Morgan-Stone Lattices versus De Morgan Lattices

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MORGAN-STONE LATTICES VERSUS DE MORGAN LATTICES

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Abstract. *Morgan-Stone (MS) lattices* are axiomatized by the constant-free identities of those axiomatizing *Morgan-Stone (MS) algebras*, in which case double negation is an endomorphism of any MS lattice onto its De Morgan lattice subalgebra, and so this point has interesting consequences concerning the issues of lattices of [quasi-]varieties of MS lattices facilitating finding these much. First, we prove that the variety of MS lattices is the quasi-variety generated by a six-element one with lattice reduct being the direct product of the three- and two-element chain lattices, in which case subdirectly-irreducible MS lattices are exactly isomorphic copies of non-one-element subalgebras of the six-element generating MS lattice with the double negation endomorphism kernel being the only non-trivial congruence of any non-simple one, and so, by a universal tool elaborated here, we get a 29-element non-chain distributive lattice of varieties of MS lattices, isomorphic to the one of sets of such subalgebras containing embedable ones, subsuming the four-/three-element chain one of "De Morgan"/Stone lattices/algebras (viz., constant-free versions of De Morgan algebras)/(more precisely, their term-wise definitionally equivalent constant-free versions, called *Stone lattices*). And what is more, we prove that any sub-quasi-variety of the *quasi-equational join* (viz., the quasi-variety generated by the union) of a finitely-generated quasi-variety of MS lattices and the variety of De Morgan lattices, including the former, is the quasi-equational join of its intersection with the latter and the former. As a consequence, using the eight-element non-chain distributive lattice L of quasi-varieties of De Morgan lattices, found earlier, we prove that the lattice of sub-quasi-varieties of "the [quasi-]equational join of the varieties of De Morgan and Stone lattices"/"the unbounded equational approximation of MS algebras (viz., the greatest variety of MS lattices without bounded members not expandable to MS algebras)", being a non-chain distributive (15/29)-element one embedable into the direct product of L and a (2/5)-element chain, is constituted by 2/5 planes, each being isomorphic to the filter F of L with least element, being the intersection of that Q of the plane and the variety of De Morgan lattices, and consisting of the quasi-equational joins of Q and elements of F.

1. Introduction

The notion of *De Morgan lattice*, being originally due to [7], has been independently explored in [5] under the term *distributive i-lattice* w.r.t. their subdirectly-irreducibles and the lattice of varieties. They satisfy so-called *De Morgan identities*. On the other hand, these are equally satisfied in *Stone algebras* (cf., e.g., [4]). This has inevitably raised the issue of unifying such varieties. Perhaps, a first way of doing it within the framework of De Morgan algebras (viz., bounded De Morgan lattices; cf., e.g., [1]) has been due to [2] under the term *Morgan-Stone (MS) algebra* providing a description of their subdirectly-irreducibles. Here, we study unbounded MS algebras naturally called *Morgan-Stone (MS) lattices*.

The rest of the work is as follows. Section 2 is a concise summary of basic set-theoretical and algebraic issues underlying the work. Next, in Section 3, we elaborate a universal tool of finding the lattice of relative sub-varieties of finite

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sub-classes of congruence-distributive varieties, consisting of finite algebras, each non-one-element non-simple subalgebra of which has an endomorphism with kernel, being the only non-trivial congruence of the subalgebra. Then, in Section 4, we apply it to finding the lattices of varieties of [bounded] MS lattices [properly subsuming MS algebras]. Likewise, in Section 5, upon the basis of the double negation endomorphism of MS lattices and the eight-element non-chain distributive lattice of quasi-varieties of De Morgan lattices found in [9], we find a (15/29)-element one of sub-quasi-varieties of "the [quasi-]equational join of De Morgan and Stone lattices"/"the unbounded equational approximation of MS algebras".

2. General background

2.1. Set-theoretical background. Non-negative integers are identified with sets/ordinals of lesser ones, their set/ordinal being denoted by ω. Unless any confusion is possible, one-element sets are identified with their elements.

Given any sets A, B, D and $\theta \subseteq A^2$, let $\wp_{(\omega)}([B,]A)$ be the set of all (finite) subsets of A [including B], $(\Delta_A|\nu_\theta) \triangleq \{\langle a, a|\theta[\{a\}]\rangle \mid a \in A\}$ and $\chi^B_A \triangleq (((A \cap B) \times \{1\}) \cup ((A \setminus B) \times \{0\}))$, A-tuples {viz., functions with domain A} being written in the sequence form $\bar{t}$ with t_a, where $a \in A$, standing for $\pi_a(\bar{t})$. Then, given any $\overline{S} \in \wp(D)^B$ and $\bar{f} \in \prod_{b\in B} S_b^A$, we have its *functional product* $(\prod^{\mathrm{F}} \bar{f}) : A \to (\prod_{b\in B} S_b), a \mapsto \langle f_b(a)\rangle_{b\in B}$ such that

$$\ker(\prod\nolimits^{\mathrm{F}} \bar{f}) = (A^2 \cap (\bigcap_{b\in B} (\ker f_b))), \tag{2.1}$$

$$\forall b \in B : f_b = ((\prod\nolimits^{\mathrm{F}} \bar{f}) \circ \pi_b), \tag{2.2}$$

$f_0 \odot f_1$ standing for $(\prod^{\mathrm{F}} \bar{f})$, whenever $B = 2$.

A *lower/upper cone of* a poset $\mathcal{P} = \langle P, \leqq\rangle$ is any $C \subseteq P$ such that, for all $a \in C$ and $b \in P$, $(a \geqq / \leqq b) \Rightarrow (b \in C)$. Then, an $a \in S \subseteq P$ is said to be *minimal/maximal in* S, if $\{a\}$ is a lower/upper cone of S, their set being denoted by $(\min/\max)_{\mathcal{P}|\leqq}(S)$, in case of the equality of which to S, this is called an *antichain of* $\mathcal{P}$.

An $X \in Y \subseteq \wp(A)$ is said to be *meet-irreducible in* Y, if $\forall Z \in \wp(Y) : ((A \cap (\bigcap Z)) = X) \Rightarrow (X \in Z)$, their set being denoted by $\mathrm{MI}(Y)$.

2.2. Algebraic background. Unless otherwise specified, we deal with a fixed but arbitrary finitary functional signature Σ, Σ-algebras/"their carriers" being denoted by /respective [multiple] capital Fraktur/Italic letters (with /same indices|suffices) "their class being denoted by A_Σ"/. Let Tm_Σ be the set of Σ-terms with variables in $\{x_i\}_{i\in\omega}$ and $\mathrm{Eq}_\Sigma \triangleq \mathrm{Tm}^2_\Sigma$, any $([\langle\Gamma,]\langle\phi,\psi\rangle[\rangle]) \in ([\wp_\omega(\mathrm{Eq}_\Sigma)\times]\,\mathrm{Eq}_\Sigma)$ being viewed as a Σ*[-quasi]-equation/-identity* $[\Gamma \to](\phi \approx \psi)$ /"identified with the universal closure of $[\bigwedge\Gamma \to](\phi \approx \psi)$, in which case, providing $\Sigma_+ \triangleq \{\wedge,\vee\} \subseteq \Sigma$, $\phi \lessapprox \psi$ stands for $(\phi \wedge \psi) \approx \phi$, while, for any Σ-algebra $\mathfrak{A}$ and $\bar{a} \in A^2$, $a_{0|1}(\leqslant|\geqslant)^{\mathfrak{A}} a_{1|0}$ means $\mathfrak{A} \models (x_0 \lessapprox x_1)[x_i/a_i]_{i\in 2}$, whereas, for any $\diamond \in L_+$ [and $n \in (\omega \setminus 1)$], $\diamond_{1[+n]}(\bar{x}_{1[+n]}) \triangleq ([\diamond_n(\bar{x}_n)\diamond]x_{0[+n]})$. The set $[\mathcal{Q}]\mathcal{E}(\mathsf{K})$ of Σ[-quasi]-identities true in a $\mathsf{K} \subseteq \mathsf{A}_\Sigma$ is called its *[quasi-]equational theory.* Given a unary $\wr \in \Sigma$ and a $\varphi \in \mathrm{Tm}_\Sigma$, (by induction on $n \in (\omega \setminus 1)$) set $\wr^{0(+n)}\varphi \triangleq (\wr\wr^{n-1})\varphi$.

A subclass of $(\mathsf{K} \subseteq)\mathsf{A}_\Sigma$ "closed under $(\mathsf{K}\cap)\mathbf{I}[\mathbf{S}]|\mathbf{S}|\mathbf{P}^{\{\mathrm{U}\}}$"/"containing every Σ-algebra with all finitely-generated subalgebras in the class" is referred to as *"(relatively) [hereditarily-]abstract|hereditary|{ultra-}multiplicative"/local* ⟨cf. [6]⟩. Given a $\mathsf{K} \subseteq \mathsf{A}_\Sigma \ni \mathfrak{A}$, set $\hom^{[\mathrm{S}]}_{(\mathrm{I})}(\mathfrak{A},\mathsf{K}) \triangleq \{h \in \hom(\mathfrak{A},\mathfrak{B}) \mid \mathfrak{B} \in \mathsf{K}[,(\mathrm{img}\,h) = B](,(\ker h) = \Delta_A)\}$ and $\mathrm{Co}_\mathsf{K}(\mathfrak{A}) \triangleq \{\theta \in \mathrm{Co}(\mathfrak{A}) \mid (\mathfrak{A}/\theta) \in \mathsf{K}\}$, in which case, for all

$\mathfrak{B} \in \mathsf{A}_\Sigma$ and $h \in \hom^{[\mathrm{S}]}(\mathfrak{A}, \mathfrak{B})$:

$$\forall \theta \in \mathrm{Co}(\mathfrak{B}) : (\ker h) \subseteq h^{-1}[\theta] \triangleq \{a \in A^2 \mid (a \circ h) \in \theta\} \in \mathrm{Co}(\mathfrak{A}) \tag{2.3}$$
$$[\forall \vartheta \in (\mathrm{Co}(\mathfrak{A}) \cap \wp(\ker h, A^2)) : h[\vartheta] \triangleq \{b \circ h \mid b \in \vartheta\} \in \mathrm{Co}(\mathfrak{B}),$$
$$\vartheta = h^{-1}[h[\vartheta]], \theta = h[h^{-1}[\theta]],$$

and so the posets $\mathrm{Co}(\mathfrak{A}) \cap \wp(\ker h, A^2)$ and $\mathrm{Co}(\mathfrak{B})$ partially ordered by inclusion are isomorphic], while, by the Homomorphism Theorem:

$$\ker[\hom^{\mathrm{S}|}(\mathfrak{A}, \mathsf{K})] = \mathrm{Co}_{(\mathbf{I}|(\mathbf{IS}))\mathsf{K}}(\mathfrak{A}), \tag{2.4}$$

and so, since, for any set I, $\overline{\mathfrak{B}} \in \mathsf{A}_\Sigma^I$ and $\bar{f} \in (\prod_{i \in I} \hom(\mathfrak{A}, \mathfrak{B}_i))$:

$$(\prod\nolimits^{\mathrm{F}} \bar{f}) \in \hom(\mathfrak{A}, \prod_{i \in I} \mathfrak{B}_i), \tag{2.5}$$

taking $I \triangleq \mathrm{Co}_{(\mathbf{I}|(\mathbf{IS}))\mathsf{K}}(\mathfrak{A})$, $\overline{\mathfrak{B}} \triangleq \langle \mathfrak{A}/i \rangle_{i \in I} \in ((\mathbf{I}|(\mathbf{IS}))\mathsf{K})^I$ and $\bar{f} \triangleq \langle \nu_i \rangle_{i \in I} \in (\prod_{i \in I} \hom^{\mathrm{S}}(\mathfrak{A}, \mathfrak{B}_i))$, by (2.1) and (2.2), we eventually get:

$$(\mathfrak{A} \in \mathbf{IP}^{\mathrm{SD}}(\{\mathbf{I}\}|(\{\mathbf{I}\}\mathbf{S}))\mathsf{K}) \Leftrightarrow ((A^2 \cap (\bigcap \ker[\hom^{\mathrm{S}|}(\mathfrak{A}, \mathsf{K})])) = \Delta_A). \tag{2.6}$$

