Model Order Reduction – Also with Many Inputs and Outputs?

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ABSTRACT

The Finite Element Method is a widely used discretization method for mechanical systems. To ensure accurate simulations on the one hand and to ensure an acceptable simulation time on the other hand, Model Order Reduction is often applied to reduce the mathematical complexity of the system. One possible projection based Model Order Reduction method for linear systems is moment matching based on Krylov subspaces. In this method, the size of the reduced order model is directly proportional to the number of inputs and outputs of the system. Therefore, tangential directions can be applied to reduce the number of inputs used for Model Order Reduction. In this contribution, we examine the suitability of the Singular Value Decomposition Model Order Reduction (SVDMOR) method for mechanical systems. SVDMOR is commonly used for electrical circuit simulation and reduces the number of inputs and outputs based on a singular value decomposition of the transfer function. The behavior of the singular values, which correspond to the error between the full order model and the model with a reduced number of inputs and outputs, as well as the tangential directions are investigated in the frequency domain on a numerical example.

1 INTRODUCTION

The increasing demand for cost reduction and material savings makes computer simulations of complex mechanical systems indispensable. The linear Finite Element Method is a widely used discretization method. This method enables the determination of the eigenfrequencies and eigenmodes of the system, but also to carry out simulations in the time and frequency domain. These results can then, for example, be used for design optimization or controller design. From a mathematical point of view, the Finite Element Method results in a set of ordinary differential equations (ODE). Depending on the required accuracy of the model, the dimension of the ODE can easily reach up to one million or even more. Model Order Reduction is a powerful tool to enormously reduce the computational time to solve the ODE while keeping the main properties of the system. The idea is to approximate the system dynamics in a lower dimensional subspace. A popular method is moment matching based on Krylov subspaces. It matches the input to output behavior between the full order model and the reduced order model at so-called expansion points in the frequency domain. The dimension of the reduced order model directly depends on the number of inputs and outputs of the system. However, the number of inputs can become very large for example for systems with surface inputs or when multiple physical domains are coupled in co-simulations. Tangential interpolation reduces the number of inputs and outputs that are used for Model Order Reduction and weakens the aforementioned interpolation condition. The Singular Value Decomposition Model Order Reduction (SVDMOR) has been developed in [10] and was successfully applied and extended for the simulation of electrical circuits with many terminals in [18, 6, 7, 21]. A detailed investigation of the suitability for Finite Element Models of mechanical systems is still pending.

This contribution aims at identifying the potential of SVDMOR in the context of Model Order Reduction of mechanical Finite Element models. It gives insights into the error approximation in the frequency domain and on the choice of the tangential directions.

This paper is structured as follows. In Section 2, we first introduce the basics about ordinary differential equations to describe Finite Element models, before we explain projection based Model Order Reduction and specifically Model Order Reduction using moment matching with Krylov subspaces and tangential interpolation, and introduce some error measures to determine the accuracy of different models. Section 3 investigates the SVDMOR to reduce the number of inputs and outputs in combination with classical Model Order Reduction based on moment matching for a mechanical example.

2 THEORETICAL BACKGROUND

In this section, the theoretical background on system description with Finite Element models, Model Order Reduction in general, and especially the method of moment matching and tangential directions using Singular Value Decomposition Model Order Reduction is explained.

2.1 System description

For complex mechanical systems it is often very difficult or mostly impossible to find an analytical solution. Therefore, these are typically spatially discretized using the linear Finite Element Method (FEM), see [4, 26] for details about FEM. The resulting linear second-order ODE

$$\begin{aligned} \boldsymbol{M}\ddot{\boldsymbol{q}}(t) + \boldsymbol{D}\dot{\boldsymbol{q}}(t) + \boldsymbol{K}\boldsymbol{q}(t) &= \boldsymbol{B}\boldsymbol{u}(t), \\ \boldsymbol{y}(t) &= \boldsymbol{C}\boldsymbol{q}(t) \end{aligned} \tag{1}$$

with initial conditions

$$\boldsymbol{q}(0) = \boldsymbol{q}_0, \quad \dot{\boldsymbol{q}}(0) = \dot{\boldsymbol{q}}_0, \quad t \geq 0 \tag{2}$$

is called equation of motion. This equation contains the time-dependent vector of nodal displacements $\boldsymbol{q}(t) \in \mathbb{R}^N$ and its first and second derivatives with respect to time, $\dot{\boldsymbol{q}}(t)$ and $\ddot{\boldsymbol{q}}(t)$. It has N degrees of freedom. The mass and stiffness matrices $\boldsymbol{M}, \boldsymbol{K} \in \mathbb{R}^{N\times N}$ contain information about the mass distribution and material properties whereas the damping matrix $\boldsymbol{D} \in \mathbb{R}^{N\times N}$ is used to model dissipative effects. A commonly used damping model is the so-called Rayleigh damping $\boldsymbol{D} = \alpha\boldsymbol{M} + \beta\boldsymbol{K}$ with the damping constants $\alpha, \beta \in \mathbb{R}^+$. The input vector $\boldsymbol{u}(t) \in \mathbb{R}^b$ is mapped to the nodal displacements by the input matrix $\boldsymbol{B} \in \mathbb{R}^{N\times b}$ whereas the output matrix $\boldsymbol{C} \in \mathbb{R}^{c\times N}$ maps the nodal displacements to the output vector $\boldsymbol{y}(t) \in \mathbb{R}^c$.

