Time Fractional Heat Conduction Problem in a Thin Hollow Circular Disk and It's Thermal Deflection

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TIME FRACTIONAL HEAT CONDUCTION PROBLEM IN A THIN HOLLOW CIRCULAR DISK AND IT'S THERMAL DEFLECTION

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Abstract. In this paper, we analysed 2D problem of thermoelasticity with the fractional order derivative of order $0 < \alpha \leq 2$ and determine temperature and thermal deflection of a circular disk. The zero initial condition are assumed. The inner and outer circular edges are kept at temperatures $f_1(z,t)$ and $f_2(z,t)$. The lower and upper surfaces are kept at temperatures $f_3(r,t)$ and $f_4(r,t)$. The solution is obtained applying Laplace, finite Fourier and Hankel transforms. Numerical results are illustrated graphically with the help of PTC Mathcad software.

Keywords: Capito Fractional Derivative, Heat Conduction, Thermal Deflection, Hollow Circular Disk, Mittag-Leffler Functions.

1. **Introduction**

Biot [1] introduced the generalization of the classical coupled thermoelasticity theory. Green and Nagdhi [2] introduced the thermoelastic material behavior without energy dissipation with linear and nonlinear theories. Povstenko [3] solved the heat conduction problem with time fractional derivative and its thermal stresses. Sherief et. al. [4] discussed the coupled and generalized theory of thermoelasticity with some limiting cases using the method of fractional calculus. Sur and Kanoria [5,8] developed the new theory of thermoelastic distribution of two temperature with new heat conduction equation and FGVM with fractional order. Gaikwad et. al. [6, 7] solved the nonhomogeneous thermoelastic problem of thermal deflection with internal heat generation and thermoelastic deformation with partially distributed heat supply in a circular disk. Raslan [9] discussed the 2D problem of fractional order thermoelasticity in a circular plate with temperature distribution is axisymmetric. Gaikwad [10] solved 2D problem of thermoelasticty under steady state temperature distribution of thin plate with internal heat generation. Gaikwad [11] have been solved axisymmetric thermoelastic temperature distribution of thin circular plate with internal heat source. Some contribution of these theory have been discussed in [12–33].

In this paper, the work of Gaikwad et. al. has been modified and prepare a new thermoelastic model with time fractional derivative of order $0 < \alpha \leq 2$ and

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use of Mittag-Leffler function and determined the temperature, thermal deflection in a thin hollow circular plate. The zero initial condition are assumed. The inner and outer circular edges are kept at temperatures $f_1(z,t)$ and $f_2(z,t)$. The lower and upper surfaces are kept at temperatures $f_3(r,t)$ and $f_4(r,t)$. The solution is obtained applying Laplace, finite Fourier and Hankel transforms. Numerical results are illustrated graphically with the help of PTC Mathcad software.

2. Formulation of The Problem

We consider a 2D problem for a thin hollow circular disk with zero initial condition occupying the region $a \le r \le b$, $0 \le z \le h$. The inner and outer circular edges are kept at temperatures $f_1(z,t)$ and $f_2(z,t)$. The lower and upper surfaces are kept at temperatures $f_3(r,t)$ and $f_4(r,t)$. The mathematical model is constructed for nonlocal Caputo type time fractional heat conduction equation of order α for a thin hollow disk and temperature and thermal deflection are required to be determined.

The Caputo type fractional derivative given by [36]

$$D^{\alpha} f(t) = \begin{cases} \dfrac{1}{\Gamma(n-\alpha)} \int_0^t \dfrac{f^n(\tau)}{(t-\tau)^{\alpha+1-n}} d\tau, & n-1 < \alpha < n; \\ \dfrac{df(t)}{dt}, & n = 1. \end{cases}$$

For finding the Laplace transform, the Caputo derivative requires information of the initial values of the function $f(t)$ and its integer derivative of the order $k = 1, 2, ..., n-1$

$$L\{D^{\alpha} f(t); s\} = s^{\alpha} F(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1} f^{(k)}(0), \qquad n-1 < \alpha < n$$

The governing heat conduction equation in the form of fractional order parameter for a thin hollow circular disk satisfies the differential equation,