According to [10], *pre-varieties* are abstract hereditary multiplicative classes of Σ-algebras, $\mathbf{ISP}\mathsf{K} = \mathbf{IP}^{\mathrm{SD}}\mathbf{S}_{(>1)}\mathsf{K}$ being the least one including a {finite} class K of {finite} Σ-algebras and so said to be *generated by* this {and *finitely-generated*}. Likewise, *[quasi-]varieties* are hereditary [ultra-]multiplicative classes closed under $\mathbf{H}^{[\mathrm{I}]}[\triangleq \mathbf{I}]$ (these are exactly model classes of sets of Σ-[quasi-]identities, and so are local and also said to be *[quasi-]equational*; cf., e.g., [6]), $\mathbf{H}^{[\mathrm{I}]}\mathbf{SP}^{[\mathrm{U}]}\mathsf{K} = \mathrm{Mod}([\mathcal{Q}]\mathcal{E}(\mathsf{K}))[\{= \mathbf{ISP}\mathsf{K}$; cf., e.g., [3, Corollary 2.3]}] being the least one including K and so said to be *generated by* this {and *finitely-generated*}. Then, intersections of a $\mathsf{K} \subseteq \mathsf{A}_\Sigma$ with [quasi-/pre-]varieties are called its *relative sub-[quasi-/pre-]varieties*, in which case, for any $\mathcal{I} \subseteq \mathrm{Eq}_\Sigma$,

$$(\mathbf{IP}^{\mathrm{SD}}(\mathsf{K}) \cap \mathrm{Mod}(\mathcal{I})) = \mathbf{IP}^{\mathrm{SD}}(\mathsf{K} \cap \mathrm{Mod}(\mathcal{I})), \tag{2.7}$$

and so $\mathsf{S} \mapsto (\mathsf{S} \cap \mathsf{K})$ and $\mathsf{R} \mapsto \mathbf{IP}^{\mathrm{SD}}\mathsf{R}$ are inverse to one another isomorphisms between the lattices of relative sub-varieties of $\mathbf{IP}^{\mathrm{SD}}\mathsf{K}$ and those of K. Furthermore, a variety $\mathsf{V} \subseteq \mathsf{A}_\Sigma$ is said to be *congruence-distributive*, if, for each $\mathfrak{A} \in \mathsf{V}$, the lattice $\mathrm{Co}(\mathfrak{A})$ is distributive. Given [quasi-]varieties $\mathsf{Q}, \mathsf{Q}' \subseteq \mathsf{A}_\Sigma$, their *[quasi-]equational join* is the [quasi-]variety $\mathsf{Q} \uplus^{[\mathrm{Q}]} \mathsf{Q}'$ generated by $\mathsf{Q} \cup \mathsf{Q}'$, the lattice of sub-[quasi-]varieties of Q (including Q') with meet/join $\cap/\uplus^{[\mathrm{U}]}$ being denoted by $\mathfrak{L}_{[\mathrm{Q}]}((\mathsf{Q}',)\mathsf{Q})$.[1]

Finally, recall that an $\mathfrak{A} \in \mathsf{A}_\Sigma$ is said to be *simple/subdirectly-irreducible*, if $\Delta_A \in (\max_\subseteq / \mathrm{MI})(\mathrm{Co}(\mathfrak{A}) \setminus (\{A^2\}/\varnothing))$, in which case $|A| \neq 1$, the class of [those of] them [which are in a $\mathsf{K} \subseteq \mathsf{A}_\Sigma$] being denoted by $(\mathrm{Si}/\mathrm{SI})[(\mathsf{K})]$ and, by (2.3), being [relatively] abstract, and so, by (2.3),

$$(\mathrm{Si} \mid \mathrm{SI})(\mathbf{IP}^{\mathrm{SD}}(\mathbf{S})\mathsf{K}') \subseteq \mathbf{I}(\mathbf{S}_{>1})\mathsf{K}', \tag{2.8}$$

for any $\mathsf{K}' \subseteq \mathsf{A}_\Sigma$. Then, varieties without non-simple subdirectly-irreducibles are said to be *semi-simple*.

[1]Though being proper classes, [quasi-]varieties, being model classes of their [quasi-]equational theories, are uniquely determined by these, so, under identification with them, are allowed to be viewed as sets and to constitute {po}sets, lattices, etc.

3. Preliminaries

A *congruence-determining endomorphism for/of* a non-simple non-one-element Σ-algebra $\mathfrak{A}$ is any $h \in \hom(\mathfrak{A}, \mathfrak{A})$ such that $\mathrm{Co}(\mathfrak{A}) = \{\Delta_A, \ker h, A^2\}$, in case of existence of which $\mathfrak{A}$ is called *endo-pre-simple*, either simple or endo-pre-simple Σ-algebras [without non-pre-simple non-one-element subalgebras] being called *[(non-trivially-)hereditarily] pre-simple* and being, clearly, subdirectly-irreducible.

Theorem 3.1. *Let K be a [finite] class of [finite] hereditarily pre-simple Σ-algebras [$\mathsf{V} \supseteq \mathsf{K}$ a congruence-distributive variety $\{$in particular, $\Sigma_+ \subseteq \Sigma$, while each member of $\mathsf{K}{\upharpoonright}\Sigma_+$ is a lattice; cf.* [8]$\}$*] and $\mathsf{K}' \subseteq \mathbf{IS}_{>1}\mathsf{K}$. [Suppose $\mathbf{S}_{>1}\mathsf{K} \subseteq \mathbf{I}\mathsf{K}'$.] Then, for all $\mathfrak{A}, \mathfrak{B} \in \mathbf{S}_{>1}\mathsf{K}$, and every (non-)injective $h \in \hom^{\mathrm{S}}(\mathfrak{A}, \mathfrak{B})$, ($\mathfrak{A}$ is not simple with a congruence-determining endomorphism h' such that* $(\ker h') = (\ker h)$ *and)* $(h^{-1}(\circ h')) \in \hom_{\mathrm{I}}(\mathfrak{B}, \mathfrak{A})$*, in which case, for every $\mathfrak{A}' \in \mathsf{K}'$, $(\mathsf{K}' \cap \mathbf{H}\mathfrak{A}') \subseteq \mathbf{IS}\mathfrak{A}'$ [and so relative sub-varieties of K' are exactly its relatively hereditarily-abstract subclasses. In particular, for any relatively hereditarily-abstract $\mathsf{K}'' \subseteq \mathsf{K}'$, i.e., $\mathsf{K}'' = (\mathsf{K}' \cap \mathbf{IS}\mathsf{K}''\langle'\rangle)\langle$ where $\mathsf{K}''' \subseteq \mathsf{K}'\rangle$, there is a $\overline{\Phi} \in (\prod_{\mathfrak{C} \in (\mathsf{K}' \setminus \mathsf{K}'')} (\mathcal{E}(\mathsf{K}'') \setminus \mathcal{E}(\mathfrak{C})))$, K'' being then the relative sub-variety of K' relatively axiomatized by* $\mathrm{img}\,\overline{\Phi}$*].*

Proof. The []-non-optional part is by the Homomorphism Theorem. [Now, consider any relatively hereditarily-abstract $\mathsf{K}'' \subseteq \mathsf{K}'$, any set I, any $\overline{\mathfrak{D}} \in (\mathsf{K}'')^I$, any subalgebra $\mathfrak{E}$ of $\mathfrak{B}' \triangleq \prod_{i \in I} \mathfrak{D}_i$, any $\mathfrak{F} \in \mathsf{K}'$ and any $g \in \hom^{\mathrm{S}}(\mathfrak{E}, \mathfrak{F})$, in which case, as $\mathfrak{F}$ is finite, there are a finitely-generated $\mathfrak{E}' \in \mathbf{S}\mathfrak{E} \subseteq \mathbf{S}\mathfrak{B}'$, a $\mathfrak{F}' \in \mathbf{S}_{>1}\mathsf{K}$ and an $e \in \hom^{\mathrm{S}}_{\mathrm{I}}(\mathfrak{F}, \mathfrak{F}')$ such that $g' \triangleq ((g{\upharpoonright}E') \circ e) \in \hom^{\mathrm{S}}(\mathfrak{E}', \mathfrak{F}')$, while, for each $i \in I$, there are some $\mathfrak{G}_i \in \mathbf{S}_{>1}\mathsf{K}$ and some $e'_i \in \hom^{\mathrm{S}}_{\mathrm{I}}(\mathfrak{D}_i, \mathfrak{G}_i)$, and so, by (2.3) and the subdirect irreducibility of $\mathfrak{F}'$, $(\ker g') \in \mathrm{MI}(\mathrm{Co}(\mathfrak{E}'))$, as well as $\mathcal{H} \triangleq \{(\pi_i{\upharpoonright}E') \circ e'_i \mid i \in I\} \subseteq \hom(\mathfrak{E}', \mathsf{K})$ is finite, for both K and all its members are so, whereas $((E')^2 \cap (\bigcap \ker[\mathcal{H}])) = ((E')^2 \cap (\bigcap_{i \in I} \ker(\pi_i{\upharpoonright}E'))) = \Delta_{E'} \subseteq (\ker g') \subsetneq (E')^2$, for $|\,\mathrm{img}\, g'| = |F'| = |F| > 1$, $\mathcal{H}$ being thus non-empty. Then, by the congruence distributivity of $\mathsf{V} \ni \mathfrak{E}'$, there is some $f \in \mathcal{H}$ such that $(\ker f) \subseteq (\ker g')$, in which case there is some $i \in I$ such that $f' \triangleq (f \circ {e'_i}^{-1}) = (\pi_i{\upharpoonright}E') \in \hom(\mathfrak{E}', \mathfrak{D}_i)$, while $(\ker f') = (\ker f) \subseteq (\ker g') \subsetneq (E')^2$, and so $(\ker f') \neq (E')^2$, i.e., $|\,\mathrm{img}\, f'| > 1$, whereas $\mathfrak{D}' \triangleq f'[\mathfrak{E}'] \in \mathbf{S}_{>1}\mathfrak{D}_i \subseteq \mathbf{S}_{>1}\mathsf{K}' \subseteq \mathbf{IS}_{>1}\mathsf{K} \subseteq \mathbf{I}\mathsf{K}'$. Take any $\mathfrak{C}' \in (\mathsf{K}' \cap \mathbf{I}\mathfrak{D}') \neq \varnothing$ and any $h'' \in \hom^{\mathrm{S}}_{\mathrm{I}}(\mathfrak{C}', \mathfrak{D}') \neq \varnothing$, in which case $\mathfrak{C}' \in (\mathsf{K}' \cap \mathbf{IS}\mathfrak{D}_i) \subseteq (\mathsf{K}' \cap \mathbf{IS}\mathsf{K}'') \subseteq \mathsf{K}''$, while, by the Homomorphism Theorem, $(h'' \circ {f'}^{-1} \circ g' \circ e^{-1}) \in \hom^{\mathrm{S}}(\mathfrak{C}', \mathfrak{F})$, and so $\mathfrak{F} \in (\mathsf{K}' \cap \mathbf{H}\mathfrak{C}') \subseteq (\mathsf{K}' \cap \mathbf{IS}\mathfrak{C}') \subseteq (\mathsf{K}' \cap \mathbf{IS}\mathsf{K}'') \subseteq \mathsf{K}''$, as required.] $\square$

4. Morgan-Stone lattices versus distributive lattices

From now on, we deal with the signatures $\Sigma^{(-)}_{+[,01]} \triangleq (\Sigma_+(\cup\{\neg\})[\cup\{\bot, \top\}])$, [bounded] lattices being supposed $\Sigma_{+[,01]}$-algebras. For any $n \in ((\omega \setminus 2)|2)$, let $\mathfrak{D}_{n[,01]}$ be the chain bounded lattice with carrier $n|\{n\}$. Recall the well-known:

Lemma 4.1. *Let $\mathfrak{A}$ be a [bounded] lattice and $F \subseteq A$. Suppose F is either a prime filter of $\mathfrak{A}$ or in $\{\varnothing, A\}[\cap\varnothing]$. Then, (unless $F \in \{\varnothing, A\}$) $\chi^F_A \in \hom^{(\mathrm{S})}(\mathfrak{A}, \mathfrak{D}_{2[,01]})$.*

Lemma 4.2. *Let $\mathfrak{A} \in \mathsf{A}_\wedge \ni \mathfrak{B}$ be a semi-lattice with bound $b \in A$ (i.e., $b = (a \wedge^{\mathfrak{A}} b)$, for all $a \in A$) and $h \in \hom^{\mathrm{S}}(\mathfrak{A}, \mathfrak{B})$. Then, $\mathfrak{B}$ is a semi-lattice with bound $h(b)$.*