Transferring the equation of motion (1) into the Laplace domain with the complex Laplace variable $s \in \mathbb{C}$ and setting the initial conditions to $\boldsymbol{q}(0) = \dot{\boldsymbol{q}}(0) = \boldsymbol{0}$ results in the transfer function

$$\boldsymbol{H}(s) = \frac{\boldsymbol{Y}(s)}{\boldsymbol{U}(s)} = \boldsymbol{C}\left(s^2\boldsymbol{M} + s\boldsymbol{D} + \boldsymbol{K}\right)^{-1}\boldsymbol{B} \in \mathbb{C}^{c\times b}. \tag{3}$$

The transfer function represents a mapping from the inputs in the Laplace domain $\boldsymbol{U}(s) \in \mathbb{C}^b$ to the outputs in the Laplace domain $\boldsymbol{Y}(s) \in \mathbb{C}^c$. As described in [20], the transfer function is often evaluated along the imaginary axis which is then called frequency response $\boldsymbol{H}(\mathrm{i}\omega)$ with the angular frequency $\omega \in \mathbb{R}$ and the imaginary unit i. For systems with multiple inputs and multiple outputs (MIMO-systems), the Frobenius norm

$$\|\boldsymbol{H}(s)\|_\mathrm{F} = \sqrt{\sum_{i=1}^{c}\sum_{j=1}^{b}|H_{i,j}(s)|^2} = \sqrt{\sum_{i=1}^{m}\sigma_i^2(s)} \tag{4}$$

or the spectral norm

$$\|\boldsymbol{H}(s)\|_2 = \sigma_1(s) \tag{5}$$

can be taken into account to have a frequency dependent scalar function of the input to output behavior of the system. Here, $\sigma_i \in \mathbb{R}$ are the singular values of the transfer matrix and σ_1 is the largest singular value. The number of singular values is $m = \min(b, c)$.

2.2 Model Order Reduction with Krylov subspaces and tangential interpolation

Depending on the complexity of the model and on the requirements on the simulation accuracy, the dimension N of the system matrices can easily become very large, potentially exceeding millions. As a consequence, the simulation time can become infeasible. In such cases, it is possible to apply Model Order

Reduction in order to achieve a compromise between the simulation time and the model accuracy. Projection based Model Order Reduction (MOR) aims at finding a reduced order model

$$\begin{aligned} \boldsymbol{M}_{\mathrm{red}}\ddot{\boldsymbol{q}}_{\mathrm{red}}(t)+\boldsymbol{D}_{\mathrm{red}}\dot{\boldsymbol{q}}_{\mathrm{red}}(t)+\boldsymbol{K}_{\mathrm{red}}\boldsymbol{q}_{\mathrm{red}}(t)&=\boldsymbol{B}_{\mathrm{red}}\boldsymbol{u}(t),\\ \bar{\boldsymbol{y}}(t)&=\boldsymbol{C}_{\mathrm{red}}\boldsymbol{q}_{\mathrm{red}}(t) \end{aligned} \tag{6}$$

with initial conditions

$$\boldsymbol{q}_{\mathrm{red}}(0)=\boldsymbol{W}^{\mathsf{T}}\boldsymbol{q}(0),\quad \dot{\boldsymbol{q}}_{\mathrm{red}}(0)=\boldsymbol{W}^{\mathsf{T}}\dot{\boldsymbol{q}}(0),\quad t\geq 0 \tag{7}$$

by approximating the nodal displacements in a low dimensional subspace $\mathcal{V}=\mathrm{span}(\boldsymbol{V})$ with

$$\boldsymbol{q}(t)\approx\boldsymbol{V}\boldsymbol{q}_{\mathrm{red}}(t). \tag{8}$$

In Equations (6) and (8), $\boldsymbol{q}_{\mathrm{red}}(t)\in\mathrm{R}^n$ is the reduced order state vector having $n\ll N$ degrees of freedom,

$$\begin{aligned} &\boldsymbol{M}_{\mathrm{red}}=\boldsymbol{W}^{\mathsf{T}}\boldsymbol{M}\boldsymbol{V},\quad \boldsymbol{D}_{\mathrm{red}}=\boldsymbol{W}^{\mathsf{T}}\boldsymbol{D}\boldsymbol{V},\quad \boldsymbol{K}_{\mathrm{red}}=\boldsymbol{W}^{\mathsf{T}}\boldsymbol{K}\boldsymbol{V},\\ &\boldsymbol{B}_{\mathrm{red}}=\boldsymbol{W}^{\mathsf{T}}\boldsymbol{B},\quad \boldsymbol{C}_{\mathrm{red}}=\boldsymbol{C}\boldsymbol{V} \end{aligned} \tag{9}$$

are the reduced order system matrices, and $\bar{\boldsymbol{y}}(t)\in\mathbb{R}^c$ is the output of the reduced order system. The matrices $\boldsymbol{V}\in\mathbb{R}^{N\times n}$ and $\boldsymbol{W}\in\mathbb{R}^{N\times n}$ are the biorthogonal (i.e. $\boldsymbol{W}^{\mathsf{T}}\boldsymbol{V}=\boldsymbol{I}$) reduction matrices. If $\boldsymbol{W}=\boldsymbol{V}$, the projection is called orthogonal projection or Galerkin projection, otherwise it is called skew projection or Petrov-Galerkin projection. Since the orthogonal projection maintains symmetry and stability properties, only orthogonal projection is used in this work. See [9] for some investigations on skew projection for mechanical systems.