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2} = \frac{1}{k}\frac{\partial^{\alpha} T}{\partial t^{\alpha}} \qquad \text{in } a \le r \le b,\ 0 \le z \le h, \text{ for } t > 0, \tag{1}$$

with the boundary conditions,

$$T = f_1(z,t), \qquad \text{at } r = a, \text{ for } t > 0, \tag{2}$$

$$T = f_2(z,t), \qquad \text{at } r = b, \text{ for } t > 0, \tag{3}$$

$$T = f_3(r,t), \qquad \text{at } z = 0, \text{ for } t > 0, \tag{4}$$

$$T = f_4(r,t), \qquad \text{at } z = h, \text{ for } t > 0. \tag{5}$$

The zero initial condition are assumed,

$$T = 0, \qquad \text{at } t = 0,\ 0 < \alpha \le 2\ , \tag{6}$$

$$\frac{\partial T}{\partial t} = 0, \qquad \text{at } t = 0,\ 1 < \alpha \le 2\ . \tag{7}$$

The differential equation satisfied the deflection function $w(r,t)$ defined in [6] as

$$\nabla^2\nabla^2 w = -\frac{1}{(1-\nu)D}\nabla^2 M_T \tag{8}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} \tag{9}$$

and M_T is the thermal moment of the disk, ν is the Poisson's ratio of the disk material, D is the flexural rigidity of the disk denoted by

$$D = \frac{Eh^3}{12(1-\nu^2)}, \tag{10}$$

The term M_T is defined in [6] as

$$M_T = a_t E \int_0^h \left(z - \frac{h}{2}\right) T(r,z,t)dz, \tag{11}$$

a_t and E are the coefficients of the linear thermal expansion and the Young modulus respectively.

Since the inner and outer edges of the hollow circular disk are clamped,

$$w = 0 \qquad \text{at } r = a \text{ and } r = b, \tag{12}$$

Initially, $T = w = 0$, at $t = 0$.
Equations (1) to (12) constitute the mathematical formulation of the problem under consideration.

3. Solution of The Heat Conduction Problem

Firstly we define the finite Fourier transform and their inverse transform over the variable z in the range $0 \le z \le h$ defined in [34] as

$$\overline{T}(r,\eta_p,t) = \int_{z'=0}^{h} K(\eta_p,z').T(r,z',t).dz' \tag{13}$$

$$T(r,z,t) = \sum_{n=1}^{\infty} K(\eta_p,z).\overline{T}(r,\eta_p,t) \tag{14}$$

where

$$K(\eta_p,z) = \sqrt{\frac{2}{h}}\sin(\eta_p z).$$

and $\eta_1, \eta_2, \ldots$ are the positive roots of the transcendental equation

$$\sin(\eta_p h) = 0, \quad p = 1,2,3,\ldots$$

i.e.

$$\eta_p = \frac{p\pi}{h}, \quad p = 1,2,3,\ldots$$

Applying the finite Fourier transform with respect to the axial coordinates z, defined in equation (13) to equation (1) and using the conditions (2)-(7), one obtains

$$\frac{\partial^2 \overline{T}}{\partial r^2} + \frac{1}{r}\frac{\partial \overline{T}}{\partial r} - \eta_p^2 \overline{T} = \frac{1}{k}\frac{\partial^\alpha \overline{T}}{\partial t^\alpha} \tag{15}$$

with

$$\overline{T} = \overline{f}_1(\eta_p, t), \qquad \text{at } r = a, \text{ for } t > 0, \tag{16}$$

$$\overline{T} = \overline{f}_2(\eta_p, t), \qquad \text{at } r = b, \text{ for } t > 0, \tag{17}$$

$$\overline{T} = 0, \qquad \text{at } t = 0,\ 0 < \alpha \le 2, \tag{18}$$

$$\frac{\partial \overline{T}}{\partial t} = 0, \qquad \text{at } t = 0,\ 0 < \alpha \le 2\ , \tag{19}$$

where $\overline{T} = T(r, \eta_p, t)$.

Secondly, we define finite Hankel transform and their inverse transform over the variable r in the range $a \le r \le b$ as defined in [34] as,

$$\overline{\overline{T}}(\beta_m, \eta_p, t) = \int_{r'=a}^{b} r'.K_0(\beta_m, r').T(r', \eta_p, t).dr' \tag{20}$$

$$\overline{T}(r, \eta_p, t) = \sum_{m=1}^{\infty} K_0(\beta_m, r).\overline{\overline{T}}(\beta_m, \eta_p, t) \tag{21}$$

