a Specker R-algebra iff each nonzero idempotent in $\mathrm{Id}(S)$ is faithful. Consequently, if S is a Specker R-algebra, then $\mathrm{Id}(S)$ is the unique faithful generating algebra of idempotents of S.*

The considerations of the previous section give rise to two functors $\mathcal{I} : \mathbf{Sp}_R \to \mathbf{BA}$ and $\mathcal{S} : \mathbf{BA} \to \mathbf{Sp}_R$. The functor $\mathcal{I}$ associates with each $S \in \mathbf{Sp}_R$ the Boolean algebra $\mathrm{Id}(S)$ of idempotents of S, and with each R-algebra homomorphism $\alpha : S \to S'$ the restriction $\mathcal{I}(\alpha) = \alpha|_{\mathrm{Id}(S)}$ of α to $\mathrm{Id}(S)$. The functor $\mathcal{S}$ associates with each $B \in \mathbf{BA}$ the Specker R-algebra $R[B]$, and with each Boolean homomorphism $\sigma : B \to B'$ the induced R-algebra homomorphism $\alpha : R[B] \to R[B']$ that sends each y_e to $y_{\sigma(e)}$.

Lemma 3.3. *The functor $\mathcal{S}$ is left adjoint to the functor $\mathcal{I}$.*

Theorem 3.4. *The functors $\mathcal{I}$ and $\mathcal{S}$ yield an equivalence of $\mathbf{Sp}_R$ and $\mathbf{BA}$ iff R is indecomposable.*

Thus, when R is indecomposable, Theorem 3.4 and Stone duality yield a dual equivalence between $\mathbf{Sp}_R$ and the category **Stone** of Stone spaces (zero-dimensional compact Hausdorff spaces).



The functors $\mathcal{I}$ and $\mathcal{S}$ compose with the functors of Stone duality to give functors between $\mathbf{Sp}_R$ and **Stone**. The resulting contravariant functor from **Stone** to $\mathbf{Sp}_R$ is the Boolean power functor $(-)^* : \mathbf{Stone} \to \mathbf{Sp}_R$ that associates with each $X \in \mathbf{Stone}$ the Boolean power $X^* = C(X, R_{\text{disc}})$, where $C(X, R_{\text{disc}})$ is the R-algebra of continuous functions from X to the discrete space R_{disc}, and with each continuous map $\varphi : X \to Y$ the R-algebra homomorphism $\varphi^* : Y^* \to X^*$ given by $\varphi(f) = f \circ \varphi$. The functor $(-)_* : \mathbf{Sp}_R \to \mathbf{Stone}$ sends the Specker R-algebra S to the Stone space of $\text{Id}(S)$ and associates with each R-algebra homomorphism $S \to T$, the continuous map from the Stone space of $\text{Id}(T)$ to the Stone space of $\text{Id}(S)$.

We next show that for an indecomposable R, the functor $(-)_* : \mathbf{Sp}_R \to \mathbf{Stone}$ has a natural interpretation, one that does not require reference to $\text{Id}(S)$. Let S be a Specker R-algebra and let $\text{Hom}_R(S, R)$ be the set of R-algebra homomorphisms from S to R. We define a topology on $\text{Hom}_R(S, R)$ by declaring $\{U_s : s \in S\}$ as a subbasis, where $U_s = \{\alpha \in \text{Hom}_R(S, R) : \alpha(s) = 0\}$. We also recall that the Stone space of a Boolean algebra B can be described as the set $\text{Hom}(B, \mathbf{2})$ of Boolean homomorphisms from B to $\mathbf{2}$, topologized by the basis $\{Z(e) : e \in B\}$, where $Z(e) = \{\sigma \in \text{Hom}(B, \mathbf{2}) : \sigma(e) = 0\}$.

Proposition 3.5. *Let R be indecomposable and let S be a Specker R-algebra. Then* $\text{Hom}_R(S, R)$ *is homeomorphic to* $\text{Hom}(\text{Id}(S), \mathbf{2})$.