There exist multiple methods for projective Model Order Reduction for linear time-invariant systems that seek to find a subspace that is as small as possible on the one hand and, on the other hand, minimizes the error between the full order model and the reduced order model in the desired domain and norm. Well-known methods are based on modal truncation, balanced truncation or rational interpolation, also called moment matching, see e.g. [2, 24] or especially [19, 23] for an overview on Model Order Reduction for mechanical systems.

In this work, we focus on moment matching methods, that are widely used for mechanical engineering problems, see [17]. Moment matching is based on matching the transfer function and, if desired, also the derivates of the transfer function with respect to the complex Laplace variable of the full order model and the reduced order model. The resulting property can be expressed as

$$\left.\frac{\partial^i\boldsymbol{H}(s)}{\partial s^i}\right|_{s=\hat{s}_k}=\left.\frac{\partial^i\boldsymbol{H}_{\mathrm{red}}(s)}{\partial s^i}\right|_{s=\hat{s}_k}, \tag{10}$$

where $s=\hat{s}_k$ are specific frequency expansion points that have to be chosen by the user or in iterative process, see e.g. [15] for point placement of expansion points or [16, 11] for automated iterative shift selection. In general, Krylov subspaces are used to produce reduction matrices that fulfil the property given in Equation (10). A second order Krylov subspace is for the expansion point $\hat{s}_k\in\mathbb{C}$ up to the moment $J_k\in\mathbb{N}$ is explained in [24, 12] and is given by

$$\begin{aligned} {}_{\hat{s}_k}\mathcal{K}_{J_k}(\boldsymbol{A}_1,\boldsymbol{A}_2,\boldsymbol{G}_1)&=\mathrm{span}(\boldsymbol{R}_0,\boldsymbol{R}_1,\ldots,\boldsymbol{R}_{J_k-1})\\ \text{with}\quad \boldsymbol{R}_0&=\boldsymbol{G}_1,\\ \boldsymbol{R}_1&=\boldsymbol{A}_1\boldsymbol{R}_0,\\ \boldsymbol{R}_j&=\boldsymbol{A}_1\boldsymbol{R}_{j-1}+\boldsymbol{A}_2\boldsymbol{R}_{j-2}\quad\text{for } j=2,3,\ldots,J_k-1,\\ \boldsymbol{A}_1&=-(\hat{s}_k^2\boldsymbol{M}+\hat{s}_k\boldsymbol{D}+\boldsymbol{K})^{-1}(2\hat{s}_k\boldsymbol{M}+\boldsymbol{D}),\\ \boldsymbol{A}_2&=-(\hat{s}_k^2\boldsymbol{M}+\hat{s}_k\boldsymbol{D}+\boldsymbol{K})^{-1}\boldsymbol{M},\\ \boldsymbol{G}_1&=-(\hat{s}_k^2\boldsymbol{M}+\hat{s}_k\boldsymbol{D}+\boldsymbol{K})^{-1}\boldsymbol{B}. \end{aligned} \tag{11}$$

Multipoint moment matching matches the transfer function and its derivatives at multiple frequency expansion points and the resulting reduction matrix

$$\mathrm{span}(\boldsymbol{V})=\mathrm{span}({}_{\hat{s}_1}\mathcal{K}_{J_1}\ \cup\ \ldots\ \cup\ {}_{\hat{s}_K}\mathcal{K}_{J_K}) \tag{12}$$

is spanned by the union of the individual Krylov subspaces. Deflation strategies such as the modified Gram Schmidt algorithm are typically used to ensure a full rank of the reduction matrix. Furthermore, it ensures that the columns of the reduction matrix are orthogonal. These subspaces are often called input Krylov subspaces, because they use the input matrix $\boldsymbol{B}$. The output Krylov subspaces to obtain the reduction matrix $\boldsymbol{W}$ can be built similarly, using the output matrix $\boldsymbol{C}$.