Let $(\diamond\|\flat)_{0|1} \triangleq ((\wedge\|\bot)|(\vee\|\top))$, $\mathcal{DM}_i \triangleq (\neg(x_0 \diamond_i x_1) \approx (\neg x_0 \diamond_{1-i} \neg x_1))$, $\mathcal{MN}_{i,j} \triangleq (\neg^{3\cdot i} x_0 \approx (\neg^{3\cdot i} x_0 \diamond_j \neg^{2-i} x_0))$ and $\mathcal{NB}_i \triangleq (\neg\flat_{1-i} \approx \flat_i)$, where $i, j \in 2$. Then, a *[bounded/] Morgan$\{$-Stone$\}$ ($\{MS\}$) lattice[/algebra]* is any $\Sigma^-_{+[,01]}$-algebra $\mathfrak{A} \in \mathrm{Mod}(\{\mathcal{DM}_i\}_{i \in (1\langle+1\rangle)} \cup \{\mathcal{MN}_{j,k}\}_{j \in (1\langle+1\rangle), k \in (2\{-1\})}[\cup\{\mathcal{NB}_l\}_{l \in ((0/1)\langle\cup\{1\}\rangle)}])$ with [bounded] distributive lattice $\mathfrak{A}{\upharpoonright}\Sigma_{+[,01]}$ /[[1] $\{$resp., [2]$\}$] and their variety denoted by $[\mathsf{B}/]\mathsf{M}\{\mathsf{S}\}(\mathsf{L}[/\mathsf{A}])$, in which case $\hbar^{\mathfrak{A}} \triangleq (\neg^2)^{\mathfrak{A}} \in \hom(\mathfrak{A}[{\upharpoonright}\Sigma^-_+/], \mathfrak{A}[{\upharpoonright}\Sigma^-_+/])$ and

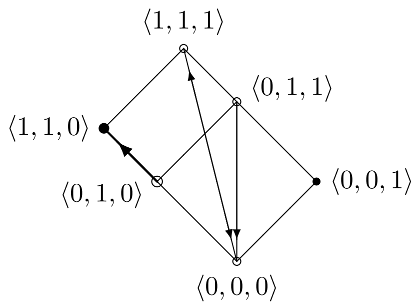

FIGURE 1. The [bounded] Morgan-Stone lattice $\mathfrak{MS}_{(6|2)[,01]}$.

$((\mathfrak{A}[\upharpoonright\Sigma^{-}_{+}/])\upharpoonright(\mathrm{img}\,\hbar^{\mathfrak{A}})) \in \mathsf{M}(\mathsf{L}[/\mathsf{A}])$, an $a \in A$ being called *(negatively) idempotent*, if $\neg^{\mathfrak{A}}(\neg^{\mathfrak{A}})a = (\neg^{\mathfrak{A}})a$, with their set $\Im^{\mathfrak{A}}_{(\neg)}(\supseteq \Im^{\mathfrak{A}})$.

4.1. **Subdirectly-irreducibles.** Let $\mathfrak{MS}_{(6|2)[,01]}$ be the [bounded] MS lattice with $\Sigma_{+[,01]}$-reduct $((\mathfrak{D}_{2[,01]} \times \mathfrak{D}_{(2|1)[,01]})\upharpoonright((2^2 \setminus \{\langle \Bbbk, 0\rangle \mid \Bbbk \in (2\setminus(1|0))\}))) \times \mathfrak{D}_{(2|0)[,01]}$ and $\neg^{\mathfrak{MS}_{(6|2)[,01]}} \triangleq \{\langle a, \langle 1 - \pi_{\min(2,3-\ell)}(a)\rangle_{\ell\in 3}\rangle \mid a \in MS_{6|2}\}$ the Hasse diagram of its lattice reduct with its (non-)idempotent elements marked by (non-)solid |large circles and arrows reflecting action of its operation $\neg$ on its non-idempotent elements |"as well as thick lines" being depicted at Figure 1. Then, $(\mathfrak{MS}_{5[,01]}|\mathfrak{MS}_{4:n[,01]}|\mathfrak{DM}_{4[,01]}|\mathfrak{K}_{3:n[,01]}|\mathfrak{S}_{3[,01]}|\mathfrak{B}_{2[,01]}) \triangleq (\mathfrak{MS}_{6[,01]}\upharpoonright((MS_6\setminus\{\langle 0,0,1\rangle\})|((MS_6 \cap \pi_2^{-1}[\{n\}]) \cup (3 \times \{1-n\}))|\hbar^{\mathfrak{MS}_6}[MS_6]|(DM_4 \cap MS_{4:n})|(MS_5 \cap MS_{4:1})|(K_{3:0} \cap K_{3:1})))$, where $n \in 2$, and members of $\mathsf{M}_{[01(+)/-]} \triangleq (\{\mathfrak{MS}_{6[,01]}\} \cup (\{\mathfrak{MS}_{2[,01]}\}[\setminus/\cap\varnothing]))$ exhaust those of $\mathsf{MS}_{[01(+)/-]} \triangleq \mathbf{S}_{>1}(\{\mathfrak{MS}_{6[,01]}\}[\cup\mathsf{M}_{01/-}])$ with isomorphic $\mathfrak{K}_{3:0[,01]}$ and $\mathfrak{K}_{3:1[,01]}$ "but without"// "being the only" isomorphic distinct members of $\mathsf{MS}_{n//2[,01(+)/-]} \triangleq (\mathsf{MS}_{[01/-]} \setminus (\{\mathfrak{K}_{3:(1-n)[,01]}\}//\varnothing))$ partially-//quasi-ordered by the embedability relation between them $\preceq_{n//[,01(+)/-]} \triangleq \{\langle \mathfrak{B}, \mathfrak{C}\rangle \in \mathsf{MS}^2_{n//[,01/-]} \mid (B \not\subseteq C) \Rightarrow (\exists m \in (\{n\}//2) : K_{3:m\|(1-m)} = \| \subseteq (B\|C))[,(B = MS_2) \Rightarrow (B = C)/]\}$ "the Hasse diagram of the poset being depicted at Figure 2"//.

Lemma 4.3. *For any* $\mathfrak{A} \in \mathbf{S}_{(>1)}\mathsf{M}_{[01]}$, $\mathrm{Co}(\mathfrak{A}) = \{\Delta_A, \ker\hbar^{\mathfrak{A}}, A^2\}$ *(in which case* $\mathfrak{A}$*, being subdirectly-irreducible, is simple iff either* $A^2 = (\ker\hbar^{\mathfrak{A}})$*, i.e.,* $A = MS_2$*, or* $\hbar^{\mathfrak{A}}$ *is injective, i.e.,* $A \subseteq DM_4$*), and so* $\{$*non-*$\}$*simple members of* $\mathsf{MS}_{n[,01]}$ *are marked by* $\{$*non-*$\}$*solid circles-nodes at Figure 2.*

Proof. Given any $I \subseteq 3$, put $\theta_I \triangleq (A^2 \cap (\bigcap_{i\in I}\ker(\pi_i\upharpoonright A)))$. Consider any $\theta \in (\mathrm{Co}(\mathfrak{A}) \setminus \{\Delta_A\}) \subseteq \mathrm{Co}(\mathfrak{A}\upharpoonright\Sigma_+)$, in which case, by the congruence-distributivity of lattices [8], the simplicity of two-element algebras, absence of their proper non-one-element subalgebras and (2.3), there is some $J \subseteq 3$ such that $\theta = \theta_J$. Take any $a \in (\theta\setminus\Delta_A) \neq \varnothing$, in which case there is some $j \in 3$ such that $\pi_j(\pi_0(a)) \neq \pi_j(\pi_1(a))$, and so $0 \notin J$, because $\theta \ni (a\circ(\neg^{(2\cdot j) \bmod 3})^{\mathfrak{A}}) \notin (\ker\pi_0)$. Then, $J \subseteq K \triangleq (3\setminus 1)$, in which case $\theta \supseteq \theta_K = (\ker\hbar^{\mathfrak{A}})$. Furthermore, unless $\theta = \theta_K$, take any $b \in (\theta \setminus \theta_K) \neq \varnothing$, in which case $(\theta \cap DM_4^2) \ni c \triangleq (b \circ (\neg^2)^{\mathfrak{A}}) = (b \circ \hbar^{\mathfrak{A}}) \notin \Delta_A$, and so there is some $k \in 3$ such that $\pi_k(\pi_0(c)) \neq \pi_k(\pi_1(c))$. In that case, since $\pi_0(\pi_l(c)) = \pi_1(\pi_l(c))$, for all $l \in 2$, no $m \in K$ is in J, because $\theta \ni (c\circ(\neg^{\max(m-\max(1,k),\max(1,k)-m)})^{\mathfrak{A}}) \notin (\ker\pi_m)$, and so $\theta = \theta_{J\cap K} = \theta_\varnothing = A^2$, as required. □

Theorem 4.4. $[\mathsf{B}/]\mathsf{MSL}[/\mathsf{A}] = \mathbf{ISP}(\{\mathfrak{MS}_{6[,01]}\}[\cup\mathsf{M}_{01/-}]) = \mathbf{IP}^{\mathrm{SD}}\mathsf{MS}_{((0|1)[,])[01/-]}$, *in which case* $\mathrm{SI}([\mathsf{B}/]\mathsf{MSL}[/\mathsf{A}]) = \mathbf{I}\mathsf{MS}_{((0|1)[,])[01/-]}$, *and so* $\mathrm{Si}([\mathsf{B}/]\mathsf{MS}(\mathsf{L}[/\mathsf{A}]) = \mathbf{I}((\{\mathfrak{MS}_{2[,01]}\}[/\varnothing]) \cup \mathbf{S}_{>1}\mathfrak{DM}_{4[,01]})$.

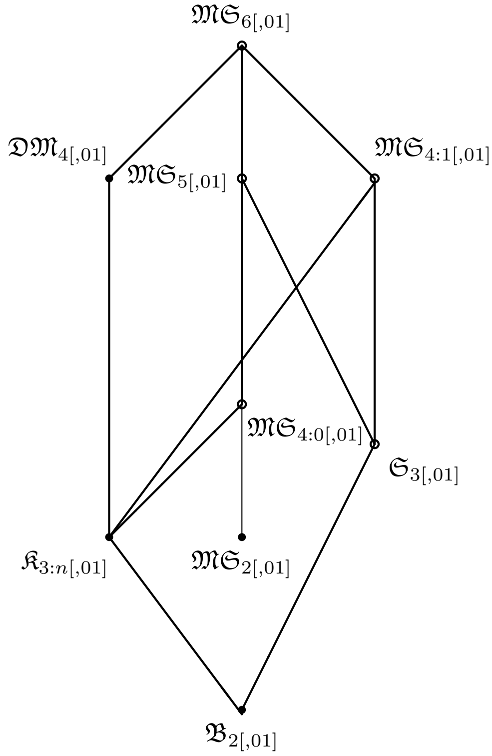

FIGURE 2. The embedability poset $\mathsf{MS}_{n[,01]}$ [with merely thick lines].

