Solutions and Challenges in Computing FBSDEs with Large Jumps for Dam and Reservoir System Operation

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Abstract. Optimal control of Lévy jump-driven stochastic differential equations plays a central role in management of resource and environment. Problems involving large Lévy jumps are still challenging due to their mathematical and computational complexities. We focus on numerical control of a real-scale dam and reservoir system from the viewpoint of forward-backward stochastic differential equations (FBSDEs): a new mathematical tool in this research area. The problem itself is simple but unique, and involves key challenges common to stochastic systems driven by large Lévy jumps. We firstly present an exactly-solvable linear-quadratic problem and numerically analyze convergence of different numerical schemes. Then, a more realistic problem with a hard constraint of state variables and a more complex objective function is analyzed, demonstrating that the relatively simple schemes perform well.

Keywords: Forward-backward stochastic differential equations· Resource and environment· Tempered stable subordinator· Stochastic maximum principle· Least-squares Monte-Carlo

1 Introduction

Uncertainties are ubiquitous in mathematical modeling and control for environmental and resource management. Stochastic differential equations (SDEs) have been principal tools for efficiently as well as rigorously describing stochastic dynamics of environments and resources [1-3]. Stochastic control based on Markovian feedback policy [4] is a well-established concept implementable in applications because it enables decision-makers to make decisions based on system observations.

Operation of dam and reservoir systems has been a major problem involving management of both resource and environment [5]. A dam and reservoir system consists of a reservoir created in a river to receive and store stochastic inflow discharge and an associated dam as a hydraulic structure to control outflow discharge [6]. Each dam and reservoir system has different operation goal depending on its construction purpose; however, it has a common principle that there exist some targeted reservoir water volume and outflow discharge. In this way, operation of a dam and reservoir system is understood as a stochastic control problem having targeted state.

Existing stochastic control models of dam and reservoir systems are based on dynamic programming approaches where finding an optimal control reduces to solving an optimality equation of a degenerate elliptic or parabolic type [6-8]. This methodology works only if the optimality equation is solvable analytically or its dimension is relatively low, one or two in most cases, such that a common numerical method like a finite difference scheme is implementable [9]. Such cases are too simple from an engineering viewpoint [10]. Furthermore, the previous study suggested that the inflow discharge follows an SDE driven by both small and large Lévy jumps [11], leading to an optimality equation of an integro-differential type having a singular integral kernel that is not necessarily easy to numerically discretize.

To tackle the above-mentioned issue in the stochastic control of dam and reservoir systems, we introduce forward-backward stochastic differential equations (FBSDEs) [4, 12] as a new mathematical tool in this research area. FBSDEs are often equivalent to the optimality equations of the degenerate elliptic and parabolic types [12-13] but are more suited to higher-dimensional problems. This is because they can be implemented using a Monte-Carlo method that can mitigate or even defeat the curse of dimensionality [14-16]. Our FBSDEs are based on a stochastic maximum principle [4] and fully couple forward and backward processes, both of which are driven by a Lévy jump process having infinite activities. To the best of our knowledge, FBSDEs, especially those driven by jumps, have not been investigated in dam and reservoir control problems. In addition, studies focusing on numerical computation of jump-driven FBSDEs are still rare except for purely theoretical ones [17-18].

The objective of this paper is thus set to be formulation and analysis of new jump-driven FBSDEs for stochastic control of dam and reservoir systems. The model proposed in this paper is simple but unique and oriented to engineering applications. A simplified linear-quadratic problem that is solvable analytically but still non-trivial is firstly derived and analyzed numerically using different least-squares Monte-Carlo methods. We use the exact discretization formula [19] to efficiently simulate the inflow discharge process driven by Lévy jumps having infinite activities. A more realistic case having constrained state variables and a more complex objective function is then numerically analyzed with a least-squares Monte-Carlo method. Through the numerical experiments conducted in this paper, we discuss difficulties and remaining challenges in modeling and computation of dam and reservoir systems using FBSDEs.

2 Stochastic Process Model

2.1 Stochastic Differential Equations

We consider a continuous-time operation problem of a dam and reservoir system having the three state variables: the inflow discharge $(I_t)_{t\geq 0}$ as a non-negative variable having the range $\Omega_I = [0,+\infty)$, the water volume $(V_t)_{t\geq 0}$ of a reservoir having the range $\Omega_V = [\underline{V},\bar{V}]$ with constants $\underline{V} < \bar{V}$, and the outflow discharge $(O_t)_{t\geq 0}$ having the range $\Omega_O = [\underline{O},\bar{O}]$ with constants $\underline{O} < \bar{O}$. Typically, we have $\underline{V} = \underline{O} = 0$.

Assume that the inflow is uncontrollable while the outflow is indirectly controllable by tuning its acceleration $(a_t)_{t\geq 0}$ having the range $A = [-\bar{a}, \bar{a}]$ with $\bar{a} > 0$. The state and control variables must be constrained in the corresponding ranges for any $t \geq 0$. Mathematically, the constants $\underline{O}, \bar{O}, \underline{V}, \bar{V}, \bar{a}$ are not necessarily bounded. We consider both bounded and unbounded cases in this paper. Clearly, the former is more realistic.

