Numerical methods for model reduction of periodic dynamical systems: Review and applications

Mohammad Sahadet Hossain and Mohammad Monir Uddin

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Mohammad Sahadet Hossain,[1,*] and M. Monir Uddin [1]
[1]Department of Mathematics and Physics, North South University
15/B Basundhara, Dhaka-1229, Bangladesh

[*]mohammad.hossain@northsouth.edu

***Abstract*—In the second-half of the past century the expeditious development of systems and control theory together with the achievements of digital control and signal processing have set the stage for a renewed interest in the study of periodic dynamical systems, specially in aerospace realm, control of industrial processes, mechanical systems, modeling of periodic time-varying filters and networks, circuit simulation, and multirate sampled-data systems, etc. These complicated systems are composed of large numbers of separate devices and they are described by very large mathematical models consisting of more and more mathematical systems with very large dimensions. Simulations of such systems can be unacceptably expensive and time-consuming due to limited computer memory and CPU consumption. The idea of model reduction is that the large models should be replaced by smaller models which are amenable to fast and efficient simulation and which still capture the devices' input-output behavior to an accepted accuracy. In this paper we review the different approaches for model reduction of time varying systems, and depict the numerical results showing the advantages and disadvantages of these approaches.**

***Index Terms*—Linear time-varying systems, Model order reduction, Krylov approximation, Lyapunov equations, Balanced truncation model reduction.**

I. INTRODUCTION

In this paper, we analyze the linear time-varying (LTV) systems for both continuous and discrete-time case, and study the concepts of model reduction for those systems. A continuous-time LTV system in general has the form

$$\begin{aligned} E(t)\dot{x}(t) &= A(t)x(t) + B(t)u(t), \quad x(t_0) = x_0 \\ y(t) &= C(t)x(t), \end{aligned} \tag{1}$$

where $x(t) \in \mathbb{R}^n$, called the state vector, $u(t)$ is the system input, $y(t)$ is the system output, and n is the system order at any given time t. The matrices $E(t)$, $A(t)$, $B(t)$, $C(t)$ are of order compatible with $x(t)$, $u(t)$, and $y(t)$ and assumed to be continuous functions of time. All the system matrices are time-varying, periodic with period $K \geq 1$ and the matrices $E(t)$ and $A(t)$ can be singular at any given time t.

Formally speaking, a reduced-order system of order r for system (1) would be a system of the form

$$\begin{aligned} \tilde{E}(t)\dot{\tilde{x}}(t) &= \tilde{A}(t)\tilde{x}(t) + \tilde{B}(t)u(t), \\ \tilde{y}(t) &= \tilde{C}(t)\tilde{x}(t). \end{aligned} \tag{2}$$

The system is of potentially smaller dimension, i.e., $r << n$, and thus lower computational cost, than the original system (1), but it is now in a form suitable for use in higher level simulation.

A linear discrete-time periodic descriptor system with time-varying dimensions has the form

$$\begin{aligned} E_k x_{k+1} &= A_k x_k + B_k u_k, \\ y_k &= C_k x_k, \qquad k \in \mathbb{Z}, \end{aligned} \tag{3}$$

where $E_k \in \mathbb{R}^{\mu_{k+1} \times n_{k+1}}$, $A_k \in \mathbb{R}^{\mu_{k+1} \times n_k}$, $B_k \in \mathbb{R}^{\mu_{k+1} \times p_k}$, $C_k \in \mathbb{R}^{q_k \times n_k}$ are time-varying, and periodic with a period $K \geq 1$. Clearly, $\sum_{k=0}^{K-1} \mu_k = \sum_{k=0}^{K-1} n_k = n$. The matrices E_k are allowed to be singular for all k.