Proof. By the Prime Ideal Theorem, for any $\mathfrak{A} \in \mathsf{MSL}$ and any $a \in (A^2 \setminus \Delta_A)$, there is a prime filter F of $\mathfrak{A}{\restriction}\Sigma_+$ such that $a \notin (\ker \chi^F_A)$, in which case, by $\mathcal{DM}_{0\|1}$, $(H|G) \triangleq ((A \setminus (\neg^{\mathfrak{A}})^{-1}[F])|(\neg^{\mathfrak{A}})^{-1}[A \setminus H])$ is either a prime filter of $\mathfrak{A}{\restriction}\Sigma_+$ or in $\{\varnothing, A\}$, and so, by $\mathcal{MN}_{0\|1,0}$ and Lemma 4.1, $h \triangleq \{\langle b, \langle \chi^F_A(b), \chi^G_A(b), \chi^H_A(b)\rangle\rangle \mid b \in A\} \in \hom(\mathfrak{A}, \mathfrak{MS}_6)$ with $a \notin (\ker h)$. Then, (2.3), (2.6), (2.8), Lemmas 4.2, 4.3 and the following equality complete the argument:

$$(\mathsf{MS}_{01} \cap \mathsf{MSA}) = \mathsf{MS}_{01-}. \tag{4.1}$$ □

4.2. **The lattice of sub-varieties.** First, by Theorem 3.1, Lemma 4.3 and (4.1), we immediately have:

Corollary 4.5. *Let* $\mathsf{K} \subseteq \mathsf{M}_{[01]}$ *and* $\mathsf{K}' \subseteq \mathbf{IS}_{>1}\mathsf{K}$. $\{$*Suppose* $\mathbf{S}_{>1}\mathsf{K} \subseteq \mathbf{I}\mathsf{K}'$.$\}$ *Then, for all* $\mathfrak{A}, \mathfrak{B} \in \mathbf{S}_{>1}\mathsf{K}$ *and every (non-)injective* $h \in \hom^{\mathrm{S}}(\mathfrak{A}, \mathfrak{B})$, $(h^{-1}(\circ\hbar^{\mathfrak{A}})) \in \hom_{\mathrm{I}}(\mathfrak{B}, \mathfrak{A})$, *in which case, for each* $\mathfrak{A}' \in \mathsf{K}'$, $(\mathsf{K}' \cap \mathbf{H}\mathfrak{A}') \subseteq \mathbf{IS}\mathfrak{A}'$ $\{$*and so relative sub-varieties of* K' *are exactly its relatively hereditarily-abstract subclasses. In particular, for any relatively hereditarily-abstract* $\mathsf{K}'' \subseteq \mathsf{K}'$, *i.e.,* $\mathsf{K}'' = (\mathsf{K}' \cap \mathbf{IS}\mathsf{K}''\langle'\rangle)\langle$ *where* $\mathsf{K}''' \subseteq \mathsf{K}'\rangle$, *there is a* $\mathbf{\Phi} \in (\prod_{\mathfrak{C} \in (\mathsf{K}' \setminus \mathsf{K}'')}(\mathcal{E}(\mathsf{K}'') \setminus \mathcal{E}(\mathfrak{C})))$, K'' *being then the relative sub-variety of* K' *relatively axiomatized by* $\operatorname{img} \mathbf{\Phi}\}$.

In this way, taking (2.7) and Theorem 4.4 into account, the lattice of varieties of [bounded/] MS lattices[/algebras] is isomorphic to the one of lower cones of the poset $\langle \mathsf{MS}_{(0|1)[,01/-]}, \preceq_{(0|1)[,01/-]} \rangle$, given by Figure 2. Though the task of finding the latter, being reduced to that of finding anti-chains of the poset involved, is to be solved rather mechanically, the one of finding relative axiomatizations of lower cones of the poset under consideration is not *so* easily solvable.

Let $\varphi_n^{i,j,k,l,m} \triangleq ((\neg^j x_i \diamond_n \neg^k x_i) \diamond_n \neg^m x_l)$, where $i,j,k,l,m \in 3$ and $n \in 2$, while $\mathcal{I}^{i,j,k,l,m}_{o,\imath,\jmath,\Bbbk,\ell} \triangleq (\varphi_0^{i,j,k,l,m} \lessapprox \varphi_1^{o,\imath,\jmath,\Bbbk,\ell})$, where $o,\imath,\jmath,\Bbbk,\ell \in 3$, whereas:

$$
\begin{aligned}
\mathcal{M}_{(\mathrm{N}|\mathrm{A})} &\triangleq \mathcal{I}^{0,2,2,0(+(0|1)),2}_{0,0,0(+(1|0)),0(+(0|1)),0(+1)},\\
\mathcal{S}_{(\mathrm{A})} &\triangleq \mathcal{I}^{0,0,1,0(+2),1(+1)}_{1,0,0,1(+1),0(+1)},\\
\mathcal{K}^{(\mathrm{W})}_{\{\mathrm{M}\}} &\triangleq \mathcal{I}^{0,0,1,0\{+2\},1\{+1\}}_{1,1,0(+2),1\{+1\},1},\\
\mathcal{T} &\triangleq \mathcal{I}^{0,2,2,0,2}_{0,1,1,0,1},\\
\mathcal{Q}_{(\mathrm{A})} &\triangleq \mathcal{I}^{0,1,2,0(+1),2}_{0,0,0,0(+1),0(+1)},\\
\mathcal{P} &\triangleq \mathcal{I}^{0,2,2,0,2}_{1,2,1,0,0}.
\end{aligned}
$$

Then, members of $[\mathsf{B}/](\mathsf{N}|\mathsf{A})\{\mathsf{D}\}\mathsf{ML}[/\mathsf{A}] \triangleq ([\mathsf{B}/]\mathsf{MSL}[/\mathsf{A}] \cap \mathrm{Mod}(\mathcal{M}_{(\mathrm{N}|\mathrm{A})}))$ are called *[bounded/] (nearly|almost) {De} Morgan lattices[/algebras]*. Likewise, ones of

$$[\mathsf{B}/](\mathsf{A})\mathsf{SL}[/\mathsf{A}] \triangleq ([\mathsf{B}/]\mathsf{MSL}[/\mathsf{A}] \cap \mathrm{Mod}(\mathcal{S}_{(\mathrm{A})}))$$

are called *[bounded/] (almost) Stone lattices[/algebras]*, those of $[\mathsf{B}/](\mathsf{A})\mathsf{BL}[/\mathsf{A}] \triangleq ([\mathsf{B}/](\mathsf{A})\mathsf{SL}[/\mathsf{A}] \cap [\mathsf{B}/](\mathsf{A})\mathsf{ML}[/\mathsf{A}])$ being called *[bounded/] (almost) Boolean lattices [/algebras]*; cf. [9, Definition 3.5] for an equivalent definition in the non-optional case. Next, members of

$$[\mathsf{B}/](\mathsf{P}|\{\mathsf{A}\}\mathsf{Q})\mathsf{SMSL}[/\mathsf{A}] \triangleq ([\mathsf{B}/]\mathsf{MSL}[/\mathsf{A}] \cap \mathrm{Mod}((\mathcal{P}|\mathcal{Q})_{|\{\mathrm{A}\}}))$$

are said to be *|{almost} pseudo-|quasi-strong*, those of

$$[\mathsf{B}/]\{\mathsf{A}\}\mathsf{SMSL}[/\mathsf{A}] \triangleq ([\mathsf{B}/]\mathsf{PSMSL}[/\mathsf{A}] \cap [\mathsf{B}/]\{\mathsf{A}\}\mathsf{QSMSL}[/\mathsf{A}])$$

being said to be *{almost} strong*. Likewise, members of $[\mathsf{B}]\mathsf{TNIMSL} \triangleq ([\mathsf{B}]\mathsf{MSL} \cap \mathrm{Mod}(\mathcal{T}))$ are said to be *totally negatively idempotent*, for their elements are *all* negatively idempotent, in view of their being models of $\{\mathcal{T}, \mathcal{MN}_0\}[x_0/\neg x_0]$. Further, members of $[\mathsf{B}/]\langle\lfloor\mathsf{A}\rfloor\mathsf{Q}\rceil\mathsf{S}\rangle(\mathsf{W})\mathsf{K}\{\mathsf{M}\}\mathsf{SL}[/\mathsf{A}] \triangleq ([\mathsf{B}/]\langle\lceil\mathsf{P}|\langle\lfloor\mathsf{A}\rfloor\mathsf{Q}\rceil\mathsf{S}\rangle(\mathsf{W})\mathsf{K}\{\mathsf{M}\}\mathsf{SL}[/\mathsf{A}] \cap \mathrm{Mod}(\mathcal{K}^{(\mathrm{W})}_{\{\mathrm{M}\}}))$ are called *[bounded/] ⟨⌈ |⌊almost⌋ pseudo-|quasi-⌉strong⟩ (weakly) Kleene-{Morgan-}Stone lattices[/algebras]*. Likewise, those of

$$[\mathsf{B}/]\{\mathsf{N}|\mathsf{A}\}(\mathsf{W})\mathsf{KL}[/\mathsf{A}] \triangleq ([\mathsf{B}/]\{\mathsf{N}|\mathsf{A}\}\mathsf{DML}[/\mathsf{A}] \cap \mathrm{Mod}(\mathcal{K}^{(\mathrm{W})}))$$

are called *[bounded/] {nearly|almost} (weakly) Kleene lattices[/algebras]*. Finally, the trivial variety of one-element $\Sigma^-_{+[,01]}$-algebras is naturally denoted by $[\mathsf{B}]\mathsf{OMSL}$.

Let $\mathcal{MS}_{[01]}\langle(\mathfrak{A})\rangle \triangleq (\{[\mathcal{NB}_0,]\mathcal{M}, \mathcal{M}_\mathrm{N}, \mathcal{M}_\mathrm{A}, \mathcal{S}, \mathcal{S}_\mathrm{A}, \mathcal{Q}, \mathcal{Q}_\mathrm{A}, \mathcal{P}, \mathcal{K}, \mathcal{K}_\mathrm{M}, \mathcal{K}^\mathrm{W}, \mathcal{K}^\mathrm{W}_\mathrm{M}, \mathcal{T}\}\langle\cap \mathcal{E}(\mathfrak{A})\rangle)$ ⟨where $\mathfrak{A} \in \mathsf{MS}_{[01]}$⟩.

Lemma 4.6. *For any $\mathfrak{A} \in \mathsf{MS}_{[01]}$, $\mathcal{MS}_{[01]}(\mathfrak{A})$ is given by Table 1.*

TABLE 1. Identities of $\mathcal{MS}_{[01]}$ true in members of $\mathsf{MS}_{[01]}$.

$\mathfrak{MS}_{6[,01]}$	$\emptyset[\cup\{\mathcal{NB}_0\}]$
$\mathfrak{MS}_{5[,01]}$	$\{[\mathcal{NB}_0,]\mathcal{P}, \mathcal{K}^\mathrm{W}, \mathcal{K}^\mathrm{W}_\mathrm{M}\}$
$\mathfrak{MS}_{4:0[,01]}$	$\{[\mathcal{NB}_0,]\mathcal{M}_\mathrm{N}, \mathcal{P}, \mathcal{K}, \mathcal{K}_\mathrm{M}, \mathcal{K}^\mathrm{W}, \mathcal{K}^\mathrm{W}_\mathrm{M}\}$
$\mathfrak{MS}_{4:1[,01]}$	$\{[\mathcal{NB}_0,]\mathcal{Q}, \mathcal{Q}_\mathrm{A}, \mathcal{K}, \mathcal{K}_\mathrm{M}, \mathcal{K}^\mathrm{W}, \mathcal{K}^\mathrm{W}_\mathrm{M}\}$
$\mathfrak{DM}_{4[,01]}$	$\mathcal{MS}_{[01]} \setminus \{\mathcal{K}, \mathcal{K}^\mathrm{W}, \mathcal{S}, \mathcal{S}_\mathrm{A}, \mathcal{T}\}$
$\mathfrak{MS}_{2[,01]}$	$\mathcal{MS}_{[01]} \setminus \{[\mathcal{NB}_0,]\mathcal{M}, \mathcal{Q}, \mathcal{S}\}$
$\mathfrak{K}_{3:0\|1[,01]}$	$\mathcal{MS}_{[01]} \setminus \{\mathcal{S}, \mathcal{S}_\mathrm{A}, \mathcal{T}\}$
$\mathfrak{S}_{3[,01]}$	$\mathcal{MS}_{[01]} \setminus \{\mathcal{M}, \mathcal{M}_\mathrm{N}, \mathcal{M}_\mathrm{A}, \mathcal{T}\}$
$\mathfrak{B}_{2[,01]}$	$\mathcal{MS}_{[01]} \setminus \{\mathcal{T}\}$