where

$$K_0(\beta_m, r) = \frac{\pi}{\sqrt{2}} \frac{\beta_m J_0(\beta_m b).Y_0(\beta_m b)}{\left[1 - \dfrac{J_0^2(\beta_m b)}{J_0^2(\beta_m a)}\right]^{1/2}} \left[\frac{J_0(\beta_m r)}{J_0(\beta_m b)} - \frac{Y_0(\beta_m r)}{Y_0(\beta_m b)}\right]$$

and $\beta_1,\ \beta_2,\ \beta_3,\ \ldots$ are the positive root of transcendental equation

$$\frac{J_0(\beta a)}{J_0(\beta b)} - \frac{Y_0(\beta a)}{Y_0(\beta b)} = 0.$$

Applying the finite Hankel transform with respect to the radial coordinate r, defined in equation (20) to equation (15) and using the conditions (16)-(20), one obtains

$$\frac{\partial^\alpha \overline{\overline{T}}(\beta_m, \eta_p, t)}{\partial t^\alpha} + k(\beta_m^2 + \eta_p^2)\overline{\overline{T}}(\beta_m, \eta_p, t) = A(\beta_m, \eta_p, t) \tag{22}$$

$$\overline{\overline{T}}(\beta_m, \eta_p, t) = 0, \qquad \text{for } t = 0, \tag{23}$$

$$\frac{\overline{\overline{T}}(\beta_m, \eta_p, t)}{\partial t} = 0, \qquad \text{for } t = 0, \tag{24}$$

where

$$\begin{aligned} A(\beta_m, \eta_p, t) = k\Bigg\{ & a\frac{dK_0(\beta_m, r)}{dr}\overline{f}_1(\eta_p, t)\Big|_{r=a} - b\frac{dK_0(\beta_m, r)}{dr}\overline{f}_2(\eta_p, t)\Big|_{r=b} \\ & + \frac{dK_0(\eta_p, z)}{dz}\overline{f}_3(\beta_m, t)\Big|_{z=0} + \frac{dK_0(\eta_p, z)}{dz}\overline{f}_4(\beta_m t)\Big|_{z=h} \Bigg\} \end{aligned} \tag{25}$$

Applying the Laplace transform with respect to time t and their inverse the solution of the equation (22) is obtained as

$$\overline{\overline{T}}(\beta_m, \eta_p, t) = \frac{A(\beta_m, \nu_n, t)}{k(\beta_m^2 + \nu_n^2)}[1 - E_\alpha(-k(\beta_m^2 + \nu_n^2)t^\alpha)] \tag{26}$$

Here $E_\alpha(.)$ represents the Mittag-Leffler function.

Finally taking inverse finite Hankel transform defined in equation (21) and inverse finite Fourier transform defined in equation (14), the expressions of the temperature $T(r, z, t)$ as

$$T(r, z, t) = \sum_{p=1}^{\infty}\sum_{m=1}^{\infty} K(\eta_p, z)K_0(\beta_m, r)\frac{1}{k(\beta_m^2 + \eta_p^2)}[1 - E_\alpha(-k(\beta_m^2 + \eta_p^2)t^\alpha)].b_{pm} \tag{27}$$

where

$$\begin{aligned} b_{pm} = \Bigg[& k.a.K_1(\beta_m, a)\int_{z'=0}^{h} K(\eta_p, z').f_1(z', t').dz' \\ & - k.b.K_1(\beta_m, b)\int_{z'=0}^{h} K(\eta_p, z').f_2(z', t').dz' \\ & + \sqrt{\frac{2}{\pi}}k.\eta_p.\int_{r'=a}^{b} r'.K_0(\beta_m, r').f_3(r', t').dr' \\ & + \sqrt{\frac{2}{\pi}}k.\eta_p.\cos(\eta_p h).\int_{r'=a}^{b} r'.K_0(\beta_m, r').f_4(r', t').dr' \Bigg] \end{aligned}$$

4. **Determination of Thermal Deflection**

Assume the solution of (8) satisfying conditions (12) as

$$w(r, t) = \sum_{m=1}^{\infty} C_m(t)\left[\frac{J_0(\beta_m r)}{J_0(\beta_m b)} - \frac{Y_0(\beta_m r)}{Y_0(\beta_m b)}\right] \tag{28}$$

where $\beta_1,\ \beta_2,\ \beta_3,\ \ldots$ are the positive root of transcendental equation

$$\frac{J_0(\beta a)}{J_0(\beta b)} - \frac{Y_0(\beta a)}{Y_0(\beta b)} = 0.$$

It can be easily shown that

$$w = 0 \qquad \text{at } r = a \text{ and } r = b, \tag{29}$$

Hence the solution (28) satisfies the condition (12).