We consider the problem in a complete probability space as in the usual setting [4]. The stochastic system dynamics we consider are formulated as follows:

$$\mathrm{d}I_t = \rho(\underline{I} - I_t)\mathrm{d}t + \int_0^\infty zN(\mathrm{d}z, \mathrm{d}t), \tag{1}$$

$$\mathrm{d}O_t = a_t \mathrm{d}t, \tag{2}$$

$$\mathrm{d}V_t = (I_t - O_t)\mathrm{d}t, \tag{3}$$

where (1), (2), and (3) describe inflow, outflow, and storage processes, respectively. Here, $\rho > 0$ is the inverse of the correlation time of the inflow, N is a Poisson random measure of a subordinator type having only positive jumps with the Lévy measure $v(\mathrm{d}z)$. Based on a recent identification result for a real river [11], set

$$v(\mathrm{d}z) = \rho a z^{-(1+\alpha)} e^{-bz} \mathrm{d}z, \quad z > 0 \tag{4}$$

with constants a, b, and $\alpha \in (0,1)$. This Lévy measure is of the infinite activities type since $\int_0^\infty v(\mathrm{d}z) = +\infty$. Its first-order moment $M_1 = \int_0^\infty z v(\mathrm{d}z)$ is bounded and is given as $M_1 = \rho a b^{1-\alpha}\Gamma(1-\alpha) > 0$ with a Gamma function Γ. The inflow discharge is more intermittent as α gets closer to 0; in real cases α is close to 0.5 [11].

We assume that the system (1)-(3) is equipped with a deterministic initial condition (I_0, O_0, V_0) belonging to the space $\Omega_I \times \Omega_O \times \Omega_V$. Furthermore, a natural filtration generated by the Poisson random measure N is denoted as $(\mathcal{F}_t)_{t\geq 0}$. We consider a Markovian setting, and the control a_t is assumed to be progressively measurable with respect to the filtration $\mathcal{F}_t$ at each $t \geq 0$. The control should be chosen so that the following constraint is satisfied as explained above: $O_t \in \Omega_O$ and $V_t \in \Omega_V$ with $a_t \in A$ for each $t \geq 0$. The earlier studies considered that the outflow discharge is directly controllable [6-8]; however, instantaneously adjusting the outflow discharge is not only technically difficult but also triggers catastrophic failures of reservoir walls [20]. We therefore consider problems without such an impulsive adjustment.

For simplicity of the analysis, we assume that the system (1)-(3) with each control $(a_t)_{t\geq 0}$ has a unique path-wise solution that is right-continuous and has left limits. This assumption itself is interesting because our problem is a of state-constrained problem: a non-classical problem requiring a careful treatment at boundaries [6, 8], but its rigorous mathematical analysis is beyond the scope of this paper.

2.2 Objective Function

The objective function to optimize the process $(a_t)_{t\geq 0}$ is formulated. We focus on a discounted case whose long-run limit is an infinite horizon case. Assume that the time horizon of the optimization problem is $\Omega_T = [0,T]$ with a terminal time $T > 0$. Our objective function ϕ is a functional of the initial condition and the process $(a_t)_{t\geq 0}$:

$$\begin{aligned}\phi\left(I_0, O_0, V_0; a_{(\cdot)}\right) &\equiv \mathbb{E}^{0,i,o,v}\left[\int_0^T f\left(I_s, O_s, V_s, a_s\right)\mathrm{d}s\right] \\ &= \mathbb{E}^{0,i,o,v}\left[\int_0^T e^{-\delta s}\left(\frac{w_1}{2}a_s^2 + \frac{w_2}{2}\left(I_s - O_s\right)^2 + \frac{w_3}{2}\left(V_s - \hat{V}\right)^2 + \frac{w_4}{2}\max\left\{\hat{O} - O_s, 0\right\}^2\right)\mathrm{d}s\right]\end{aligned}. \quad (5)$$

Here, $\mathbb{E}^{t,i,o,v}$ is the conditional expectation using the information $(t, I_t, O_t, V_t) = (t, i, o, v)$, $\delta > 0$ is the discount rate, $\hat{V} \in \Omega_V$ is the target water volume, and $\hat{O} \in \Omega_O$ is the threshold discharge, and $w_1, w_2, w_3, w_4 \geq 0$ are weighting coefficients satisfying $w_1 w_2 w_3 w_4 \neq 0$. The parameters in (5) are allowed to be time-dependent if necessary.