Proof. Clearly, for any line of Table 1, the identities of the second column of it are true in the algebra of the first one. Conversely,

$$\begin{aligned}
\mathfrak{MS}_{(5|6)[,01]} &\not\models \mathcal{K}^{|\mathrm{W}}_{\{\mathrm{M}\}}[x_i/\langle 1-\min(1,i), 1|\max(1-i,i-1),\\
&\quad \min(1,i)\rangle]_{i\in(2\{+1\})},\\
\mathfrak{S}_{3[,01]} &\not\models \mathcal{M}_{(\mathrm{N}|\mathrm{A})}[x_i/\langle i,1,1\rangle]_{i\in(1(+(0|1)))},\\
\mathfrak{DM}_{4[,01]} &\not\models \mathcal{K}^{(\mathrm{W})}[x_i/(\langle i,i,1-i\rangle]_{i\in 2},\\
\mathfrak{MS}_{4:1[,01]} &\not\models \mathcal{P}[x_0/\langle 0,1,1\rangle, x_1/\langle 0,0,1\rangle],\\
\mathfrak{MS}_{4:0[,01]} &\not\models \mathcal{Q}_{\mathrm{A}}[x_i/\langle i,1,i\rangle]_{i\in 2},\\
\mathfrak{K}_{3:0[,01]} &\not\models \mathcal{S}_{(\mathrm{A})}[x_i/\langle \max(1-i,i-1), \max(1-i,i-1),\\
&\quad \max(0,i-1)\rangle]_{i\in(2(+1))},\\
(\mathfrak{B}|\mathfrak{MS})_{2[,01]} &\not\models (\mathcal{T}|(\mathcal{M}\|\mathcal{Q}\|\mathcal{S}[\|\mathcal{NB}_0]))[x_i/\langle 1|(i\|i\|(1-i)[\|1]),\\
&\quad 1, 1|0\rangle]_{i\in(1|(1\|1\|2[\|0]))}.
\end{aligned}$$

Then, the fact that varieties are abstract and hereditary ends the proof. □

Theorem 4.7. *Sub-varieties of* $[\mathsf{B}/]\mathsf{MS}(\mathsf{L}[/\mathsf{A}])$ *form the non-chain distributive lattice with* $29[(+11)/(-9)]$ *elements, embedable into* $(\mathfrak{D}_{4[+(3/0)]}\times\mathfrak{D}_{4[+(3/0)-1]})\times\mathfrak{D}_4$*, whose Hasse diagram with [either thick or] thin lines is depicted at Figure 3, any (non-)solid circle-node of it being marked by a (non-)semi-simple variety* $\mathsf{V}\subseteq[\mathsf{B}/]\mathsf{MS}(\mathsf{L}[/\mathsf{A}])$*, numbered from* $1[+(0/20)]$ *to* $29[+11]$ *according to Table 2 with* $\mathsf{K}\triangleq(\{\mathfrak{MS}_{2[,01]}\}[/\varnothing])$*,* $i\in 2$*,* $\mathsf{MS}_{\mathsf{V},i[,01/-]}\triangleq\max_{\preceq_{i[,01/-]}}(\mathsf{MS}_{i[,01/-]}\cap\mathsf{V})$*, given by the third column, and* $\mathbb{k}\triangleq(9\cdot(1[/0]))$ *[as well as* $\ell\triangleq(29\cdot(0/1))$*], in which case* $\mathrm{SI}(\mathsf{V})=\mathbf{IS}_{>1}\mathsf{MS}_{\mathsf{V},i[,01/-]}$*, and so* V *is the (pre-*$\|$*quasi-)variety generated by* $\mathsf{MS}_{\mathsf{V},i[,01/-]}$*,* $[\mathsf{B}]\mathsf{SMSL}$ *being that generated by* $\{\mathrm{SI}\}([\mathsf{B}]\mathsf{DML}\cup[\mathsf{B}]\mathsf{SL})$*.*

Proof. Clearly, the sets appearing in the third column of Table 2 are exactly all anti-chains of the poset $\langle\mathsf{MS}_{i[,01/-]},\preceq_{i[,01/-]}\rangle$. Then, (2.7), Theorem 4.4, Corollary 4.5 and Lemma 4.6 complete the argument. □

Table 2. Maximal subdirectly-irreducibles of varieties of [bounded/] Morgan-Stone lattices[/algebras].

$1[+\ell]$	$[\mathsf{B}]\mathsf{MS}(\mathsf{L}[/\mathsf{A}])$	$\{\mathfrak{MS}_{6[,01]}\}[\cup\mathsf{K}]$
$2[+\ell]$	$[\mathsf{B}]\mathsf{PS}\langle\mathsf{WK}\rangle\mathsf{MS}(\mathsf{L}[/\mathsf{A}])$	$\{\mathfrak{MS}_{5[,01]},\mathfrak{DM}_{4[,01]}\}[\cup\mathsf{K}]$
$3\lceil+1\rceil[+\ell]$	$[\mathsf{B}]\mathsf{WK}\lceil\mathsf{M}\rceil\mathsf{S}(\mathsf{L}[/\mathsf{A}])$	$\{\mathfrak{MS}_{5[,01]},\mathfrak{MS}_{4:1[,01]}\lceil,\mathfrak{DM}_{4[,01]}\rceil\}[\cup\mathsf{K}]$
$5[+\ell]$	$[\mathsf{B}]\mathsf{PSWKS}(\mathsf{L}[/\mathsf{A}])$	$\{\mathfrak{MS}_{5[,01]}\}[\cup\mathsf{K}]$
$6\lceil+1\rceil[+\ell]$	$[\mathsf{B}]\mathsf{K}\lceil\mathsf{M}\rceil\mathsf{S}(\mathsf{L}[/\mathsf{A}])$	$\{\mathfrak{MS}_{4:j[,01]}\mid j\in 2\}\lceil\cup\{\mathfrak{DM}_{4[,01]}\}\rceil[\cup\mathsf{K}]$
$8\lceil+1\rceil[+\ell]$	$[\mathsf{B}]\mathsf{PSK}\lceil\mathsf{M}\rceil\mathsf{S}(\mathsf{L}[/\mathsf{A}])$	$\{\mathfrak{MS}_{4:0[,01]},\mathfrak{S}_{3[,01]}\lceil,\mathfrak{DM}_{4[,01]}\rceil\}[\cup\mathsf{K}]$
$10[+\ell]$	$[\mathsf{B}]\mathsf{NDM}(\mathsf{L}[/\mathsf{A}])$	$\{\mathfrak{MS}_{4:0[,01]},\mathfrak{DM}_{4[,01]}\}[\cup\mathsf{K}]$
$11[+\ell]$	$[\mathsf{B}]\mathsf{N}\{\mathsf{W}\}\mathsf{K}(\mathsf{L}[/\mathsf{A}])$	$\{\mathfrak{MS}_{4:0[,01]}\}[\cup\mathsf{K}]$
12	$[\mathsf{B}]\mathsf{TNIMSL}$	$\{\mathfrak{MS}_{2[,01]}\}$
$22\lfloor-\mathbb{k}\rfloor$	$[\mathsf{B}/]\lfloor\mathsf{A}\rfloor\mathsf{QS}\langle\{\mathsf{W}\}\mathsf{K}\rangle\mathsf{MS}(\mathsf{L}[/\mathsf{A}])$	$\{\mathfrak{MS}_{4:1[,01]},\mathfrak{DM}_{4[,01]}\}\lfloor\cup\mathsf{K}\rfloor$
$23\lfloor-\mathbb{k}\rfloor$	$[\mathsf{B}/]\lfloor\mathsf{A}\rfloor\mathsf{QS}\{\mathsf{W}\}\mathsf{KS}(\mathsf{L}[/\mathsf{A}])$	$\{\mathfrak{MS}_{4:1[,01]}\}\lfloor\cup\mathsf{K}\rfloor$
$24\lfloor-\mathbb{k}\rfloor$	$[\mathsf{B}/]\lfloor\mathsf{A}\rfloor\mathsf{SMS}(\mathsf{L}[/\mathsf{A}])$	$\{\mathfrak{S}_{3[,01]},\mathfrak{DM}_{4[,01]}\}\lfloor\cup\mathsf{K}\rfloor$
$25\lfloor-\mathbb{k}\rfloor$	$[\mathsf{B}/]\lfloor\mathsf{A}\rfloor\mathsf{DM}(\mathsf{L}[/\mathsf{A}])$	$\{\mathfrak{DM}_{4[,01]}\}\lfloor\cup\mathsf{K}\rfloor$
$26\lfloor-\mathbb{k}\rfloor$	$[\mathsf{B}/]\lfloor\mathsf{A}\rfloor\mathsf{S}\{\mathsf{W}\}\mathsf{KS}(\mathsf{L}[/\mathsf{A}])$	$\{\mathfrak{S}_{3[,01]},\mathfrak{K}_{3:i[,01]}\}\lfloor\cup\mathsf{K}\rfloor$
$27\lfloor-\mathbb{k}\rfloor$	$[\mathsf{B}/]\lfloor\mathsf{A}\rfloor\{\mathsf{W}\}\mathsf{K}(\mathsf{L}[/\mathsf{A}])$	$\{\mathfrak{K}_{3:i[,01]}\}\lfloor\cup\mathsf{K}\rfloor$
$28\lfloor-\mathbb{k}\rfloor$	$[\mathsf{B}/]\lfloor\mathsf{A}\rfloor\mathsf{S}(\mathsf{L}[/\mathsf{A}])$	$\{\mathfrak{S}_{3[,01]}\}\lfloor\cup\mathsf{K}\rfloor$
$29\lfloor-\mathbb{k}\rfloor$	$[\mathsf{B}/]\lfloor\mathsf{A}\rfloor\mathsf{B}(\mathsf{L}[/\mathsf{A}])$	$\{\mathfrak{B}_{2[,01]}\}\lfloor\cup\mathsf{K}\rfloor$
21	$[\mathsf{B}]\mathsf{OMSL}$	$\varnothing$

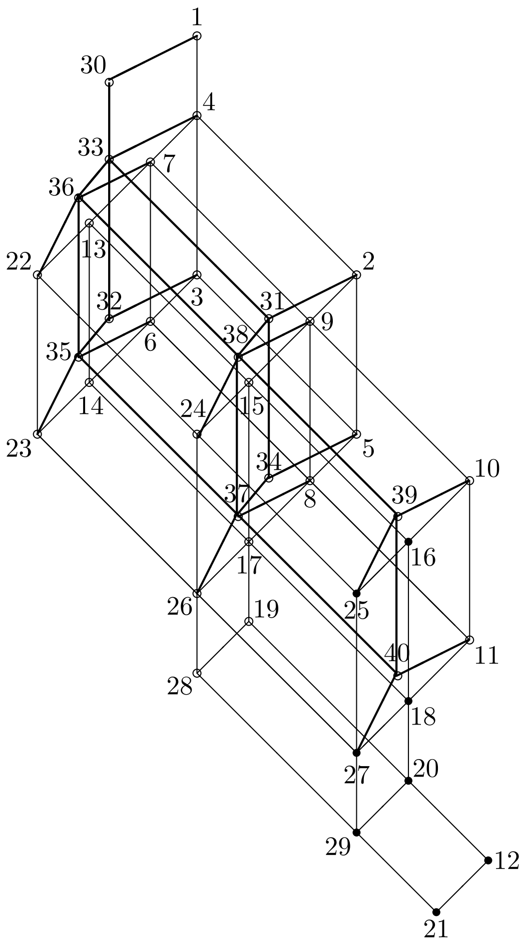

FIGURE 3. The lattice of varieties of [bounded/] Morgan-Stone lattices[/algebras].

Thus, it is rather SMSL/A than MSL/A that is the right "abstraction" of De Morgan and Stone lattices/algebras. Likewise, QSMSL, being the greatest variety of MS lattices disjoint with $(\mathsf{BMSL} \setminus \mathsf{MSA}){\restriction}\Sigma^-_+$, is to be viewed as "the unbounded equational approximation of MS algebras".

5. On quasi-varieties of Morgan-Stone lattices

5.1. **Non-idempotencity versus two-valued Boolean homomorphisms.** Given any $\mathsf{K} \subseteq [\mathsf{B}]\mathsf{MSL}$, NIK stands for the relative sub-quasi-variety of K, relatively axiomatized by the Σ^-_+-quasi-identity:

$$(\neg x_0 \approx x_0) \to (x_0 \approx x_1), \tag{5.1}$$

members of which are said to be *non-idempotent*. Conversely, those of $\mathsf{IK} \triangleq (\mathsf{K} \setminus \mathsf{NIK})$ are said to be *idempotent*. Clearly, for any $\mathcal{Q} \subseteq (\wp_\omega(\mathrm{Eq}_{\Sigma^-_{+[,01]}}) \times \mathrm{Eq}_{\Sigma^-_{+[,01]}})$,

$$(\mathsf{NIK} \cup (\mathsf{K} \cap \mathrm{Mod}(\mathcal{Q}))) = (\mathsf{K} \cap \mathrm{Mod}(\{(\{\neg x_0 \approx x_0\} \cup \Gamma) \to \Phi \mid (\Gamma \to \Phi) \in (\mathcal{Q}[x_i/x_{i+1}]_{i\in\omega}\}). \tag{5.2}$$

Likewise,

$$\mathsf{NI}[\mathsf{B}]\mathsf{TNIMSL} = [\mathsf{B}]\mathsf{OMSL}. \tag{5.3}$$

Given any more $\mathsf{K}' \subseteq [\mathsf{B}]\mathsf{MSL}$, set $(\mathsf{K} \otimes \mathsf{K}') \triangleq \{\mathfrak{A} \times \mathfrak{B} \mid \langle \mathfrak{A}, \mathfrak{B} \rangle \in (\mathsf{K} \times \mathsf{K}')\}$.