Now,

$$\nabla^2\nabla^2 w = \left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r}\right)^2 \sum_{m=1}^{\infty} C_m(t)\left[\frac{J_0(\beta_m r)}{J_0(\beta_m b)} - \frac{Y_0(\beta_m r)}{Y_0(\beta_m b)}\right] \tag{30}$$

Using the well-known result

$$\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r}\right) J_0(\beta_m r) = -\beta_m^2 J_0(\beta_m r) \tag{31}$$

$$\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r}\right) Y_0(\beta_m r) = -\beta_m^2 Y_0(\beta_m r) \tag{32}$$

in equation (30), one obtains

$$\nabla^2\nabla^2 w = \sum_{m=1}^{\infty} C_m(t)\beta_m^4 \left[\frac{J_0(\beta_m r)}{J_0(\beta_m b)} - \frac{Y_0(\beta_m r)}{Y_0(\beta_m b)}\right] \tag{33}$$

Using equation (27) in equation (11), one obtains

$$\begin{aligned} M_T = -\sqrt{\frac{2}{h}} a_t E h \sum_{p=1}^{\infty}\sum_{m=1}^{\infty} \frac{[\cos(\eta_p h)+1]}{\eta_p} K_0(\beta_m, r) \\ \frac{1}{k(\beta_m^2+\eta_p^2)}[1 - E_\alpha(-k(\beta_m^2+\eta_p^2)t^\alpha)].b_{pm} \end{aligned} \tag{34}$$

Now,

$$\begin{aligned} \nabla^2 M_T = -\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r}\right) \frac{1}{\sqrt{2\pi}} a_t E h \sum_{p=1}^{\infty}\sum_{m=1}^{\infty} \frac{[\cos(\eta_p h)+1]}{\eta_p} K_0(\beta_m, r) \\ \frac{1}{k(\beta_m^2+\eta_p^2)}[1 - E_\alpha(-k(\beta_m^2+\eta_p^2)t^\alpha)].b_{pm} \end{aligned} \tag{35}$$

solving equation (35), one obtains

$$\begin{aligned} \nabla^2 M_T = \frac{1}{\sqrt{2\pi}} a_t E h \sum_{p=1}^{\infty}\sum_{m=1}^{\infty} \frac{[\cos(\eta_p h)+1]}{\eta_p} \beta_m^2 K_0(\beta_m, r) \\ \frac{1}{k(\beta_m^2+\eta_p^2)}[1 - E_\alpha(-k(\beta_m^2+\eta_p^2)t^\alpha)].b_{pm} \end{aligned} \tag{36}$$

Substituting equation (33) and (36) into equation (8), one obtains

$$\begin{aligned} \sum_{m=1}^{\infty} C_m(t)\beta_m^4 \left[\frac{J_0(\beta_m r)}{J_0(\beta_m b)} - \frac{Y_0(\beta_m r)}{Y_0(\beta_m b)}\right] = -\frac{1}{\sqrt{2\pi}} a_t E h \frac{1}{(1-\nu)D} \sum_{p=1}^{\infty}\sum_{m=1}^{\infty} \frac{[\cos(\eta_p h)+1]}{\eta_p} \beta_m^2 \\ \times \frac{\pi}{\sqrt{2}} \frac{\beta_m J_0(\beta_m b).Y_0(\beta_m b)}{\left[1 - \frac{J_0^2(\beta_m b)}{J_0^2(\beta_m a)}\right]^{1/2}} \left[\frac{J_0(\beta_m r)}{J_0(\beta_m b)} - \frac{Y_0(\beta_m r)}{Y_0(\beta_m b)}\right] \\ \frac{1}{k(\beta_m^2+\eta_p^2)}[1 - E_\alpha(-k(\beta_m^2+\eta_p^2)t^\alpha)].b_{pm} \end{aligned} \tag{37}$$

Solving equation (37), one obtains

$$\begin{aligned} C_m(t) = -\frac{1}{\sqrt{2\pi}} a_t E h \frac{1}{(1-\nu)D} \sum_{p=1}^{\infty}\sum_{m=1}^{\infty} \frac{[\cos(\eta_p h)+1]}{\eta_p} \frac{1}{\beta_m^2} \\ \times \frac{\pi}{\sqrt{2}} \frac{\beta_m J_0(\beta_m b).Y_0(\beta_m b)}{\left[1 - \frac{J_0^2(\beta_m b)}{J_0^2(\beta_m a)}\right]^{1/2}} \\ \frac{1}{k(\beta_m^2+\eta_p^2)}[1 - E_\alpha(-k(\beta_m^2+\eta_p^2)t^\alpha)].b_{pm} \end{aligned} \tag{38}$$