In the expectation of (5), the first term penalizes sudden changes of the outflow discharge; the second term penalizes the deviation from the run-of-river condition ($I_t = O_t$) not only for sustainable operation of the reservoir with smaller impacts against the downstream river but also for continuous hydropower generation [21]; the third term penalizes deviation of the water volume from the prescribed target value $\hat{V}$; and the fourth term penalizes outflow discharges smaller than the prescribed threshold value $\hat{O}$ below which the downstream environment is severely threatened [6]. The discount rate δ is the inverse of the effective time length of the decision-making, meaning that the decision-maker, an operator of the dam and reservoir system, controls the system considering future events of the time δ^{-1} ahead in the mean. The objective function is simple but is oriented to concurrent management of resource and environment considering multiple objectives using a scalarization technique. It reduces to be quadratic if $w_4 = 0$.

We introduce the dynamic counterpart of (5) to optimize the process $(a_t)_{t\geq 0}$ using a maximum principle approach based on FBSDEs:

$$\hat{\phi}\left(t, I_t, O_t, V_t; a_{(\cdot)}\right) = \mathbb{E}^{t,i,o,v}\left[-\int_t^T e^{\delta t} f\left(I_s, O_s, V_s, a_s\right)\mathrm{d}s\right] \quad (6)$$

and set its dynamically optimized counterpart, the value function, as

$$\Phi(t, i, o, v) = \sup_{a_{(\cdot)}} \hat{\phi}\left(t, I_t, O_t, V_t; a_{(\cdot)}\right). \quad (7)$$

Clearly, (6) is equivalent to (5) when $t = 0$. Taking the minus sign in (6) is simply to follow the existing argument of the stochastic maximum principle [4]. A maximizing control of (7) (and also minimizing (5)) is called an optimal control and is denoted as $(a_t^*)_{t\geq 0}$. We explore Markovian optimal feedback controls, with an abuse of notations, of the form $a_t^* = a_t(t, I_t, O_t, V_t)$. This is a dynamic and thus adaptive control based on the observation process $(t, I_t, O_t, V_t)_{t\geq 0}$ up to the current time.

2.3 Stochastic Maximum Principle

Stochastic maximum principle approach [4] reduces the optimization problem (6) to initial and terminal value problems of FBSDEs. It is a standard machinery for analyzing stochastic control problems in finance, economics, insurance, and related research fields. Our problem does not fall into them, but the approach is still applicable. Remarkable advantages of the approach based on FBSDEs over that using the dynamic programming principle is that the former can naturally handle high-dimensional problems and that both optimal controls and the controlled processes are derived simultaneously without any postprocessing. On the other hand, its disadvantage is that the FBSDEs are computationally inefficient for low-dimensional problems having one or two state variables. Another advantage of using FBSDEs is the capability to manage non-Markovian problems. Among these advantages, our modeling and computation benefit from the capability to manage high-dimensional problems and the characteristic that is able to simulate both the forward and backward processes simultaneously.

In our case, if we neglect the constraint on the state variables for simplicity, a lengthy but straightforward calculation [e.g., Theorem 5.4 of 4] with (6) leads to the following backward system of adjoint equations to be coupled with (1)-(3):

$$\mathrm{d}p_t^{(I)} = \left\{ w_2 \left(I_t - O_t \right) + \left(\rho + \delta \right) p_t^{(I)} - p_t^{(V)} \right\} \mathrm{d}t + \int_0^\infty \theta^{(I)}(t,z) \left\{ N(\mathrm{d}z, \mathrm{d}t) - v(\mathrm{d}z)\mathrm{d}t \right\}, \quad (8)$$

$$\mathrm{d}p_t^{(O)} = \left\{ \begin{matrix} w_2 \left(O_t - I_t \right) + \delta p_t^{(O)} + p_t^{(V)} \\ -w_4 \max\left\{ \hat{O} - O_t, 0 \right\} \end{matrix} \right\} \mathrm{d}t + \int_0^\infty \theta^{(O)}(t,z) \left\{ N(\mathrm{d}z, \mathrm{d}t) - v(\mathrm{d}z)\mathrm{d}t \right\}, \quad (9)$$

$$\mathrm{d}p_t^{(V)} = \left\{ w_3 \left(V_t - \hat{V} \right) + \delta p_t^{(V)} \right\} \mathrm{d}t + \int_0^\infty \theta^{(V)}(t,z) \left\{ N(\mathrm{d}z, \mathrm{d}t) - v(\mathrm{d}z)\mathrm{d}t \right\} \quad (10)$$

with the terminal condition $p_T^{(I)} = p_T^{(O)} = p_T^{(V)} = 0$. The triplet $\left(p_t^{(I)}, p_t^{(O)}, p_t^{(V)} \right)_{t\geq 0}$ is the adjoint variables to find an ("candidate of", see also **Remark 1**) optimal control as

$$a_t^*(t, I_t, O_t, V_t) = \min\left\{ \bar{a}, \max\left\{ -\bar{a}, p_t^{(O)} \right\} \right\} \text{ if } O_t \in \left(\underline{O}, \bar{O} \right) \text{ and } V_t \in \left(\underline{V}, \bar{V} \right), \quad (11)$$

otherwise it is replaced by a_t with the smallest $|a_t|$ to guarantee the constraints $O_t \in \Omega_O$ and $V_t \in \Omega_V$.