Lemma 5.1. *Any (non-one-element)* $\mathfrak{A} \in [\mathsf{B}]\mathsf{MSL}$ *is non-idempotent if(f)* $\hom(\mathfrak{A}, \mathfrak{B}_{2[,01]}) \neq \varnothing$. *In particular,* $\mathsf{NIMS}_{[01]} = \mathbf{S}\mathfrak{S}_3 = \{\mathfrak{S}_{3[,01]}, \mathfrak{B}_{2[,01]}\}$*, while any variety* $\mathsf{V} \subseteq [\mathsf{B}]\mathsf{MSL}$ *with* $\mathsf{NIV} \nsubseteq [\mathsf{B}]\mathsf{OMSL}$ *contains* $\mathfrak{B}_{2[,01]}$.

Proof. The "if" part is by the equality $\Im^{\mathfrak{B}_{2[,01]}} = \varnothing$. (Conversely, assume $\mathfrak{A}$ is non-idempotent, in which case $\mathfrak{B} \triangleq ((\mathfrak{A}\upharpoonright\Sigma_+^-)\upharpoonright(\operatorname{img}\hbar^{\mathfrak{A}})) \in \mathsf{DML}$ is neither idempotent nor one-element, and so, by (2.6) and [9, Proposition 4.2], there is an $h \in \hom(\mathfrak{B}, \mathfrak{B}_2)$. Then, [by Lemma 4.2 and absense of proper subalgebras of $\mathfrak{B}_2$] $(\hbar^{\mathfrak{A}} \circ h) \in \hom(\mathfrak{A}, \mathfrak{B}_{2[,01]})$.) Finally, the fact that $\hbar^{\mathfrak{S}_{3[,01]}} \in \hom(\mathfrak{S}_{3[,01]}, \mathfrak{B}_{2[,01]})$ completes the argument. □

Lemma 5.2. $\mathfrak{B}_2$ *is embedable into any* $\mathfrak{A} \in (\mathsf{MSL} \setminus \mathsf{TNIMSL}) \supseteq ((\mathsf{NIMSL} \cup [\mathsf{Q}]\mathsf{SMSL}) \setminus \mathsf{OMSL})$.

Proof. Take any $a \in (A \setminus \Im_{\neg}^{\mathfrak{A}}) \neq \varnothing$, in which case $\{\langle i, i, i, \neg^{\mathfrak{A}} a \diamond_i^{\mathfrak{A}} \neg^{\mathfrak{A}}\neg^{\mathfrak{A}} a\rangle \mid i \in 2\} \in \hom_{\mathrm{I}}(\mathfrak{B}_2, \mathfrak{A})$, and so Theorem 4.7 and (5.3) complete the argument. □

Though this not expandable to the bounded case, because $\mathfrak{B}_{2,01}$ is not embedable into $\mathfrak{A} = (\mathfrak{MS}_{2,01} \times \mathfrak{B}_{2,01}) \in (\mathsf{BMSL} \setminus (\mathsf{BTNIMSL} \cup \mathsf{MSA}))$, since, by Lemma 4.6, $(\mathfrak{MS}|\mathfrak{B})_{2,01} \notin (\mathsf{MSA}|\mathsf{BTNIMSL})$, we clearly have:

$$\{\langle i, i, i, \flat_i^{\mathfrak{A}}\rangle \mid i \in 2\} \in \hom_{\mathrm{I}}(\mathfrak{B}_{2,01}, \mathfrak{A}), \tag{5.4}$$

for all $\mathfrak{A} \in (\mathsf{MSA} \setminus \mathsf{BOMSL})$. This, by (2.1), (2.5), (2.6), (5.2), (5.3), Lemmas 5.1, 5.2 and Theorem 4.7, immediately yields, subsuming [9, Propositions 4.2, 4.5 and Corollary 4.4]:

Theorem 5.3. *Let* P *be the pre-variety generated by a* $\mathsf{K} \subseteq [\mathsf{B}]\mathsf{MSL}$. *Suppose* $((\mathsf{K}[\cap\mathsf{MSA}]) \setminus ([\mathsf{B}]\mathsf{TNIMSL}[\cap\mathsf{BOMSL}])) \neq \varnothing$. *Then,* NIP *is the pre-variety generated by* $(\mathsf{IK} \otimes \{\mathfrak{B}_{2[,01]}\}) \cup \mathsf{NIK}$*, in which case, for any varieties* $\mathsf{U} \subseteq \mathsf{V} \subseteq [\mathsf{B}]\mathsf{MSL}$ *such that* $\mathsf{V} \subseteq | \nsubseteq [\mathsf{B}]\mathsf{TNIMSL}$ *and* $i \in 2$*,* $\mathsf{NIV} \cup \mathsf{U}$ *is the pre-/quasi-variety generated by* $(\varnothing|((\mathsf{MS}_{\mathsf{V},i[,01]} \setminus \mathbf{S}\mathfrak{S}_{3[,01]}) \otimes \{\mathfrak{B}_{2[,01]}\}) \cup (\mathsf{MS}_{\mathsf{V},i[,01]} \cap \mathbf{S}\mathfrak{S}_{3[,01]})) \cup \mathsf{MS}_{\mathsf{U},i[,01]}$*, and so* $\mathsf{NI}[\mathsf{B}]\{\lfloor\mathsf{Q}\rfloor\mathsf{S}\}(\mathsf{M}\|\mathsf{K})\{\mathsf{S}\}\mathsf{L}\lceil\cup(\{\langle\mathsf{S}\rangle\}\mathsf{K}\{\langle\mathsf{S}\rangle\}\mathsf{L}\|\varnothing)\rceil$ *is the one generated by* $\{((\mathfrak{DM})\|\mathfrak{K})_{4\|(3:i)[,01]}^{\|\{\lfloor 0\rfloor\}} \times \mathfrak{B}_{2[,01]}\{,\lfloor\mathfrak{M}\rfloor\mathfrak{S}_{3\lfloor+1:1\rfloor[,01]}\lfloor\times\mathfrak{B}_{2[,01]}\rfloor\}\lceil,\mathfrak{K}_{3:i}^{\|0}\rceil\}$.

5.2. Quasi-varieties of quasi-strong Morgan-Stone lattices.

Lemma 5.4. *Let* K *be a (finite) class of (finite) MS lattices,* $\mathsf{P}[']\triangleq \mathbf{ISP}(\mathsf{K}[\cup\mathsf{DML}])$*,* $\mathsf{S} \subseteq | \supseteq (\mathsf{P}'|(\mathsf{P} \cup \mathbf{ISPS}))$ *and* $\{\mathsf{K}' \subseteq\}\mathsf{S}' \triangleq (\mathsf{S} \cap \mathsf{DML})\{= \mathbf{ISP}\mathsf{K}'\}$. *Then,* $\mathsf{S} = \mathbf{ISP}((\mathsf{S}'\{\langle\cap\mathsf{K}\rangle\}) \cup (\mathsf{P}\|\mathsf{K}))$*,* S' *being finitely-generated (and so being* $\mathsf{S} = (\mathsf{S}' \uplus^{\mathrm{Q}} \mathsf{P})$*).*

Proof. Consider any $\mathfrak{A} \in \mathsf{S}$ and any $\langle a, b\rangle \in (A^2 \setminus \Delta_A)$, in which case $(\mathfrak{A}\upharpoonright(\operatorname{img}\hbar^{\mathfrak{A}})) \in \mathsf{S}' \supseteq \mathbf{ISPS}'$, and so $\hbar^{\mathfrak{A}} \in \hom(\mathfrak{A}, \mathsf{S}')$, while, by (2.6), there are some $\mathfrak{B} \in (\mathsf{K}\cup\mathsf{DML})$ and $h \in \hom(\mathfrak{A}, \mathfrak{B})$ such that $h(a) \neq h(b)$, and so $\mathfrak{B} \in \mathsf{K}$, whenever $\hbar^{\mathfrak{A}}(a) = \hbar^{\mathfrak{A}}(b)$, for, otherwise, $h(a/b) = h(\hbar^{\mathfrak{A}}(a/b))$, (2.6) and [9] completing the argument. □

5.2.1. *Morgan-regularity versus regularity.* The sub-quasi-variety of any quasi-variety $\mathsf{Q} \subseteq [\mathsf{B}]\mathsf{MSL}$, relatively axiomatized by $(\mathcal{M})\mathcal{R} \triangleq (\{\neg x_0 \lessapprox x_0, (x_0 \wedge \neg x_1) \lessapprox (\neg x_0 \vee x_1)\} \to \mathfrak{I}_{1(+1),0,0,1,0}^{1(+1),1(+1),1(+1),1,1})$, is denoted by $(\mathsf{M})\mathsf{RQ}$, its members being said to to be *(Morgan-)regular*; cf. [9, Definition 4.6] for the non-optional case. As a matter of fact, the conception of (Morgan-)regularity has a sense only within (Morgan-)non-idempotent Kleene(-Morgan) framework, members of $\mathsf{NI}[\mathsf{B}]\mathsf{MSL} \cup \mathsf{DML}$ being said to be *Morgan-non-idempotent.* More precisely, we have both:

Lemma 5.5. $(\mathsf{M})\mathsf{R}[\mathsf{B}]\mathsf{MSL} \subseteq [\mathsf{B}]\mathsf{K}(\mathsf{M})\mathsf{SL}$.

Proof. Consider any $\mathfrak{A} \in (\mathsf{M})\mathsf{R}[\mathsf{B}]\mathsf{MSL}$ and $a, b(, c) \in A$. Let $(d|e) \triangleq ((a|b) \vee^{\mathfrak{A}} \neg^{\mathfrak{A}}(a|b))$ and $f \triangleq (d \wedge^{\mathfrak{A}} e)$, in which case, by $\mathcal{DM}_1$, we have $\neg^{\mathfrak{A}}(d|e) = (\neg^{\mathfrak{A}}(a|b) \wedge^{\mathfrak{A}} \neg^{\mathfrak{A}}\neg^{\mathfrak{A}}(a|b)) \leqslant^{\mathfrak{A}} \neg^{\mathfrak{A}}(a|b) \leqslant^{\mathfrak{A}} (d|e)$, and so, since, by $\mathcal{DM}_0$, $\neg^{\mathfrak{A}} f = (\neg^{\mathfrak{A}} d \vee^{\mathfrak{A}} \neg^{\mathfrak{A}} e)$, get $(d \wedge^{\mathfrak{A}} \neg^{\mathfrak{A}} f) \leqslant^{\mathfrak{A}} (d \wedge^{\mathfrak{A}} (\neg^{\mathfrak{A}} d \vee^{\mathfrak{A}} e)) = (\neg^{\mathfrak{A}} d \vee^{\mathfrak{A}} f)$. Then, by $\mathcal{MN}_{0,0}$, we eventually get $((a \wedge^{\mathfrak{A}} \neg^{\mathfrak{A}} a)(\wedge^{\mathfrak{A}}\neg^{\mathfrak{A}}\neg^{\mathfrak{A}}c)) \leqslant^{\mathfrak{A}} (\neg^{\mathfrak{A}} d(\wedge^{\mathfrak{A}}\neg^{\mathfrak{A}}\neg^{\mathfrak{A}}c)) \leqslant^{\mathfrak{A}} (\neg^{\mathfrak{A}} f(\wedge^{\mathfrak{A}}\neg^{\mathfrak{A}}\neg^{\mathfrak{A}}c)) \leqslant^{\mathfrak{A}} (f(\vee^{\mathfrak{A}}c)) \leqslant^{\mathfrak{A}} (e(\vee^{\mathfrak{A}}c))$, as required. □

Corollary 5.6. *Let* $\mathsf{K} \triangleq (\varnothing(\cup[\mathsf{B}]\mathsf{DML}))$. *Then,* $([\mathsf{B}]\mathsf{SL} \cup \mathsf{K}) \subseteq \mathsf{Q} \triangleq (\mathsf{M})\mathsf{R}[\mathsf{B}]\mathsf{MSL} \subseteq (\mathsf{NI}[\mathsf{B}]\mathsf{MSL} \cup \mathsf{K})$. *In particular,* $[\mathsf{B}]\mathsf{SMSL} \subseteq \mathsf{MR}[\mathsf{B}]\mathsf{MSL}$.