Obviously, the rank of the reduction matrix $\boldsymbol{V}$, and, therefore, the size of the reduced order model, depends on the number of expansion points, the number of matched moments at each expansion point and on the number of system inputs. Especially the number of inputs is crucial since the number of inputs can easily become very large, for example for systems with a surface input. To overcome this issue, the concept of tangential interpolation has been developed by [13] and is also explained in [3, 5, 8, 22]. The idea is, to introduce left tangential directions $\boldsymbol{l} \in \mathbb{C}^c$ and right tangential directions $\boldsymbol{r} \in \mathbb{C}^b$ such that the transfer function

$$\boldsymbol{H}(\hat{s}_i)\boldsymbol{r}_i = \boldsymbol{H}_{\mathrm{red}}(\hat{s}_i)\boldsymbol{r}_i \tag{13}$$

$$\boldsymbol{l}_j^{\mathsf{T}}\boldsymbol{H}(\hat{s}_j) = \boldsymbol{l}_j^{\mathsf{T}}\boldsymbol{H}_{\mathrm{red}}(\hat{s}_j) \tag{14}$$

is only interpolated along these specific directions. The resulting subspaces are built with the reduced input matrix $\widetilde{\boldsymbol{B}} = \boldsymbol{B}\boldsymbol{r} \in \mathbb{C}^N$ and reduced output matrix $\widetilde{\boldsymbol{C}} = \boldsymbol{l}^{\mathsf{T}}\boldsymbol{C} \in \mathbb{C}^{1\times N}$, respectively. Multiple methods exist to construct the tangential directions. In [8, 22] randomly chosen tangential directions as well as tangential directions based on the eigenvalues of the system were investigated.

A well-known method in Model Order Reduction of microelectronic systems with many input ports is Singular Value Decomposition Model Order Reduction (SVDMOR) [10] and the Extended SVDMOR (ESVDMOR) [18]. The SVDMOR approach is based on the singular value decomposition of the transfer function

$$\boldsymbol{H}(s) = \boldsymbol{U}(s)\boldsymbol{\Sigma}(s)\boldsymbol{P}^{\mathrm{H}}(s) \tag{15}$$

with a unitary matrix of left singular vectors $\boldsymbol{U}(s) \in \mathbb{C}^{c\times c}$, a unitary matrix of right singular vectors $\boldsymbol{P}(s) \in \mathbb{C}^{b\times b}$ with the superscript $^{\mathrm{H}}$ indicating the conjugate transpose of the matrix and a matrix $\boldsymbol{\Sigma}(s) = \mathrm{diag}(\sigma_1(s), \ldots, \sigma_m(s)) \in \mathbb{R}^{c\times b}$ with the singular values $\sigma_i(s) \in \mathbb{R}$ on the diagonal in descending order sorted by their dominance. These matrices are now structured in two parts such that

$$\boldsymbol{U}(s) = \begin{bmatrix}\boldsymbol{U}_1(s) & \boldsymbol{U}_2(s)\end{bmatrix}, \quad \boldsymbol{\Sigma}(s) = \begin{bmatrix}\boldsymbol{\Sigma}_1(s) & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{\Sigma}_2(s)\end{bmatrix}, \quad \boldsymbol{P}(s) = \begin{bmatrix}\boldsymbol{P}_1(s) & \boldsymbol{P}_2(s)\end{bmatrix} \tag{16}$$

with $\boldsymbol{U}_1(s) \in \mathbb{C}^{c\times r}$ being the first r left singular vectors and $\boldsymbol{P}_1(s) \in \mathbb{C}^{b\times r}$ are the first r right singular vectors. Using these matrices, the transfer function can be represented as a low-rank approximation

$$\boldsymbol{U}(s)\boldsymbol{\Sigma}(s)\boldsymbol{P}^{\mathrm{H}}(s) \approx \boldsymbol{U}_1(s)\boldsymbol{\Sigma}_1(s)\boldsymbol{P}_1^{\mathrm{H}}(s) = \widehat{\boldsymbol{H}}(s) \in \mathbb{C}^{c\times b}. \tag{17}$$

The ESVDMOR method is an extension of the SVDMOR method where also higher order moments of the transfer function are considered and the singular value decomposition is performed separately for input and output moments.

With $\boldsymbol{U}_1(s)$ and $\boldsymbol{P}_1(s)$ as tangential directions, the modified input and output matrices

$$\widetilde{\boldsymbol{B}}(s) = \boldsymbol{B}\boldsymbol{P}_1(s) \in \mathbb{C}^{N\times r} \quad \text{and} \quad \widetilde{\boldsymbol{C}}(s) = \boldsymbol{U}_1^{\mathrm{H}}(s)\boldsymbol{C} \in \mathbb{C}^{r\times N} \tag{18}$$

have only r inputs and outputs. The input output reduced transfer function becomes

$$\widetilde{\boldsymbol{H}}(s) = \widetilde{\boldsymbol{C}}(s)\left(s^2\boldsymbol{M} + s\boldsymbol{D} + \boldsymbol{K}\right)^{-1}\widetilde{\boldsymbol{B}}(s) \in \mathbb{C}^{r\times r}, \tag{19}$$

which can be projected back to the full input output space resulting in the projected transfer function

$$\widehat{\boldsymbol{H}}(s) = \boldsymbol{U}_1(s)\widetilde{\boldsymbol{C}}(s)\left(s^2\boldsymbol{M} + s\boldsymbol{D} + \boldsymbol{K}\right)^{-1}\widetilde{\boldsymbol{B}}(s)\boldsymbol{P}_1^{\mathrm{H}}(s) \in \mathbb{C}^{c\times b}. \tag{20}$$

It should be noted that, since $\boldsymbol{U}_1(s)$ and $\boldsymbol{P}_1(s)$ are frequency dependent, the reduced input and output matrices $\widetilde{\boldsymbol{B}}(s)$ and $\widetilde{\boldsymbol{C}}(s)$ also become frequency dependent. The classical Model Order Reduction to reduce the dimension of the system matrices as explained in Equations (6) to (9) is carried out with the input reduced system $\widetilde{\boldsymbol{H}}(s)$, which then results in the reduced order model with reduced number of inputs $\widetilde{\boldsymbol{H}}_{\text{red}}(s) \in \mathbb{C}^{r\times r}$. This reduced model can finally be projected back to the full input space, equivalent to Equation (20), resulting in $\widehat{\boldsymbol{H}}_{\text{red}}(s) \in \mathbb{C}^{c\times b}$.