Finally, substituting equation (38) in equation (28), one obtains the expression for the quasi-static thermal deflection $w(r,t)$ as

$$w(r,t) = -\frac{1}{\sqrt{2\pi}} a_t E h \frac{1}{(1-\nu)D} \sum_{p=1}^{\infty}\sum_{m=1}^{\infty} \frac{[\cos(\eta_p h)+1]}{\eta_p}\frac{1}{\beta_m} \times \frac{\pi}{\sqrt{2}} \frac{J_0(\beta_m b).Y_0(\beta_m b)}{\left[1-\dfrac{J_0^2(\beta_m b)}{J_0^2(\beta_m a)}\right]^{1/2}} \left[\frac{J_0(\beta_m r)}{J_0(\beta_m b)} - \frac{Y_0(\beta_m r)}{Y_0(\beta_m b)}\right] \frac{1}{k(\beta_m^2+\eta_p^2)}[1-E_\alpha(-k(\beta_m^2+\eta_p^2)t^\alpha)].b_{pm} \tag{39}$$

5. Numerical Results and Discussion

Setting
$f_1(z,t) = f_2(z,t) = (z^2-h^2)^2.e^{-At}$
$f_3(r,t) = f_4(r,t) = (r^2-a^2)^2(r^2-b^2)^2.e^{-At}$
where $A > 0$.

Dimension

The constants associated with the numerical calculation are taken as
Inner radius of a circular disk $a = 1$ m,
Outer radius of a circular disk $b = 2$ m,
Thickness of circular disk $h = 0.1$ m,

Roots of the transcendental equation

The first five positive root of the transcendental equation $\frac{J_0(\beta a)}{J_0(\beta b)} - \frac{Y_0(\beta a)}{Y_0(\beta b)} = 0$ as defined in [34] are $\beta_1 = 3.1965$, $\beta_2 = 6.3123$, $\beta_3 = 9.4445$, $\beta_4 = 12.5812$, $\beta_5 = 15.7199$.

Material Properties

The cooper material was chosen for purpose of numerical calculation for a thin circular hollow disk as
Thermal diffusivity $k = 4.42$ m^2/s
Density $\rho = 558$ kg/m^3
Specific heat $c_p = 0.091$ J/(kg K)
Poisson ratio $\nu = 0.36$
Coefficient of linear thermal expansion $a_t = 16.5\times10^{-6}$ /K
Young's modulus $E = 117$ GP$_a$

The numerical calculation are carried out according to the values of parameter α reflecting the characteristic features of the solution for various order of the fractional derivative. There distinguishing values of the parameter α are considered, $0 < \alpha < 1$, $\alpha = 1$ and $0 < \alpha \leq 2$ depicting weak, normal and strong conductivity.

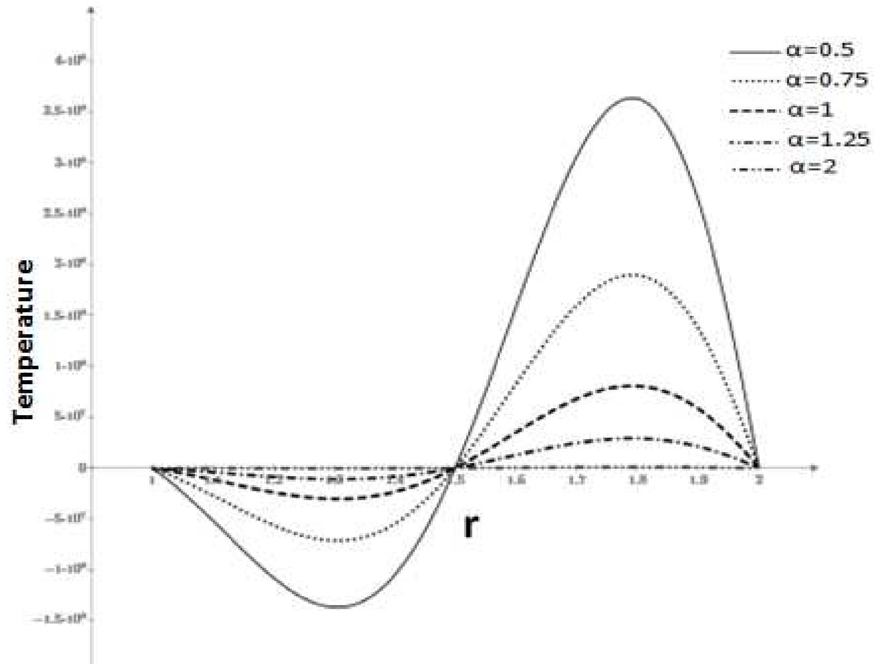

Figure 1. Temperature distribution for different values of α.