The right-hand side of (11) should be understood as a Markovian variable determined by $\left(t, I_t, O_t, V_t\right)$ at each t. The triplet $\left(\theta^{(I)}(t,\cdot), \theta^{(O)}(t,\cdot), \theta^{(V)}(t,\cdot)\right)_{t\geq 0}$ represents the predictable stochastic fields such that jump integral terms (8)-(10) exist. Consequently, the FBSDEs consist of the forward system (1)-(3), the backward system (8)-(10), the control law (11), and the corresponding initial and terminal conditions. The integrated system is a jump-driven fully-coupled FBSDEs not well-studied previously.

Finally, we introduce Markovian representations of (8)-(10) that are employed in numerical discretization of the FBSDEs:

$$p_t^{(I)} = \mathbb{E}^{t,I_t,O_t,V_t}\left[\int_t^T \left\{-w_2\left(I_s - O_s\right) - \left(\rho + \delta\right) p_s^{(I)} + p_s^{(V)}\right\} \mathrm{d}s\right], \tag{12}$$

$$p_t^{(O)} = \mathbb{E}^{t,I_t,O_t,V_t}\left[\int_t^T \left\{-w_2\left(O_s - I_s\right) + w_4 \max\left\{\hat{O} - O_s, 0\right\} - \delta p_s^{(O)} - p_s^{(V)}\right\} \mathrm{d}s\right], \tag{13}$$

$$p_t^{(V)} = \mathbb{E}^{t,I_t,O_t,V_t}\left[\int_t^T \left\{-w_3\left(V_s - \hat{V}\right) - \delta p_s^{(V)}\right\} \mathrm{d}s\right]. \tag{14}$$

Remark 1 Recall that we have neglected the state constraint in the derivation of the backward system (8)-(10). Incorporating a state-constraint to the stochastic maximum principle results in FBSDEs having state constraints in both forward and backward systems [22]. In our case, neglecting the constraint implies that the FBSDEs give only sub-optimal controls that are inferior to optimal ones. Nevertheless, we numerically show that the feasible controls based on the presented FBSDEs serve well.

2.4 An Exact Solution

The derived FBSDEs are analytically solvable under the unconstrained case because it then reduces to a linear-quadratic problem. The following proposition states that a solution to the backward system (8)-(10) is derived as a function of the (controlled) forward process $\left(t, I_t, O_t, V_t\right)$. This analytical solution is used to verifying our numerical schemes. Hereafter, A_t' represents $\frac{\mathrm{d}A_t}{\mathrm{d}t}$ etc.

Proposition 1 *Assume $w_4 = 0$ and $\Omega_O, \Omega_V, A = \mathbb{R}$. Then the backward system (8)-(10) admits a unique square-integrable solution of the following parametric form*

$$p_t^{(I)} = A_t I_t + B_t O_t + C_t V_t + D_t, \tag{15}$$

$$p_t^{(O)} = B_t I_t + F_t O_t + G_t V_t + H_t, \tag{16}$$

$$p_t^{(V)} = C_t I_t + G_t O_t + L_t V_t + P_t, \tag{17}$$

with the following backward system of Riccati type ODEs for $t < T$:

$$A_t' = w_2 + (2\rho + \delta)A_t - 2C_t - w_1^{-1}B_t^2 \,,\; A_T = 0 \,, \tag{18}$$

$$B_t' = -w_2 + (\rho + \delta)B_t + C_t - G_t - w_1^{-1}B_t F_t \,,\; B_T = 0 \,, \tag{19}$$

$$C_t' = (\rho + \delta)C_t - L_t - w_1^{-1}B_t G_t \,,\; C_T = 0 \,, \tag{20}$$

$$D_t' = -(\rho \underline{I} + M_1)A_t + (\rho + \delta)D_t - P_t - w_1^{-1}B_t H_t \,,\; D_T = 0 \,, \tag{21}$$

$$F_t' = w_2 + \delta F_t + 2G_t - w_1^{-1}F_t^2 \,,\; F_T = 0 \,, \tag{22}$$

$$G_t' = \delta G_t + L_t - w_1^{-1}F_t G_t \,,\; G_T = 0 \,, \tag{23}$$

$$H_t' = -(\rho \underline{I} + M_1)B_t + \delta H_t + P_t - w_1^{-1}F_t H_t \,,\; H_T = 0 \,, \tag{24}$$

$$L_t' = w_3 + \delta L_t - w_1^{-1}G_t^2 \,,\; L_T = 0 \,, \tag{25}$$

$$P_t' = -(\rho \underline{I} + M_1)C_t - w_3 \hat{V} + \delta P_t - w_1^{-1}G_t H_t \,,\; P_T = 0 \,. \tag{26}$$

The proof of **Proposition 1** is omitted since it uses a direct substitution of (15)-(17) into the FBSDEs. The uniqueness and optimality follow from proofs similar to the existing ones [Theorem 5.4 of 4, Theorem 3.1.1 of 12].