Proof. Consider any $\mathfrak{A} \in \mathsf{Q}$ and any $a \in \Im^{\mathfrak{A}}$, in which case, for all $b \in A$, each $\Phi \in \pi_0((\mathcal{M})\mathcal{R})$ is true in $\mathfrak{A}$ under $[x_0/a, x_1/b]$, and so, for each $i \in 2$, $\Psi_i \triangleq (\pi_1((\mathcal{M})\mathcal{R}[x_1/\neg^i x_1])$ is true in $\mathfrak{A}$. Then, in the non-optional case, by $\mathcal{MN}_{0,0}$, $\Im^{\mathfrak{A}} = A$, so, by $\mathcal{DM}_0$, any $d \in A$ is equal to a, for $(d \wedge^{\mathfrak{A}} a) = \neg^{\mathfrak{A}}(d \wedge^{\mathfrak{A}} a) = (d \vee^{\mathfrak{A}} a)$, $\mathfrak{A}$ being thus non-idempotent. (Likewise, by (2.7), Theorem 4.7 and Lemma 5.5, unless $\mathfrak{A}$ is non-idempotent, it is in $\mathbf{IP}^{\mathrm{SD}}(\mathsf{K}' \cap \mathrm{Mod}(\Psi_0))$, where $\mathsf{K}' \triangleq \mathbf{S}_{>1}(\{\mathfrak{MS}_{4:j[,01]} \mid j \in 2\} \cup \{\mathfrak{DM}_{4[,01]}[,\mathfrak{MS}_{2,01]}]\})$. On the other hand, by Lemma 4.6, $(\mathsf{K}' \setminus [\mathsf{B}]\mathsf{DML}) = (\{\mathfrak{MS}_{4:j[,01]} \mid j \in 2\} \cup \{\mathfrak{S}_{3[,01]}, \mathfrak{MS}_{2[,01]}\})$, while $\mathbf{S}\mathfrak{MS}_{4:1\{-1\}} \ni \{\mathfrak{M}\}\mathfrak{S}_{3\{-1\}} \not\models \Psi_0[x_{k+1}/\langle 0, k\{\cup 1\}, k\{\cap 0\}\rangle]_{k \in 2}$, whereas $\Psi_0 \in \mathrm{Eq}_{\Sigma_+^-}$, in which case $(\mathsf{K}' \cap \mathrm{Mod}(\Psi_0)) \subseteq [\mathsf{B}]\mathsf{DML}$, and so $\mathfrak{A} \in (\mathsf{NI}[\mathsf{B}]\mathsf{MSL} \cup [\mathsf{B}]\mathsf{DML})$.) Finally, Theorem 4.7 and the regularity of $\mathfrak{S}_{3[,01]}$ complete the argument. □

Let $\boldsymbol{\mu} \triangleq (\neg x_0 \vee \neg\neg x_0)$, $\boldsymbol{\pi} \triangleq ((x_0 \vee \neg x_1) \wedge x_1)$ and, for any $\tau \in \{\boldsymbol{\mu}\{, x_0\}\}$ (and $i \in \omega$) $\boldsymbol{\iota}_{\{\tau,\}1(+i+1)} \triangleq ((x_0([x_0/\boldsymbol{\pi}]))[x_0/(\tau[x_0/x_{0(+i+1)}])(, x_1/\boldsymbol{\iota}_{\{\tau,\}i+1})])$, in which case, by $\mathcal{DM}_j$ and $\mathcal{MN}_{j,0}$ with $j \in 2$, the Σ_+^--quasi-identities:

$$(\varnothing|\{\neg x_0 \approx \neg\neg x_0\}) \quad \to \quad (\neg\boldsymbol{\mu} \lessapprox | \approx \neg\neg\boldsymbol{\mu}), \tag{5.5}$$

$$\{\neg x_k \lessapprox | \approx \neg\neg x_k, \neg x_{1-k} \lessapprox \neg\neg x_{1-k}\} \quad \to \quad (\neg\boldsymbol{\pi} \lessapprox | \approx \neg\neg\boldsymbol{\pi}), \tag{5.6}$$

where $k \in 2$, are true in $[\mathsf{B}]\mathsf{MSL}$, and so are:

$$\begin{aligned} &((\varnothing|\{\neg((x_0\{[x_0/\tau]\})[x_0/x_l]) \approx \neg\neg((x_0\{[x_0/\tau]\})[x_0/x_l])\}) \\ &\quad \{\cup\{\neg(\tau[x_0/x_n]) \lessapprox \neg\neg(\tau[x_0/x_n]) \mid n \in (m \setminus (\varnothing|\{l\}))\}\}) \to \\ &\qquad\qquad (\neg\boldsymbol{\iota}_{\{\tau,\}m} \lessapprox | \approx \neg\neg\boldsymbol{\iota}_{\{\tau,\}m}), \end{aligned} \tag{5.7}$$

where $l \in m \in (\omega \setminus 1)$, to be shown by induction on m.

Clearly, $\mathfrak{K}_{5:1[,01]} \triangleq ((\mathfrak{MS}_{4:1[,01]} \times \mathfrak{B}_{2[,01]}) \upharpoonright ((MS_{4:1} \times B_2) \setminus (\{\langle \bar{a}, \bar{b} \rangle \mid \bar{a} \in MS_{4:1}, \bar{b} \in B_2, (1 - b_2) = a_2\}))$ is regular.

Theorem 5.7. *Let* $\mathsf{Q} \triangleq (\mathsf{M})\mathsf{R}[\mathsf{B}]\mathsf{QS}\{\mathsf{W}\}\mathsf{K}\langle\mathsf{M}\rangle\mathsf{SL}$. *Then,* $\lceil\mathsf{NI}\rceil\mathsf{Q}$ *is the pre-/quasi-variety generated by* $\{\mathfrak{K}_{5:1[,01]}(,(\mathfrak{DM}_{4[,01]} \upharpoonright (K_{3:(0|1)} \langle \cup DM_4 \rangle))\lceil \times \mathfrak{B}_{2[,01]}\rceil)\}$.

Proof. Consider any non-one-element finitely-generated $\mathfrak{A} \in (\mathsf{Q}(\setminus[\mathsf{B}]\mathsf{DML}))$ and any $h \in (\hom(\mathfrak{A}, \mathfrak{MS}_{4:1[,01]})(\setminus \hom(\mathfrak{A}, [\mathsf{B}]\mathsf{DML})))$, in which case there are some $n \in (\omega \setminus 1)$ and $\bar{a} \in A^n$ such that $\mathfrak{A}$ is generated by $B \triangleq (\mathrm{img}\,\bar{a})$, and so, by $\mathcal{MN}_{0\|1,0}$ and (5.7), $b \triangleq \neg^{\mathfrak{A}}\neg^{\mathfrak{A}}\boldsymbol{\iota}_n^{\mathfrak{A}}(\bar{a}) \geqslant^{\mathfrak{A}} \neg^{\mathfrak{A}} b$, while, by Lemma 5.1 and Corollary 5.6, $\mathcal{G} \triangleq \hom(\mathfrak{A}, \mathfrak{B}_{2[,01]}) \neq \varnothing$ is finite, for $m \triangleq |\mathcal{G}| \leqslant |2^B| \in \omega$, and so there is a bijection $\bar{g}$ from $m \in (\omega \setminus 1)$ onto $\mathcal{G}$ (whereas $(\mathrm{img}\,h) \nsubseteq DM_4$, and so there is some $c \in A$ such that $h(c) = \langle 0, 1, 1 \rangle$). Prove that $\mathcal{A} \triangleq (\prod_{i \in m}((h \circ \pi_2)^{-1}[1] \cap (g_i \circ \pi_2)^{-1}[2 \setminus 1])) = \varnothing$, by contradiction. For suppose there is some $\bar{d} \in \mathcal{A}$, in which case $e \triangleq (\vee_m^{\mathfrak{A}} \bar{d}) \in (h \circ \pi_2)^{-1}[1]$, and so $\pi_0(h(\neg^{\mathfrak{A}} e(\wedge^{\mathfrak{A}}\neg^{\mathfrak{A}}\neg^{\mathfrak{A}}c))) = 1 \nleqslant 0 = \pi_0(h(e(\vee^{\mathfrak{A}}c)))$. Then, $(\neg^{\mathfrak{A}} e(\wedge^{\mathfrak{A}}\neg^{\mathfrak{A}}\neg^{\mathfrak{A}}c)) \nleqslant^{\mathfrak{A}} (e(\vee^{\mathfrak{A}}c))$, for $(h \circ \pi_0) \in \hom(\mathfrak{A} \upharpoonright \Sigma_+, \mathfrak{D}_2)$. Now, consider any $\mathfrak{C} \in (\{\mathfrak{MS}_{6[,01]}\}[\cup \mathsf{M}_{01}])$, any $f \in \hom(\mathfrak{A}, \mathfrak{C})$ and the following complementary cases:

- $(\mathrm{img}\, f) \subseteq S_3$,
  in which case $f' \triangleq (f \circ \hbar^{\mathfrak{S}_3}) \in \hom(\mathfrak{A}, \mathfrak{B}_{2[,01]})$, while $\mathfrak{C} = \mathfrak{MS}_{6[,01]}$ [since $(S_3 \cap MS_2) = \varnothing \neq (\mathrm{img}\, f)$, as $A \neq \varnothing$], and so $f' = g_j$, for some $j \in m$. Then, $1 = \pi_2(f'(d_j)) \leqslant \pi_2(f'(e)) \leqslant 1$, in which case $\pi_{1\|2}(f(e)) = 1$, and so $f(b \wedge^{\mathfrak{A}} \neg^{\mathfrak{A}} e) = \langle 0, 0, 0\rangle \leqslant^{\mathfrak{C}} f(\neg^{\mathfrak{A}} b \vee^{\mathfrak{A}} e)$.
- $(\mathrm{img}\, f) \nsubseteq S_3$,
  in which case, for some $k \in m$, $f(a_k) \in (C \setminus S_3) = \Im^{\mathfrak{C}}_{\neg}$, and so, by $\mathcal{MN}_{0\|1,0}$ and (5.7), $f(b) \in \Im^{\mathfrak{C}}$. Then, $f(b \wedge^{\mathfrak{A}} \neg^{\mathfrak{A}} e) \leqslant^{\mathfrak{C}} f(b) = f(\neg^{\mathfrak{A}} b) \leqslant^{\mathfrak{C}} f(\neg^{\mathfrak{A}} b \vee^{\mathfrak{A}} e)$.