Analogous to the continuous-time case, a reduced-order system of dimension r for system (3) would be a system of the form

$$\begin{aligned} \tilde{E}_k \tilde{x}_{k+1} &= \tilde{A}_k \tilde{x}_k + \tilde{B}_k u_k, \\ \tilde{y}_k &= \tilde{C}_k \tilde{x}_k, \qquad k \in \mathbb{Z}, \end{aligned} \tag{4}$$

where for each k, $\tilde{x}_k$ is an r_k-dimensional vector, $\sum_{k=0}^{K-1} r_k = r$ and $r \ll n$. Apart from having much smaller state-space dimension, the reduced-order system preserves some essential and important characteristics of the original system.

II. PRELIMINARIES

Model reduction using projection formulation has become a popular and well accepted technique in the field of signal analysis and electrical interconnections. Today, the best choices for these projection subspaces, in model reduction of LTV continuous-time systems, are Krylov subspaces. In this approach, the lower order model is obtained such that some of the first moments (and/or markov parameters) [1], [5], [12], [13], [16] of the original and reduced systems are matched where the moments are the coefficient's of the Taylor series expansion of the transfer function at a suitable point. Methods based on multipoint rational approximations [11], [13] are also efficient for particular cases.

However, model reduction for time-varying systems is much more complex in that projections approach. Balanced truncation methods [20], [7], [4] have been applied for model reduction of LTV discrete-time case. Iterative techniques [2], [19], [21] are also introduced for large scale problems. Therefore, huge talents have been worked on developing the techniques of model reduction using rational approximations and the projection formulations.

III. KRYLOV-SUBSPACE BASED PROJECTION METHODS

Nowadays, moment matching using Krylov subspaces is one of the best choices in order reduction of large scale systems and it was first proposed in [17]. In this approach, the lower order model is obtained by matching the moments (and/or Markov parameters) of the original and reduced-order systems where the moments are the coefficients of the Taylor series expansion of the transfer function about a suitable expansion point. When the expansion point tends to infinity, the coefficients are called Markov parameters. Well established algorithms, such as Arnoldi [18], or two-sided Arnoldi [5] can be used to compute a projection framework for the reduced-order system. A very recent release of the Krylov subspaces based order reduction technique is global Arnoldi [19], which approximates the large, sparse systems (specially MIMO systems) to significantly small order.

For simplicity, let us consider the system (1) in linear time-invariant (LTI) form

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + Bu(t), \quad x(0) = x_0, \\ y(t) &= Cx(t), \end{aligned} \tag{5}$$

Assume that the matrices E, A present is system (5) are nonsingular. The output function for u to y for system (5) can be defined as $\bar{y}(s) = H(s)\bar{u}(s)$, where $\bar{u}(s)$ and $\bar{y}(s)$ are the Laplace transforms of u(t) and y(t) with $x(0) = 0$, and $H(s) = C(sE - A)^{-1}B$ is called the *transfer function* of the system. Using Krylov subspace method, the reduced-order model is computed applying suitable projections to system (5) given by

$$\begin{aligned} \tilde{E}(t) &= V^T(t)E(t)U(t), \ \tilde{A}(t) = V^T(t)A(t)U(t), \\ \tilde{B}(t) &= V^T(t)B(t), \ \tilde{C}(t) = C(t)\,U(t), \end{aligned} \tag{6}$$

where the matrices V and U are referred to as the left projection matrix and the right projection matrix, respectively. The projection matrices are calculated via Krylov subspaces, defined in the following:

Definition 1: The order m Krylov subspace is the space defined as

$$\mathcal{K}_m(A, b) = span\{b, Ab, A^2b, ..., A^{m-1}b\}, \tag{7}$$

where $A \in \mathbb{R}^{n\times n}$ and $b \in \mathbb{R}^n$ is called the starting vector. If b is not a single-dimensional vector, but $B = [b_1, b_2, \ldots, b_p]$, then the Krylov subspace define above is called block Krylov subspace.