The following list gives a structured overview of the resulting models:

- $\boldsymbol{H}(s) \in \mathbb{C}^{c\times b}$: full order model with all inputs and outputs (initial model),
- $\widetilde{\boldsymbol{H}}(s) \in \mathbb{C}^{r\times r}$: full order model with reduced number of inputs and outputs,
- $\widehat{\boldsymbol{H}}(s) \in \mathbb{C}^{c\times b}$: full order model with inputs and outputs projected back to full dimension,
- $\boldsymbol{H}_{\text{red}}(s) \in \mathbb{C}^{c\times b}$: reduced order model with all inputs and outputs (reduced initial model),
- $\widetilde{\boldsymbol{H}}_{\text{red}}(s) \in \mathbb{C}^{r\times r}$: reduced order model with reduced number of inputs and outputs,
- $\widehat{\boldsymbol{H}}_{\text{red}}(s) \in \mathbb{C}^{c\times b}$: reduced order model with inputs and outputs projected back to full dimension.

2.3 Error measure

To evaluate the quality of the low-rank approximation as well as the Model Order Reduction, it is important to have some decent error measures. According to the Eckhart–Young theorem [14], the spectral norm error between the transfer function $\boldsymbol{H}(s)$ and its singular value decomposition based low-rank approximation $\widehat{\boldsymbol{H}}(s)$ is defined as

$$\varepsilon_{\text{out}}(s) = \|\boldsymbol{H}(s) - \widehat{\boldsymbol{H}}(s)\|_2 = \sigma_{r+1}(s) \tag{21}$$

and the Frobenius norm error is

$$\|\boldsymbol{H}(s) - \widehat{\boldsymbol{H}}(s)\|_{\text{F}} = \sqrt{\sum_{i=r+1}^{m} \sigma_i^2(s)}. \tag{22}$$

This is only valid in the special case of low-rank approximation with the singular value decomposition. If the number of inputs or outputs is reduced with another method, the error cannot be calculated in this way. The relative approximation error is

$$\varepsilon_{\text{out,rel}}(s) = \frac{\varepsilon_{\text{out}}(s)}{\|\boldsymbol{H}(s)\|_2}. \tag{23}$$

Due to the invariance of the spectral norm under unitary transformation, the reduction error

$$\begin{aligned} \varepsilon_{\text{red}}(s) &= \|\widehat{\boldsymbol{H}}(s) - \widehat{\boldsymbol{H}}_{\text{red}}(s)\|_2 \\ &= \|\boldsymbol{U}_1(s)\widetilde{\boldsymbol{H}}(s)\boldsymbol{P}_1^{\text{H}}(s) - \boldsymbol{U}_1(s)\widetilde{\boldsymbol{H}}_{\text{red}}(s)\boldsymbol{P}_1^{\text{H}}(s)\|_2 \\ &= \|\widetilde{\boldsymbol{H}}(s) - \widetilde{\boldsymbol{H}}_{\text{red}}(s)\|_2 \end{aligned} \tag{24}$$

can be calculated either with the transfer functions projected back to the full input output space or with the transfer function with the reduced number of inputs and outputs. Additionally, the relative reduction error is calculated by

$$\varepsilon_{\text{red,rel}}(s) = \frac{\varepsilon_{\text{red}}(s)}{\|\widetilde{\boldsymbol{H}}(s)\|_2}. \tag{25}$$

According to [7] the total error, i.e. the error between the initial full order model and the reduced order model with inputs and outputs projected back to full dimension, can be bounded by the sum of the previously derived errors and results in

$$\varepsilon_{\text{tot}}(s) = \|\boldsymbol{H}(s) - \widehat{\boldsymbol{H}}_{\text{red}}(s)\|_2 \leq \underbrace{\|\boldsymbol{H}(s) - \widehat{\boldsymbol{H}}(s)\|_2}_{\varepsilon_{\text{out}}(s)} + \underbrace{\|\widetilde{\boldsymbol{H}}(s) - \widetilde{\boldsymbol{H}}_{\text{red}}(s)\|_2}_{\varepsilon_{\text{red}}(s)}. \tag{26}$$

This relation is useful because the reduction of the input and output dimension and the Model Order Reduction can be applied independently.