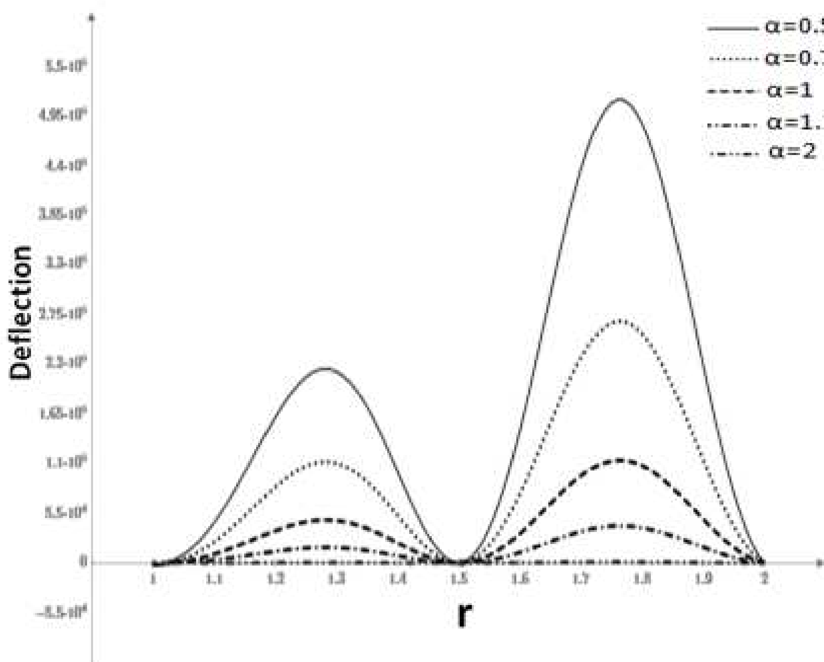

Figure 2. Thermal Deflection for different values of α.

Figure 1, shows the variation of temperature along radial direction for the different values of fractional order parameters $\alpha = 0.5, 0.75, 1, 1.25, 2$. The temperature decreases within the annular region $1 \leq r \leq 1.5$ and increases within the circular region $1.5 \leq r \leq 2$. The temperature is zero at both the inner and outer radii $r = 1$ and $r = 2$. Also it is zero at the center $r = 1.5$ of the annular disk. We observe that, the speed of propagation of the thermal signals is directly proportional to the values of fractional order parameter α within the annular region $1 \leq r \leq 1.5$ and inversely proportional to the circular region $1.5 \leq r \leq 2$.

Figure 2, shows the variation of thermal deflection along radial direction for the different values of fractional order parameters $\alpha = 0.5, 0.75, 1, 1.25, 2$. It is clear that, the thermal deflection is maximum within the regions $1 \leq r \leq 1.5$ and $1.5 \leq r \leq 2$. The temperature is zero at both the inner and outer edges $r = 1$ and $r = 2$. Also it is zero at the center $r = 1.5$ of the annular disk. We observe that, the speed of propagation of the thermal signals is inversely proportional to the values of fractional order parameter α.

6. **Conclusion**

We investigate the temperature and thermal deflection in a thin hollow circular disk in a theory of thermoelasticity based on fractional heat conduction with the Caputo time-fractional derivative of order $0 < \alpha < 2$. The present method is based on the direct method, using the finite Hankel transform, the generalized finite Fourier transform and Laplace transform.The numerical results shows the significant influence of the order of time derivative on the temperature as well as thermal deflection with radial coordinate. The time fractional order parameter α within the range $0 < \alpha < 1$ and $1 < \alpha < 2$ represent the weak and strong conductivity, while $\alpha = 1$ represents the normal conductivity. The results presented here will be more useful in studying the thermal characteristics of circular bodies in real-life engineering problems, mathematical biology by considering the fractional derivative in the field equations.

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