In most application related models, choosing a suitable basis for the concerned Krylov subspace is the most crucial task, since it guarantees the better approximation of the reduced-order model. In one-sided methods, the most popular algorithm is the Arnoldi algorithm which finds an orthonormal basis for a Krylov subspace [5], [18]. The classical Arnoldi method finds a set of orthonormal vectors that can be considered as a basis for a given Krylov subspace with one starting vector. There are several approaches of Krylov projection method. The list of very recent and well known methods includes the multipoint Krylov approximation technique, and the iterative rational Krylov algorithm (IRKA), singular value decomposition based Krylov (SVD-Krylov) technique [2], [6]. In multipoint Krylov approximation technique [8], [13], [15], the projection matrices are computed using the block krylov subspaces at multi-frequency point. A recycle technique in the Krylov subspace is applied in [8] which enables to use the same Krylov subspace to find the projection matrices at multiple frequency points. The advantage of the multipoint Krylov approximation technique is that it uses the recycle Krylov approximation scheme and hence demands no extra cost to obtain the projector at multiple frequency-points. An efficient reduced-order model can be approximated within few iteration steps due to this recycle strategy. The main disadvantage is the limitation of the frequency range and no a-priori error bounds can be suggested.

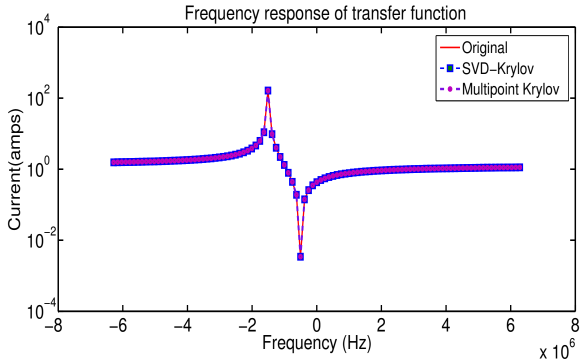

Fig. 1. Frequency response of transfer function: exact system versus reduced-order system of order $r = 3$ (RF circuit).

On the other hand, the iterative rational Krylov algorithm (IRKA) technique [6] is a more efficient approach where the reduced-order model is obtained via the Krylov based multiple interpolations. The SVD-Krylov is a two sided projection technique which combines the SVD and the Krylov subspace method [14]. This method first apply a simple descretization scheme on the original periodic model, and reformulate an equivalent time-invariant form. It then reduces it with a two sided projection technique, where one side of this reduction technique is SVD based Gramian approximation, and the other side is based on Krylov based projection. The advantage of the SVD-Krylov approximation is that it uses the recycle Krylov scheme but an a-priori error bounds can be suggested.

IV. Numerical Results of Krylov-subspace based projection methods

In this section, we consider numerical results of a simple example where the data is obtained from a small radio frequency (RF) circuit simulator as per [7]. We compare the similar results coming from different Krylov projection techniques of model reduction. The circuit system consists of 5 nodes, and is excited by a local oscillator (LO) at 2 KHz driving the mixer. The corresponding time-varying system is obtained around a steady state of the circuit at the oscillatory frequency; a total of $M = 129$ timesteps are used to describe the steady-state waveform. For the model reduction procedure, the input function $B(t)$ is a constant column vector, corresponding to the continuous small-signal input. To analyze the circuit, we consider a period of $T = 1ms$ for the steady state analysis. The final discretized model is a real LTI system of order $N = 645$.

The reduced-order model is generated by matching four moments of the Krylov subspace generated for every expansion point. We use the rank revealing QR factorization for the formulation of the projected matrix with tolerance, $tol = 10^{-5}$.

We obtain a reduced-order model of order $r = 3$. Fig. 1 shows a very nice matching of the baseband transfer functions $H_{TD}(s)$ and $\tilde{H}_{TD}(s)$, and the relative error in Fig. 2 is very small which reflects the efficiency of the reduced-order model.

In Fig 3 we depict the bode plots of the original and the reduced-order models. They show a very nice matching and very negligible error.