3 SVDMOR FOR A MECHANICAL SYSTEM

The investigated plate structure is shown in Figure 1. It is modeled with solid elements in Ansys. Due to the clamping of all three degrees of freedom of every node on the left side of the structure, the resulting mass- and stiffness matrices have $N = 3900$ degrees of freedom. Rayleigh damping with $\alpha = 3.86$ and $\beta = 2.25 \cdot 10^{-5}$ is added to the system. At every node on the upper surface of the structure, an input is defined in vertical direction. All inputs are considered independent of each other, resulting in a Boolean input matrix $\boldsymbol{B}$ with $b = 651$ inputs and as many columns. The outputs are defined equivalently resulting in $\boldsymbol{C}^\mathsf{T} = \boldsymbol{B}$. The norm of the transfer function of the system from 0 Hz to 1500 Hz is depicted in Figure 2. There are 14 eigenfrequencies in the frequency spectrum shown. As can also be seen in Equations (4) and (5), the spectral norm is always smaller than or equal to the Frobenius norm.

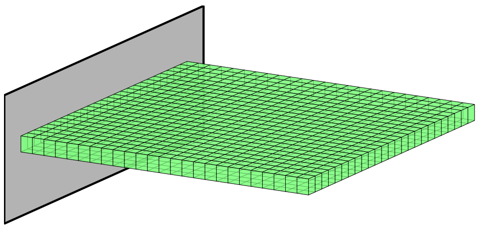

Figure 1. Finite Element plate model with 3900 degrees of freedom. 651 inputs and outputs are defined in vertical direction on all nodes of the top surface of the plate.

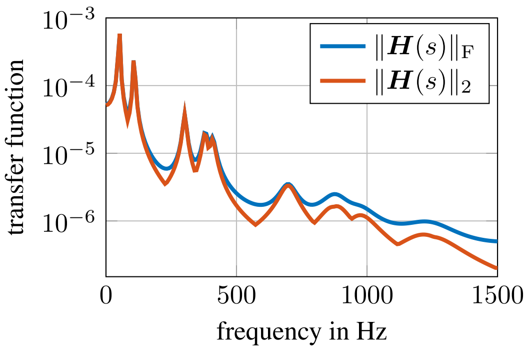

Figure 2. Spectral norm and Frobenius norm of the transfer function of the plate model.

When applying a classical moment matching based Model Order Reduction without any input reduction to the given model, every new expansion point, that matches only the transfer function but doesn't match higher order moments, increases the dimension of the reduced order model by roughly $2b = 1302$. The factor 2 occurs because the system is damped and a real-valued matrix with twice as many columns is created from the initially complex-valued projection matrix, see [12]. The actual dimension is slightly smaller due to deflation. Very often, the reduction error becomes significantly smaller with every new expansion point, but it's not always desired to have a comparatively small error and a relatively large reduced order model. For some applications, larger errors with smaller models are more preferable. For this reason, we first examine which tangential directions are dominant in the frequency range of interest.

Figure 3 shows the spectrum of the singular values of the transfer function in the frequency range mentioned above. The first singular value is in the range from $2 \cdot 10^{-7}$ to $6 \cdot 10^{-4}$ for all evaluated frequencies. The size of this interval decreases with higher numbers of the singular values. This also means that the approximation error $\varepsilon_\mathrm{out}(s)$ is strongly dependent on the frequency if only few tangential directions are used, but the error becomes almost constant if more tangential directions are taken into account. It is also noticeable that the decay rate becomes much smaller for higher numbers of singular values. While the approximation error can be reduced by approximately two orders of magnitude by increasing the number of used tangential directions from 10 to 100, the approximation error cannot further be reduced significantly by using more than 100 tangential directions.

The behavior of the singular values in the frequency range can be seen in Figure 4. A specific color represents the singular value that is associated with a specific tangential direction. Following for example the dark green line shows that the associated tangential direction is most dominant in the lower frequencies and looses importance for higher frequency range. Many singular values that are less dominant at low frequencies remain almost constant within the frequency range. This is equivalent to what was shown in Figure 3. The figure only shows the first 100 singular values, but singular values with a higher number also remain almost constant in the given frequency range. The data in the figure is closely related to the Complex Mode Indicator Function, see e.g. [25], that can be used to identify modal parameters from measurement data.

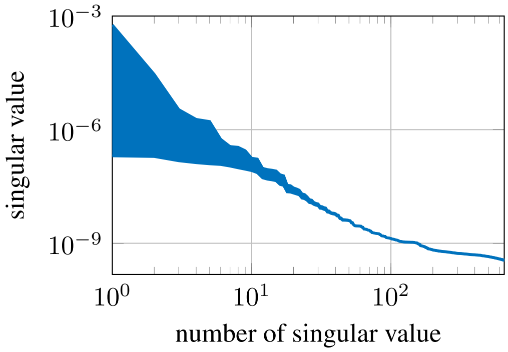

Figure 3. Range of the singular values in the frequency spectrum from 0 Hz to 1500 Hz.

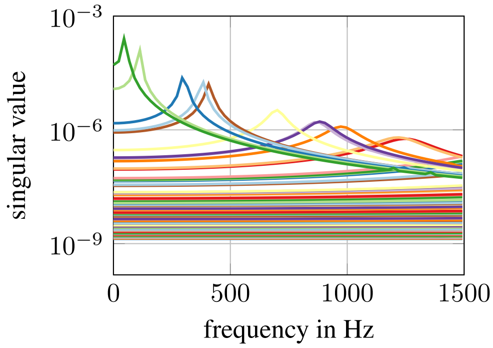

Figure 4. Frequency dependency of the singular values.