3 Numerical Experiments

3.1 Discretization

Numerical computation of the FBSDEs consists of explicit discretization of the forward system (1)-(3) using (11) and an explicit or semi-implicit discretization of the backward system (8)-(10). We use a Monte-Carlo method with $n >> 1$ sample paths and temporal resolution $\Delta t = k^{-1}T$ with $k \in \mathbb{N}$. The SDEs (2)-(3) are discretized with a classical forward-Euler scheme, while the SDE (1) is with the exact sampling scheme [19] free from temporal discretization errors: $I_{j\Delta t} = Y_{j\Delta t} + \underline{I}$ and at each time step

$$Y_{(j+1)\Delta t} = e^{-\rho\Delta}Y_{j\Delta t} + \eta_0 + \sum_{l=1}^{\bar{N}(\Delta t)} \eta_l \quad (j = 0,1,2,\dots),\; Y_0 = I_0 - \underline{I} \tag{27}$$

Here, η_0 is a tempered stable variable with the exponent b based on the stable one:

$$\left(\frac{a\left(1-e^{-\alpha\rho\Delta t}\right)\Gamma(1-\alpha)}{\alpha\cos(V)}\right)^{\frac{1}{\alpha}} \sin\left(\alpha\left(V+\pi/2\right)\right)\left(\frac{\cos\left(V-\alpha\left(V+\pi/2\right)\right)}{\mathrm{e}}\right)^{\frac{1-\alpha}{\alpha}}, \tag{28}$$

where V follows a uniform distribution in $(0,1)$, e follows an exponential distribution with the intensity 1, $\bar{N}(\Delta t)$ is a Poisson process with the intensity $-a\left(1-e^{-\alpha\lambda\Delta t}\right)\Gamma(-\alpha)b^{\alpha}$, and each η_l ($l=1,2,3,...$) are independent random variables generated by the probability density function

$$p(\eta)=\frac{1}{\left(1-e^{\alpha\lambda\Delta t}\right)\Gamma(-\alpha)b^{\alpha}}\eta^{-(1+\alpha)}\left(e^{-b\eta}-e^{-be^{\lambda\Delta t}\eta}\right),\ \eta>0\,. \tag{29}$$

The independent random variables appearing the above-presented formula can be easily generated using a common rejection sampling method. Excellent theoretical performance of the scheme compared with classical Euler-Maruyama type scheme has been demonstrated in Kawai and Masuda [19]; especially, the scheme works stably for arbitrary Δt without blowing up. This stability is important in engineering applications of the proposed model because it is not always possible to choose sufficiently small Δt under limited computational resources. This is the reason why we do not apply the classical Euler-Maruyama scheme to (1). To the best of the author's knowledge, this special scheme has not been used for computing FBSDEs.

The backward system (8)-(10) is discretized explicitly or semi-implicitly. Both schemes are represented in a unified manner as follows:

$$p_{j\Delta t}^{(I)}=\mathbb{E}^{j\Delta t,I_{j\Delta t},O_{j\Delta t},V_{j\Delta t}}\left[p_{(j+1)\Delta t}^{(I)}+\Delta t\left\{-w_2\left(I_{j\Delta t}-O_{j\Delta t}\right)-(\delta+\rho)p_{(j+S)\Delta t}^{(I)}+p_{(j+1)\Delta t}^{(V)}\right\}\right], \tag{30}$$

$$p_{j\Delta t}^{(O)}=\mathbb{E}^{j\Delta t,I_{j\Delta t},O_{j\Delta t},V_{j\Delta t}}\left[p_{(j+1)\Delta t}^{(O)}+\Delta t\left\{\begin{array}{l}-w_2\left(O_{j\Delta t}-I_{j\Delta t}\right)+w_4\max\left\{\hat{O}-O_{j\Delta t},0\right\}\\-\delta p_{(j+S)\Delta t}^{(O)}-p_{(j+1)\Delta t}^{(V)}\end{array}\right\}\right], \tag{31}$$

$$p_{j\Delta t}^{(V)}=\mathbb{E}^{j\Delta t,I_{j\Delta t},O_{j\Delta t},V_{j\Delta t}}\left[p_{(j+1)\Delta t}^{(V)}+\Delta t\left\{-w_3\left(V_{j\Delta t}-\hat{V}\right)-\delta p_{(j+S)\Delta t}^{(V)}\right\}\right], \tag{32}$$

where $S=0$ corresponds to the semi-implicit scheme while $S=1$ to the explicit scheme since we are dealing with a time-backward system. Implicit and semi-implicit numerical schemes usually require some iterative evaluation of the system of the form (30)-(32); however, the adjoint variables in the conditional expectations of (30)-(32) are of the affine form and we do not need such an iteration. For example, in the semi-implicit scheme, (32) can be rewritten to directly find $p_{j\Delta t}^{(V)}$:

$$p_{j\Delta t}^{(V)}=\left(1+\delta\Delta t\right)^{-1}\mathbb{E}^{j\Delta t,I_{j\Delta t},O_{j\Delta t},V_{j\Delta t}}\left[p_{(j+1)\Delta t}^{(V)}-\Delta t w_3\left(V_{j\Delta t}-\hat{V}\right)\right]. \tag{33}$$