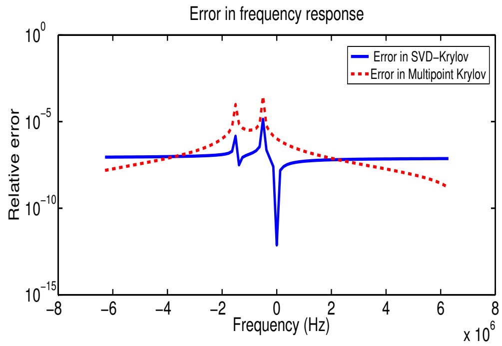

Fig. 2. Error in the frequency response of transfer function of reduced-order system (RF circuit).

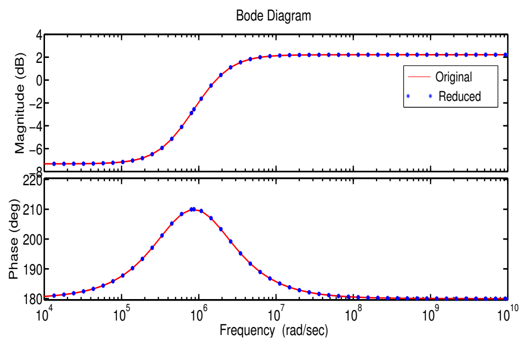

Fig. 3. Bode plots for the exact system and the reduced-order system of order $r = 3$.

V. Balanced-truncation projection method

A popular model reduction technique for LTV systems in discrete-form is the Balanced-truncation (BT) approach where the original state space system is transformed into a balanced form whose reachability and observability Gramians become diagonal and equal [2], [3], [4], [20]. The balanced truncation method truncates all those states of the balanced system that are both difficult to reach and to observe. An important property of this method is that asymptotic stability is preserved in the reduced order system and an a priori error bound can be computed. Model order reduction of periodic systems using balanced-truncation has been considered in [3], [20].

We assume that the set of periodic matrix pairs $\{(E_k, A_k)\}_{k=0}^{K-1}$ is periodic stable [3]. The periodic matrix equations that arise in the context of periodic control and model order reduction (MOR) of periodic systems can be represented by the following generalized periodic discrete-time Lyapunov equations (PDALEs) [2], [3]

$$A_k X_k A_k^T - E_k X_{k+1} E_k^T = -B_k B_k^T, \tag{8}$$

and

$$A_k^T Q_{k+1} A_k - E_k^T Q_k E_k = -C^T C_k, \tag{9}$$

where P_k, Q_k are called the periodic controllability Gramian, and the observability Gramian of the periodic system, respectively. Here $P_K = P_0$, $Q_K = Q_0$ are periodic.

The numerical solutions of (8) and (9) have been considered in [3] for time-varying matrix coefficients in descriptor form. The method proposed there is based on an initial reduction of the periodic matrix pairs $\{(E_k, A_k)\}_{k=0}^{K-1}$ to the generalized periodic Schur form [21], [10] and on solving the resulting generalized periodic Sylvester and Lyapunov equations. As a result, the method is computationally expensive and not suitable for large scale problem.

An efficient approach which works with the cyclic lifted representation of (3) and the corresponding lifted form of (8) has been considered in [7]. In practice, one should avoid these direct methods for large-scale problems because the computational complexity because they require computational complexity of $\mathcal{O}(Kn_{max}^3)$, where $n_{max} = max((n_k)$. Iterative solutions of (8) and (9) using their corresponding lifted structures have been considered in [2]. A generalized version of the alternating direction implicit (ADI) method and the Smith method are proposed there for the solutions of (8) and (9) using their corresponding lifted structures, respectively. The main focus of this iterative computation is to preserve the block diagonal structure of the approximate solution at each iterative step. Unfortunately, this is not observed at each iteration steps of the generalized ADI method proposed in [2].