In Figure 5 the tangential directions at 0 Hz and at 1500 Hz are compared using the Modal Assurance Criterion

$$\mathrm{MAC}_{i,j} = \frac{\left| \boldsymbol{u}_i^{\mathrm{H}} \boldsymbol{w}_j \right|^2}{\left(\boldsymbol{u}_i^{\mathrm{H}} \boldsymbol{u}_i \right) \left(\boldsymbol{w}_j^{\mathrm{H}} \boldsymbol{w}_j \right)}. \tag{27}$$

The Modal Assurance Criterion gives information about the consistency between a vector $\boldsymbol{u}_i$ and a vector $\boldsymbol{w}_i$ and returns values between 0 and 1 where 1 indicates full consistency, see [1]. The figure shows the 40 most dominant tangential directions. Since $\boldsymbol{C}^{\mathrm{T}} = \boldsymbol{B}$ in this example, $\boldsymbol{H}(s)$ is symmetric and the left tangential directions are equal to the right tangential directions. Comparing the lines and the columns in Figure 5 it can clearly be seen that all tangential directions that occur at 0 Hz also occur at 1500 Hz, even though the order, i.e. the dominance or importance, of the first 17 tangential directions has changed. This shows that the tangential directions do not have to be calculated for every new expansion point. Consequently, we would have to take at least 17 tangential directions to have the most dominant tangential directions in the whole frequency range. If only the first 5 tangential directions at 0 Hz were taken into account, this would not lead to reasonable results at 1500 Hz because other tangential directions are dominant there.

Although it is sufficient to calculate the tangential directions only once and apply it at every expansion point if enough tangential directions are taken into account, it is not possible to obtain information about the required number of tangential directions and the resulting approximation error by carrying out the singular value decomposition only once. Therefore, we suggest using the singular value decomposition at least twice, e.g. at the lowest and at the highest frequency of interest, and compare the obtained right singular vectors using the Modal Assurance Criterion from Equation (27) resulting in a representation like Figure 5. This shows for which singular vectors the dominance fluctuates the most. At least all singular vectors that have a different dominance in both frequencies, e.g. singular vectors 1 to 17 in Figure 5 should be considered as tangential directions. As can be seen from Figure 3, using at least those 17 singular vectors as tangential directions, the error over the whole frequency range stays here almost constant and can thus be calculated sufficiently well by using the singular vectors from the previously performed singular value decomposition.

For the given example, the following results are obtained. The number of reduced inputs and outputs to ensure the relative approximation error $\varepsilon_{\mathrm{out,rel}}(s)$ to be below $1\,\%$ is $r = 72$. In comparison to the initial model, this reduces the number of used inputs by a factor of 9. The resulting absolute approximation error $\varepsilon_{\mathrm{out}}(s)$ can be seen in red in Figure 6. Applying a classical moment matching based Model Order Reduction to the obtained input reduced model and requesting the relative reduction error $\varepsilon_{\mathrm{red,rel}}(s)$ to be lower than $1\,\%$ in the given frequency range results in a reduced order model with one expansion point at 750 Hz with $n = 144$ degrees of freedom. The absolute reduction error is shown in green. The dashed yellow line shows the sum of the input reduction error and the model reduction error and is approximately

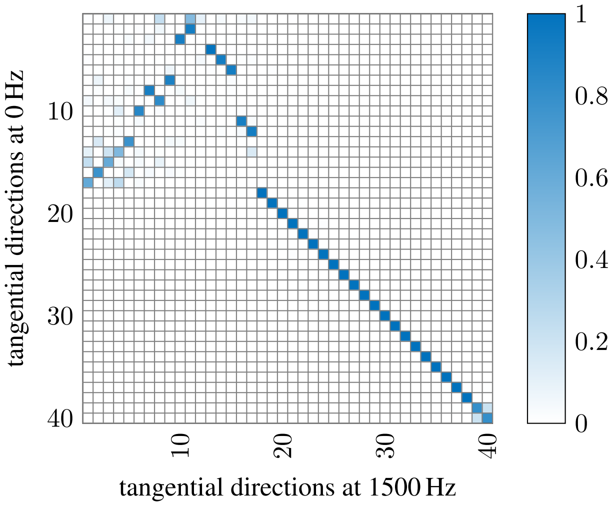

Figure 5. Modal Assurance Criterion of the 40 most dominant tangential directions at 0 Hz and 1500 Hz.

the total error $\varepsilon_{\text{tot}}(s)$. In comparison, a Model Order Reduction scheme without any input reduction, i.e. using all $b = c = 651$ inputs and outputs with a maximum error of $2\,\%$, which is the sum of the allowed error for the input reduction and for the dimensionality reduction, results also in a reduced order model with one expansion point at $750\,\text{Hz}$ with $n = 1302$ degrees of freedom, which is a factor of 9 larger than with SVDMOR. The absolute reduction error for the model without input reduction is depicted as a blue line. This clearly shows the benefit of tangential directions and specifically of SVDMOR also for the application on mechanical systems. While the reduced order model with tangential directions is by a factor of 9 smaller than without tangential directions, the resulting error $\varepsilon_{\text{out}}(s)$ is within the given tolerance, although it is some orders of magnitude larger than the error of the one-step Model Order Reduction without SVDMOR. It is not possible to bring the error of the reduced order model without SVDMOR closer to the given bounds because it is not possible to further reduce the number of shifts.