It follows that for an indecomposable R, the dual space $\text{Hom}_R(S, R)$ of a Specker R-algebra S is homeomorphic to the Stone space of $\text{Id}(S)$. This allows us to describe the contravariant functor $(-)_* : \mathbf{Sp}_R \to \mathbf{Stone}$ as follows. Associate with each $S \in \mathbf{Sp}_R$ the Stone space $S_* = \text{Hom}_R(S, R)$, and with each R-algebra homomorphism $\alpha : S \to T$, the continuous map $\alpha_* : T_* \to S_*$ given by $\alpha_*(\delta) = \delta \circ \alpha$ for each $\delta \in T_* = \text{Hom}_R(T, R)$. Thus, we have a description of $(-)_*$ that does not require passing to idempotents.

We conclude this section by giving a module-theoretic characterization of Specker R-algebras for an indecomposable R, which strengthens a result of Bergman [2, Cor. 3.5].

Theorem 3.6. *Let R be indecomposable and let S be an idempotent generated commutative R-algebra. Then the following are equivalent.*

1. *S is a Specker R-algebra.*
2. *S is a free R-module.*
3. *S is a projective R-module.*

4 Specker algebras over a domain

When R is an integral domain, Theorem 3.6 can be strengthened as follows.

Proposition 4.1. *Let R be a domain and let S be an idempotent generated commutative R-algebra. Then S is a Specker R-algebra iff S is a torsion-free R-module.*

We recall the well-known definition of a Baer ring and a weak Baer ring in the case of a commutative ring.

Definition 4.2. A commutative ring R is a *Baer ring* if the annihilator ideal of each subset of R is a principal ideal generated by an idempotent, and R is a *weak Baer ring* if the annihilator ideal of each element of R is a principal ideal generated by an idempotent.

Theorem 4.3. *Let S be a Specker R-algebra. Then S is Baer iff S is weak Baer and* $\mathrm{Id}(S)$ *is a complete Boolean algebra.*

Corollary 4.4. *Let R be indecomposable and let S be a Specker R-algebra. Then S is Baer iff R is a domain and* $\mathrm{Id}(S)$ *is a complete Boolean algebra.*

Theorem 4.5. *If R is a domain and S is a Specker R-algebra, then S_* is homeomorphic to the space* $\mathrm{Min}(S)$ *of minimal prime ideals of S.*

Let $\mathbf{BSp}_R$ be the full subcategory of $\mathbf{Sp}_R$ consisting of Baer Specker R-algebras, let **cBA** be the full subcategory of **BA** consisting of complete Boolean algebras, and let **ED** be the full subcategory of **Stone** consisting of extremally disconnected spaces.

Theorem 4.6.

1. *When R is a domain, the categories* $\mathbf{BSp}_R$ *and* **cBA** *are equivalent.*
2. *When R is a domain, the categories* $\mathbf{BSp}_R$ *and* **ED** *are dually equivalent.*

Since injectives in **BA** are exactly the complete Boolean algebras, as an immediate consequence of Theorem 4.6, we obtain:

Corollary 4.7. *When R is a domain, the injective objects in* $\mathbf{Sp}_R$ *are the Baer Specker R-algebras.*

References

[1] B. Banaschewski and E. Nelson, *Boolean powers as algebras of continuous functions*, Dissertationes Math. (Rozprawy Mat.) **179** (1980), 51.

[2] G. M. Bergman, *Boolean rings of projection maps*, J. London Math. Soc. **4** (1972), 593–598.

[3] G. Bezhanishvili, P. J. Morandi, and B. Olberding, *Bounded Archimedean ℓ-algebras and Gelfand-Neumark-Stone duality*, Theory and Applications of Categories, 2013, to appear.

[4] P. Conrad, *Epi-archimedean groups*, Czechoslovak Math. J. **24 (99)** (1974), 192–218.

[5] A. L. Foster, *Generalized "Boolean" theory of universal algebras. I. Subdirect sums and normal representation theorem*, Math. Z. **58** (1953), 306–336.

[6] ______, *Functional completeness in the small. Algebraic structure theorems and identities*, Math. Ann. **143** (1961), 29–58.

[7] G.-C. Rota, *The valuation ring of a distributive lattice*, Proceedings of the University of Houston Lattice Theory Conference (Houston, Tex., 1973), Dept. Math., Univ. Houston, Houston, Tex., 1973, pp. 574–628.