The generalized reflexive inverses of periodic descriptor systems via the corresponding lifted representation has been considered in [21]. Model reduction of periodic descriptor system (3) exploiting the generalized reflexive inverses of periodic matrix pairs has been considered in [9].

Let us assume that P_k, and Q_k be full rank Gramians for $k = 0, 1, \dots, K-1$ of system (3). Then the Cholesky factorizations of the Gramians are given by

$$P_k = R_k R_k^T,\ Q_k = L_k L_k^T.$$

We then compute the singular value decomposition

$$L_k^T E_{k-1} R_k = [U_{k,1}, U_{k,2}] \begin{bmatrix} \Sigma_{k,1} & \\ & \Sigma_{k,2} \end{bmatrix} [V_{k,1}, V_{k,2}]^T, \tag{10}$$

where $[U_{k,1}, U_{k,2}]$, $[V_{k,1}, V_{k,2}]$ are orthogonal, $\Sigma_{k,1} = \mathrm{diag}(\sigma_{k,1}, \dots, \sigma_{k,r_k})$, $\Sigma_{k,2} = \mathrm{diag}(\sigma_{k,r_k+1}, \dots, \sigma_{n_k})$, with $\sigma_{k,1} \geq \cdots \geq \sigma_{k,r_k} > \sigma_{k,r_k+1} \geq \dots \geq \sigma_{k,n_k} > 0$ are called the *Hankel singular values*, of system (3), for $k = 0, 1, \dots, K-1$. It should be noted that for a balanced system, truncation of states related to the small causal Hankel singular values does not change system properties essentially [7].

We then compute the projection matrices as

$$S_{k,r} = [\,L_{k+1} U_{k+1,1} \Sigma_{k+1,1}^{-1/2}], \quad T_{k,r} = [\,R_k V_{k,1} \Sigma_{k,1}^{-1/2}],$$

and finally a reduced-order system of dimension $\bar{r} = (r_0, r_1, \dots, r_{K-1})$ of (3) can be found as [3]

$$\begin{aligned} \tilde{E}_k &= S_{k,r}^T E_k T_{k+1,r}, & \tilde{A}_k &= S_{k,r}^T A_k T_{k,r}, \\ \tilde{B}_k &= S_{k,r}^T B_k, & \tilde{C}_k &= C_k T_{k,r}, \end{aligned} \tag{11}$$

where $\tilde{E}_k \in \mathbf{R}^{r_{k+1} \times r_{k+1}}$, $\tilde{A}_k \in \mathbf{R}^{r_{k+1} \times r_k}$, $\tilde{B}_k \in \mathbf{R}^{r_{k+1} \times m_k}$, $\tilde{C}_k \in \mathbf{R}^{p_k \times r_k}$ are K-periodic matrices, $\sum_{k=0}^{K-1} r_k = r$, and $r_k \leq n_k$, $n \ll r$.

Let $\mathcal{H}(z) = \mathcal{C}(z\mathcal{E} - \mathcal{A})^{-1}\mathcal{B}$ be the transfer function of the original lifted system, and $\tilde{\mathcal{H}}(z)$ be the transfer function of the corresponding reduced-order lifted system [1], [2]. Then the reduced order model has the following $\mathbf{H}_\infty$-norm error bound (see, e.g. [20], [1])

$$\|\mathcal{H} - \tilde{\mathcal{H}}\|_{\mathbf{H}_\infty} \leq 2\,\mathrm{trace}\,(\mathrm{diag}(\Sigma_{0,2}, \dots, \Sigma_{K-1,2})), \tag{12}$$

where $\Sigma_{k,2}$, $k = 0, 1, \dots, K-1$, contains the truncated causal Hankel singular values.