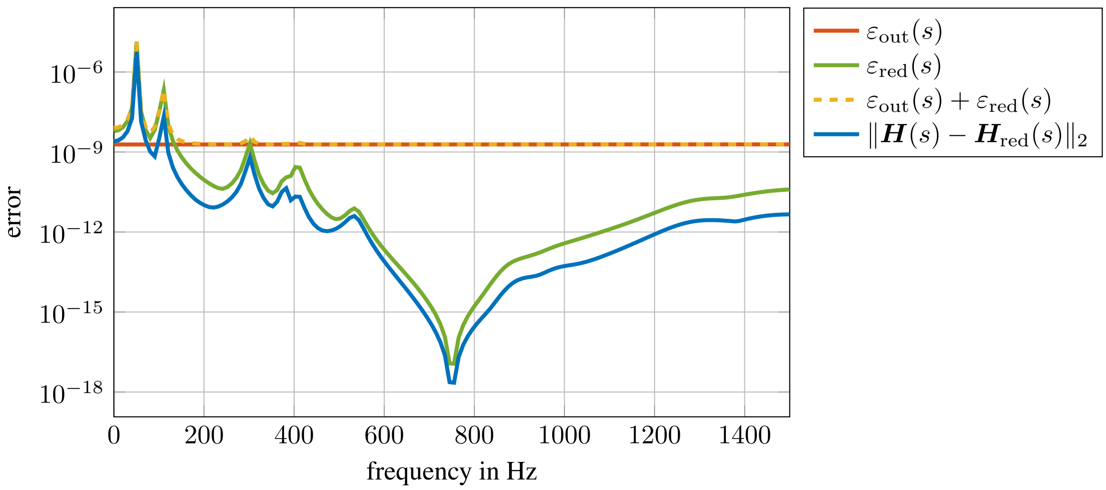

Figure 6. Comparing the reduction error of a classical Model Order Reduction scheme without any input reduction with a two-step approach using SVDMOR to reduce the number of inputs and outputs and moment matching to reduce the system dimension.

Note, that the number of inputs and outputs that are necessary for an appropriate approximation highly depends on the frequency range of interest. Figure 7 shows, how many inputs are necessary to have an approximation error $\varepsilon_{\mathrm{out,rel}}(s)$ that is below $1\,\%$ depending on the frequency range of interest. If we just want static accordance at $0\,\mathrm{Hz}$, already $r = 5$ inputs are required. The number of required input rises very fast and reaches all 651 inputs if we are interested in the frequency range from $0\,\mathrm{Hz}$ to $3470\,\mathrm{Hz}$. For a larger frequency interval it is no longer feasible to reduce the number of required inputs while ensuring an error below $1\,\%$.

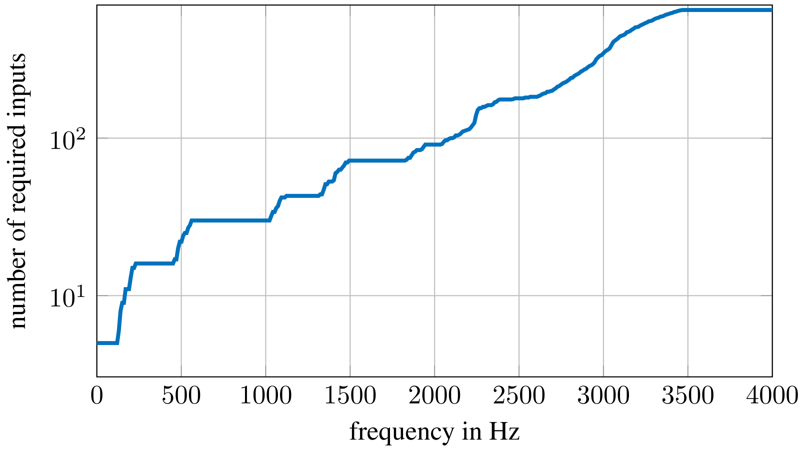

Figure 7. Number of required inputs to reach an approximation error $\varepsilon_{\mathrm{out,rel}}(s)$ below $1\,\%$ depending on the frequency range of interest.

4 CONCLUSION

In this contribution, we examined the Singular Value Decomposition Model Order Reduction method for a mechanical system modeled with the Finite Element Method. It was shown that SVDMOR can be applied successfully for mechanical systems, but it is important to gain enough information about the singular values and the singular vectors. With the help of the Model Assurance Criterion the most important singular vectors in the given frequency domain were identified. In total, 72 singular vectors were used as tangential directions to reduce the number of inputs and outputs by a factor of 9. With the resulting model, a moment matching based Model Order Reduction was conducted to get a dimensionality reduced order model with 144 degrees of freedom instead of 1302 for the reduced order model that is obtained with all inputs and outputs. However, the benefit gained from SVDMOR strongly depends on the desired frequency interval to be examined. The number of required tangential directions increases significantly as the frequency spectrum increases.

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