Each conditional expectation in (30)-(32) must be evaluated numerically for implementing the schemes. We employ a least-squares Monte-Carlo method [14] using monomial and piece-wise smooth basis functions. The method itself is quite standard in numerical computation of FBSDEs as explained in Chassagneux et al. [14] but the admissible set $\mathcal{S}$ of basis are specialized for the proposed model:

$$\mathcal{S}=\left\{1, i^{\lambda_i} o^{\lambda_o} v^{\lambda_v}, i^{\lambda_i} \max\left\{\hat{O}-o,0\right\}^{\lambda_o} v^{\lambda_v} \,\middle|\, \lambda_i\lambda_o\lambda_v \neq 0, \lambda_i, \lambda_o, \lambda_v = 0,1,2,\ldots\right\}. \quad (34)$$

This is a collection of constant, monomial, and modified monomial functions considering the functional form of the fourth term of (5) that is inherited in the driver of (9).

After discretizing the conditional expectations of (30)-(32), the multiplication coefficient of each base is computed using a least-square procedure [14] with a classical conjugate gradient method. Other numerical solvers for inverting linear systems can be equally used as well if preferred. The optimal control (11) at each time step is implemented using the corresponding basis representation of $p_{j\Delta t}^{(O)}$.

The forward and backward systems are computed in an alternating manner using a Picard algorithm [Chapter4 of 14]. The convergence criterion of the Picard iteration in this paper is the following: the candidate of numerical solution obtained at the i th Picard iteration ($i \in \mathbb{N}$) is a numerical solution if the updated increment of $p_0^{(O)}$, which is a deterministic value because we are assuming a deterministic initial condition, between the i th and the $(i-1)$ th iteration becomes smaller than a threshold value $\varepsilon(=10^{-6})$. The candidate of numerical solution at the 0 th iteration is the initial guess of the iteration. The iteration procedure starts from simulating forward processes using the initial guesses $p_t^{(I)} \equiv 0$, $p_t^{(O)} \equiv 0$, $p_t^{(V)} \equiv 0$. In our computation below, the iteration is terminated at most 200 steps. It is less than 10 steps for the unbounded cases.

3.2 Unconstrained Case

We use the following parameter values of an existing dam and reservoir system in Japan [11]: $\bar{V}=6\times 10^7$ (m^3), $\underline{I}=0$ (m^3/s), $\alpha=0.5$ (-), $\rho=0.696$ (1/day), $a=0.195$ (m$^{1.5}$/s$^{0.5}$), $b=0.007$ (s/m^3), $I_0=O_0=\underline{I}$ (m^3/s), $V_0=\bar{V}$ (m^3). The parameter values for the objective function are determined so that each penalization is balanced: $\delta=0.1$ (1/day) assuming a decision-maker having a daily perspective, $w_1=7{,}000$, $w_2=10$, $w_3=1.2\times 10^{-3}$, $w_4=0$, $\hat{V}=0.5\bar{V}$ (m^3), $\hat{O}=5$ (m^3/s). These values are used unless otherwise specified. Statistical moments of the modelled inflow are as follows: average 5.130 (m^3/s), standard deviation 17.71 (m^3/s), skewness 12.84 (-), and kurtosis 259.1 (-), agreeing well with the observation [11]. Set $T=30$ (day) for the unconstrained case and $T=120$ (day) for the constrained case. Set $\Omega_O, \Omega_V, A=\mathbb{R}$ in the unconstrained case.

We firstly numerically analyze the unconstrained case having the exact solution of **Proposition 1**. The coefficients of the adjoint variables were obtained using a forward-Euler method with a sufficiently fine time resolution: 0.0005 (day). This numerical solution is considered as a reference solution with which performance of the explicit and semi-implicit schemes is discussed. Here, we focus on the adjoint variable $p_t^{(O)}$, namely its coefficients B_t, F_t, G_t, H_t, because they directly determine the optimal control (11). The basis employed for the computation here are $1, i, o, v$. This is the exact choice by **Proposition 1**. Hence, we can analyze statistical and temporal errors.

Fig. 1 compares these coefficients for the reference solution, numerical solution with the explicit scheme, and numerical solution with the semi-implicit scheme. The total number of sample paths is $n = 10{,}000$ and the time increment is $1/24$ (day). Both schemes capture characteristics of the reference solution despite n is not large, especially the semi-implicit scheme performs better with smaller bias. Sharp transitions of the coefficients are captured in the numerical solutions despite they have remarkably varied sizes. Both schemes have oscillatory coefficients near the initial time $t = 0$, which is considered due to using a deterministic initial condition with which least-squares Monte-Carlo methods become more ill-conditioned near $t = 0$. One may be able to avoid this issue by using some probabilistic initial condition or by adding some regularization term to the least-squares procedure.