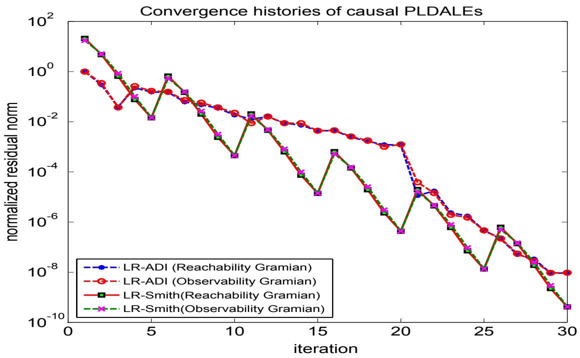

Fig. 4. Normalized residual norms for the causal reachability and observability lifted projected Lyapunov equations.

VI. Numerical Results of Balancing based projection methods

We consider an artificial periodic discrete-time descriptor system on index-1 from [2, Example 1], which is reformulated from its original model in [3, Example 1]. In this reformulation, the periodic descriptor system has $\mu_k = n_k = 404$, $m_k = 2$ and $p_k = 3$ for the periodicity $K = 10$, i.e., $k = 0, 1, \ldots, 9$. The original lifted system has order $n = 4040$.

The normalized residual norms at each iteration steps for solving the controllability and observability lyapunov equations are shown in Fig. 4. A reduced-order model was obtained by truncating the states corresponding to the smallest Hankel singular values satisfying $\sigma_{k,j} < 10^{-4}$. The computed reduced-order model has subsystems of orders $(9, 9, 9, 9, 9, 9, 9, 9, 10, 9)$. Note that stability is preserved in the reduced-order system.

Fig. 5 (a) shows the norms of the frequency responses $\mathcal{H}(e^{i\omega})$ and $\tilde{\mathcal{H}}(e^{i\omega})$ of the original and reduced-order lifted systems for a frequency range $[0, 2\pi]$. We observe a good match of the system norms. Finally, in Fig. 5(b), we display the absolute error $\|\mathcal{H}(e^{i\omega}) - \tilde{\mathcal{H}}(e^{i\omega})\|_\infty$ and the error bound given in (12).

VII. Conclusion

This article reviews the model reduction techniques for time varying systems, both continuous and discrete-time cases. We also analyze the advantages and disadvantages of different methods and their suitabilities for different model's framework. Comparative numerical results are also presented to show the efficiencies of different MOR approaches.

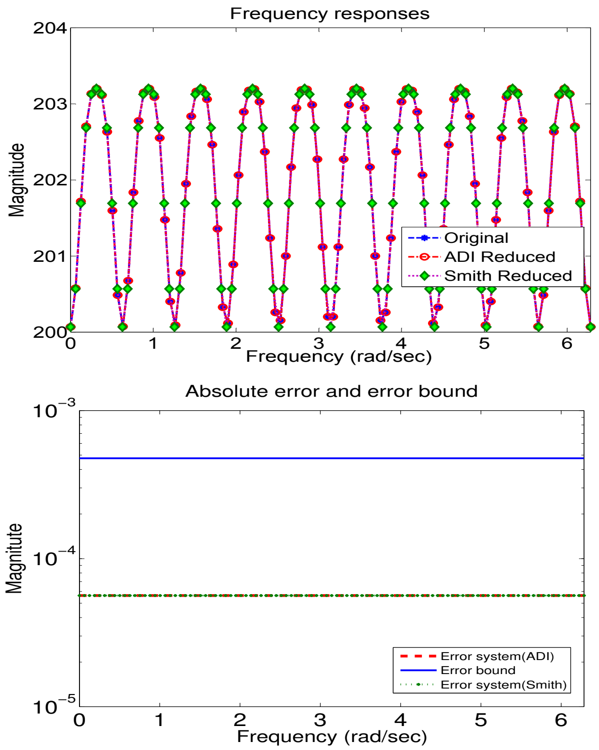

Fig. 5. (a)The frequency responses of the original and the reduced-order lifted systems (top); (b) Absolute error and error bound (down).

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