Considering the superior performance of the semi-implicit scheme, only this scheme is analyzed in the rest of this paper. **Fig. 2** compares the coefficient F_t among

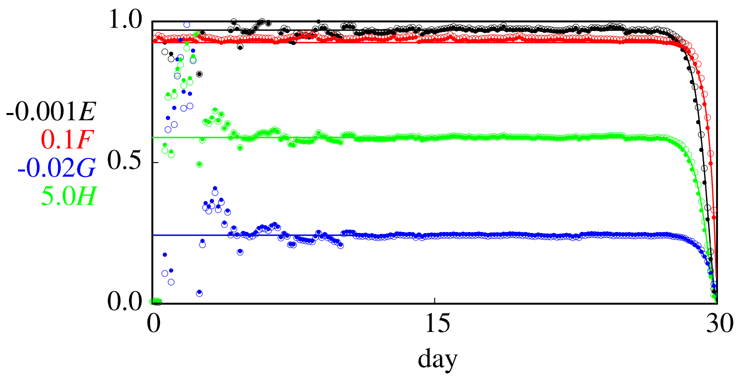

Fig. 1. Comparison of the coefficients B_t (black), F_t (red), G_t (blue), and H_t (green). Curve represent reference solution; unfilled circles (○) numerical solution with the explicit scheme; filled circles (●) numerical solution with the semi-implicit scheme.

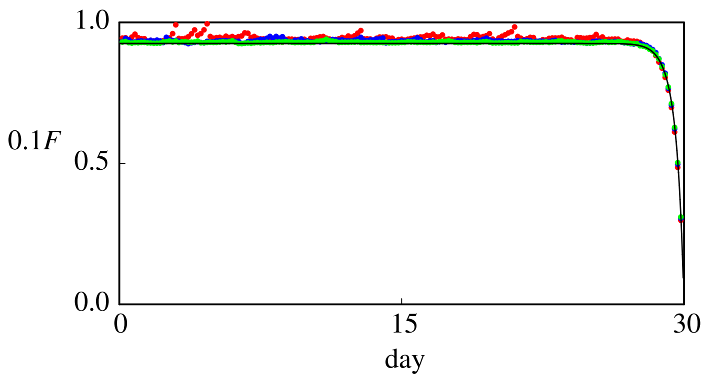

Fig. 2. Comparison of the coefficient F_t among the reference solution (black curve) and numerical solutions (unfilled circles) of the semi-implicit scheme for different computational resolution: $N = 2{,}500$ (red), $N = 10{,}000$ (blue), and $N = 40{,}000$ (green).

the reference solution (black curve) and numerical solutions (unfilled circles) of the semi-implicit scheme for different computational resolution: $n = 2{,}500$ (red),

$n = 10,000$ (blue), and $n = 40,000$ (green), where the time increment depends on n as $\Delta t = \sqrt{n}/100$ based on the basic error analysis result of FBSDEs with Monte-Carlo method [e.g., Chapter 14 of 4]. Increasing n gives less oscillatory numerical solutions closer to the reference one. The spiky oscillation visible for smaller n is considered due to the lack of total number of samples to be used in the least-squares Monte-Carlo method. Similar problems would be encountered in simulating FBSDEs driven by large jumps but have not been reported so far. Sampling large jumps are insufficient for the small n because they are rare. The l^2-errors of the coefficient F for $n = 2,500$, $n = 10,000$, and $n = 40,000$ in $(0,T)$ are 0.2204, 0.0992, and 0.0496, respectively, suggesting an almost first-order convergence of the scheme in Δt.

3.3 Constrained Case

We compare the impacts of basis in computing more realistic cases having the state constraint. We choose $w_4 = 5$ to consider a case that is not a linear-quadratic type, and set $w_1 = 3,000$ and $w_3 = 1.2 \times 10^{-4}$ to allow for larger acceleration of the outflow discharge. The semi-implicit scheme is employed here, and the two sets of basis functions are considered for approximating the adjoint variables. The first set of basis is $\mathcal{S}_1 = \{1, i, o, v\}$ that is considered to be too simple for the constrained case because of considering the constraint as well as $w_4 > 0$. The second set of basis is $\mathcal{S}_2 = \left\{1, i, o, v, i^{\lambda_I} \max\{O - o, 0\}^{\lambda_O} v^{\lambda_V}\right\}$ where $(\lambda_I, \lambda_O, \lambda_V)$ is $(0,2,0)$, $(1,1,0)$, and $(0,1,1)$. We also examined other choices of the basis such as $\mathcal{S}_2$ equipped with the base $(\lambda_I, \lambda_O, \lambda_V) = (1,0,1)$, but they did not converge. We empirically found that using too correlated basis or monomials having a too high order does not converge. However, at this stage, criterion to choose basis to guarantee both stability and convergence of numerical solutions has not been found. In addition, we artificially multiply 3 by the last term of (27) to emulate stronger jumps like floods under severe climate changes that would be more challenging to manage.

Fig. 3 compares performance of both sets of the basis against the same sample path of the inflow discharge. The computational results are not critically different between the two sets of the basis, but there is a visible difference between the controlled water volumes near $t = T$. The fact that the set $\mathcal{S}_1$, which is a too simple to approximate the adjoint variables, reasonably works suggests usefulness of sub-optimal controls in analyzing the constrained problem. Depletion of water ($V_t = 0$) was not observed in both cases, but its probability is theoretically positive although it would be small. Concerning the optimized ϕ of (5), we get 15278.2 with $\mathcal{S}_2$ and 15278.5 with $\mathcal{S}_1$, showing that the former performs better but the difference is small. Balancing sparsity and representability would be necessary for stable and convergent numerical computation of FBSDEs of dam and reservoir systems.

Finally, the numerical computation here does not cover more challenging problems to manage extreme floods and draughts possibly encountered once every few decades. Impacts of the state constraint can be more critical in such extreme cases.

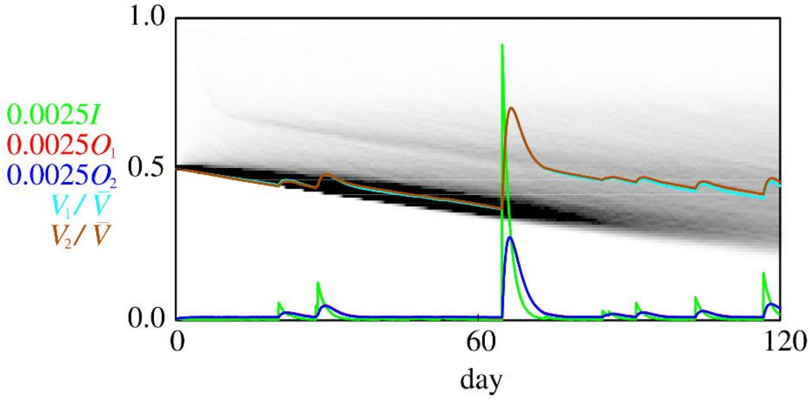

Fig. 3. The inflow discharge (green), the corresponding outflow discharges, and the water volumes. Colors of the legends correspond to the colored curves. The sub-scripts 1 and 2 represent the set of basis $\mathcal{S}_1$ and $\mathcal{S}_2$. The black and white contour plots are proportional to the probability density of the water volume with $\mathcal{S}_2$; $\mathcal{S}_1$ gives a similar result.

4 Conclusion

We analyzed a new jump-driven stochastic control problem of dam and reservoir systems receiving stochastic inflow. The FBSDEs to find the optimal acceleration of outflow discharge were derived based on a maximum principle. The FBSDEs were solvable analytically in the unbounded state. The exact solution itself is useful because it serves as a benchmark for evaluating accuracy of numerical schemes and can also be used as an initial guess of iteration schemes for solving extended problems in future.

For the unconstrained case, the least-squares Monte-Carlo methods generated reasonable numerical solutions except near the initial time $t = 0$ at which the deterministic initial condition was specified. The computational results suggested that using too complicated basis do not converge, suggesting importance of analyzing mathematical structure of the problem, especially regularity of solutions to the FBSDEs. Indeed, the computational results of the second case where the state variables are constrained suggested that using basis considering regularity of the coefficients in the objective function works well. Although the true solution of the FBSDEs in this case was not found, we numerically demonstrated that the controlled outflow tracks the inflow while effectively following the targeted states. Solving the forward SDEs via parallel computing, which was not used here, is a valuable option for more efficient computation.

Our contribution in this paper is only a starting point for modeling and control of dam and reservoir systems based on FBSDEs, and there remain a number of challenges. Firstly, the full well-posedness of the FBSDEs considering the constraints of the adjoint state variables, as implied in **Remark 1**, should be addressed theoretically. We expect that this issue is resolved by a singular control approach [23]. For an engineering implementation, we must construct approximation sequences of singular control variables, but this is an unresolved issue in general. We can somehow manage this issue if a closed-form solution to the corresponding FBSDEs is found.

Secondly, not only water quantity dynamics but also water quality dynamics are important because both critically affect environmental and ecological conditions of the reservoir and further its downstream river [24]. Other factors affecting the operation goal, such as flood mitigation, should also be considered as well when necessary. Adding these factors to the problem as new state variables is straightforward, but the size of the system and thus computational cost increase. Establishment of a massive computational environment is necessary to numerically investigate such extended problems. The forward SDEs can be simulated in a parallel manner, while the backward SDEs are not necessarily so unless the special basis having disjoint domains are employed [25]. Exploring the existence issue of such useful basis for jump-driven FBSDEs is interesting both from mathematical and engineering standpoints. To construct sparse as well as well-functioning basis is also important. We are currently addressing this issue based on a regularized least-squares Monte-Carlo approach.

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