Thus, anyway, $f(b \wedge^{\mathfrak{A}} \neg^{\mathfrak{A}} e) \leqslant^{\mathfrak{C}} f(\neg^{\mathfrak{A}} b \vee^{\mathfrak{A}} e)$, in which case, by (2.6) and Theorem 4.4, $(b \wedge^{\mathfrak{A}} \neg^{\mathfrak{A}} e) \leqslant^{\mathfrak{A}} (\neg^{\mathfrak{A}} b \vee^{\mathfrak{A}} e)$, and so $\mathfrak{A} \not\models (\mathcal{M})\mathcal{R}[x_0/b, x_1/e(, x_2/c)]$. This contradiction to the (Morgan-)regularity of $\mathfrak{A}$ shows that there is some $l \in m$ such that $\pi_2[g_l[(h \circ \pi_2)^{-1}[1]]] \subseteq 1$, in which case, by (2.1) and (2.5), $h' \triangleq (h \odot g_l) \in \hom(\mathfrak{A}, \mathfrak{K}_{5:1})$ with $(\ker h') \subseteq (\ker h)$, and so (2.6), Theorems 4.4, 5.3, Lemmas 5.1, 5.5, Corollary 5.6, the locality of quasi-varieties and the quasi-equationality of finitely-generated pre-varieties complete the argument. □

5.2.2. *Embedability lemmas and the lattices of quasi-varieties.*
5.2.2.1. Quasi-varieties of strong Morgan-Stone lattices. First, by Lemma 4.6, Theorem 4.7 and the distributivity of lattice reducts of MS lattices, we, clearly, have:

Lemma 5.8. *Let* $\mathfrak{A} \in \langle\mathsf{QS}\rangle\mathsf{MSL}$, $a \in A$, $c \triangleq (a \vee^{\mathfrak{A}} \neg^{\mathfrak{A}} a)$ *and* $d \triangleq (\neg^{\mathfrak{A}}\neg^{\mathfrak{A}} a \vee^{\mathfrak{A}} \neg^{\mathfrak{A}} a)$. *(Suppose* $(c\langle\wedge^{\mathfrak{A}} a\rangle) \neq (d\langle\wedge^{\mathfrak{A}}\neg^{\mathfrak{A}}\neg^{\mathfrak{A}} a\rangle)$.*) Then,* $(c \neq)b \triangleq \neg^{\mathfrak{A}} c = \neg^{\mathfrak{A}} d \leqslant^{\mathfrak{A}} c \leqslant^{\mathfrak{A}} d = \neg^{\mathfrak{A}} b$ *(in which case* $\{\langle 0,0,0,b\rangle, \langle 0,1,1,c\rangle, \langle 1,1,1,d\rangle\} \in \hom_{\mathrm{I}}(\mathfrak{S}_3, \mathfrak{A})$*), so* $\mathfrak{S}_3 \in \mathsf{K} \triangleq ([\mathsf{A}]\{\mathsf{Q}\}\mathsf{SMSL} \setminus [\mathsf{A}]\mathsf{DML})$ *is embedable into any* $\mathfrak{B} \in (\mathsf{MSL} \setminus \mathsf{NDML}) \supseteq \mathsf{K}$.

This, by Theorem 4.7, (5.2), Lemma 5.4, Corollary 5.6 and [9], yields:

Theorem 5.9. *Let* $\mathsf{P} \subseteq \mathsf{SMSL}$ *be a pre-variety and* $(\mathsf{K} \subseteq)\mathsf{P}' \triangleq (\mathsf{P} \cap \mathsf{DML})$. *Suppose* $\mathsf{P} \nsubseteq \mathsf{DML}$ *(and* P' *is the pre-variety generated by* K*). Then,* P *is the pre-variety generated by* $\mathsf{P}' \cup \mathsf{SL}$ *(in which case it is the one generated by* $\mathsf{K} \cup \{\mathfrak{S}_3\}$*),* P' *being a finitely-generated quasi-variety, and so being* $\mathsf{P} = (\mathsf{P}' \uplus^{\mathrm{Q}} \mathsf{SL})$. *In particular,* $f : L_{\mathrm{Q}}(\mathsf{SL}, \mathsf{SMSL}) \to L_{\mathrm{Q}}(\mathsf{BL}, \mathsf{DML}), \mathsf{Q} \mapsto (\mathsf{Q} \cap \mathsf{DML})$ *and* $g : L_{\mathrm{Q}}(\mathsf{BL}, \mathsf{DML}) \to L_{\mathrm{Q}}(\mathsf{SL}, \mathsf{SMSL}) : \mathsf{Q}' \mapsto (\mathsf{Q}' \uplus^{\mathrm{Q}} \mathsf{SL})$ *are inverse to one another isomorphisms between* $\mathfrak{L}_{\mathrm{Q}}(\mathsf{SL}, \mathsf{SMSL})$ *and* $\mathfrak{L}_{\mathrm{Q}}(\mathsf{BL}, \mathsf{DML})$, *in which case for any* $\mathsf{Q} \in L_{\mathrm{Q}}(\mathsf{SL}, \mathsf{SMSL}) = (L_{\mathrm{Q}}(\mathsf{SMSL}) \setminus L_{\mathrm{Q}}(\mathsf{DML}))$ *and* $\mathsf{Q}' \in L_{\mathrm{Q}}(\mathsf{DML})$, $(\mathsf{Q} \cap \mathsf{Q}') = (f(\mathsf{Q}) \cap \mathsf{Q}')$ *and* $(\mathsf{Q} \uplus^{\mathrm{Q}} \mathsf{Q}') = (\mathsf{Q} \uplus^{\mathrm{U}} g(\mathsf{Q}'))$, *so* $\{\langle \mathsf{S} \cap \mathsf{DML}, 1 - \chi^{L_{\mathrm{Q}}(\mathsf{DML})}_{L_{\mathrm{Q}}(\mathsf{SMSL})}(\mathsf{S})\rangle \mid \mathsf{S} \in L_{\mathrm{Q}}(\mathsf{SMSL})\}$ *is an embedding of* $\mathfrak{L}_{\mathrm{Q}}(\mathsf{SMSL})$ *into* $\mathfrak{L}_{\mathrm{Q}}(\mathsf{DML}) \times \mathfrak{D}_2$, *the former having* $(|L_{\mathrm{Q}}(\mathsf{DML})| + |L_{\mathrm{Q}}(\mathsf{BL}, \mathsf{DML})|) = (8 + 7) = 15$ *elements and Hasse diagram depicted at Figure 4 with thick lines, the latter being embedable into the distributive lattice* $(\mathfrak{D}_5 \times \mathfrak{D}_3) \times \mathfrak{D}_2$.

Let $\mathfrak{K}_4$ be the Kleene lattice with Σ_+-reduct $\mathfrak{D}_4$ and $\neg^{\mathfrak{K}_4} \triangleq \{\langle i, 3 - i\rangle \mid i \in 4\}$. Then, Corollary 5.6, Theorems 4.7, 5.9 and [9, Proposition 4.7] immediately yield:

Corollary 5.10. $\mathsf{R}[\mathsf{S}]\mathsf{K}[\mathsf{S}]\mathsf{L}$ *is the pre-/quasi-variety generated by* $\{\mathfrak{K}_4[, \mathfrak{S}_3]\}$.

5.2.2.2. Quasi-varieties of Morgan-regular quasi-strong Morgan-Stone lattices.

Lemma 5.11. $\mathfrak{K}_{5:1}$ *is embedable into any* $\mathfrak{A} \in ((\mathsf{NIQSMSL} \cup \mathsf{MRQSMSL}) \setminus [\mathsf{P}]\mathsf{SMSL})$.

Proof. Take any $a, e \in A$ such that $\mathfrak{A} \not\models \mathcal{P}[x_0/a, x_1/e]$, in which case $\neg^{\mathfrak{A}}\neg^{\mathfrak{A}} a \nleqslant^{\mathfrak{A}} (a \vee^{\mathfrak{A}} f)$ with $f \triangleq (\neg^{\mathfrak{A}} e \vee^{\mathfrak{A}} \neg^{\mathfrak{A}}\neg^{\mathfrak{A}} e)) \geqslant^{\mathfrak{A}} \neg^{\mathfrak{A}} f$, in view of $\mathcal{DM}_1$, and so $\neg^{\mathfrak{A}}\neg^{\mathfrak{A}} a \neq a$. On the other hand, by Theorems 4.7, 5.3, 5.7 and (5.2), $\mathsf{NIQSMSL} \cup \mathsf{MRQSMSL}$ is the pre-variety generated by $\mathsf{K} \triangleq \{\mathfrak{MS}_{4:1} \times \mathfrak{B}_2, \mathfrak{DM}_4\}$, in which case, by (2.6), there are some $\mathfrak{C} \in \mathsf{K}$ and $h \in \hom(\mathfrak{A}, \mathfrak{C})$ such that $h(\neg^{\mathfrak{A}}\neg^{\mathfrak{A}} a) \nleqslant^{\mathfrak{C}} h(a \vee^{\mathfrak{A}} f)$, and so $\mathfrak{C} = (\mathfrak{MS}_{4:1} \times \mathfrak{B}_2)$, while $\pi_1(h(f)) = \langle 1, 1, 1\rangle$, whereas $\pi_0(h((a|(e\|f)))) = \langle 0, 1|0, 1\rangle$. Let $b, c, d \in A$ be as in Lemma 5.8 and $g \triangleq \{\langle 2 \times \{3 \times 1\}, b \wedge^{\mathfrak{A}} \neg^{\mathfrak{A}} f\rangle, \langle 0, 0, 1, 3 \times$

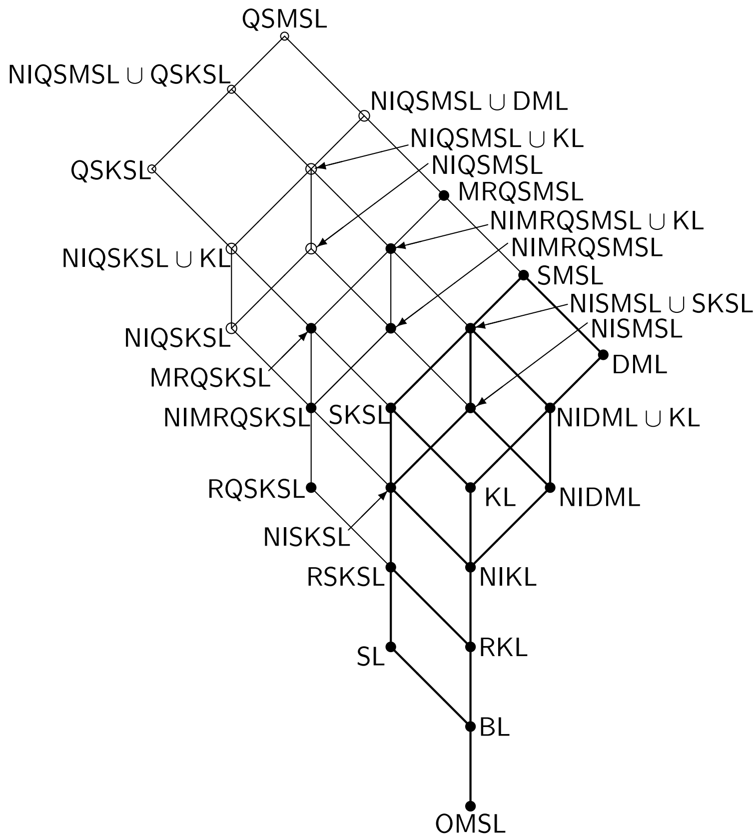

FIGURE 4. The lattice of pre-/quasi-varieties of quasi-strong Morgan-Stone lattices.

$1, \neg^{\mathfrak{A}} f\rangle, \langle 0,0,1,3\times\{1\}, f\rangle, \langle 0,1,1,3\times\{1\}, c\vee^{\mathfrak{A}} f\rangle, \langle 2\times\{3\times\{1\}\}, d\vee^{\mathfrak{A}} f\rangle\} : K_{5:1} \to A$, in which case, for all $\bar{\imath}, \bar{\jmath} \in K_{5:1}$, $(\bar{\imath} \leqslant^{(\mathfrak{D}_2^3)^2} \bar{\jmath}) \Rightarrow (g(\bar{\imath}) \leqslant^{\mathfrak{A}} g(\bar{\jmath}))$ and $h(g(\bar{\imath})) = \bar{\imath}$, and so, as $\mathfrak{K}_{5:1}{\restriction}\Sigma_+$ is a chain lattice, by $\mathcal{DM}_{0|1}$ and $\mathcal{MN}_{0|1,0}$, $g \in \mathrm{hom}_{\mathrm{I}}(\mathfrak{K}_{5:1}, \mathfrak{A})$. □

This, by Theorems 4.7, 5.7, 5.9, Lemma 5.4, Corollary 5.6, (5.2) and [9], yields:

Corollary 5.12. *Let* $\mathsf{P} \subseteq \mathsf{MRQSMSL}$ *be a pre-variety and* $(\mathsf{K} \subseteq)\mathsf{P}' \triangleq (\mathsf{P} \cap \mathsf{DML})$. *Suppose* $\mathsf{P} \nsubseteq \mathsf{SMSL}$ *(and* P' *is the pre-variety generated by* K*). Then,* P *is the pre-variety generated by* $\mathsf{P}' \cup \mathsf{RQSKSL}$ *(in which case it is the one generated by* $\mathsf{K} \cup \{\mathfrak{K}_{5:1}\}$*),* P' *being a finitely-generated quasi-variety, and so being* $\mathsf{P} = (\mathsf{P}' \uplus^{\mathrm{Q}} \mathsf{RQSKSL})$. *In particular,* $f[']: L_{\mathrm{Q}}(\mathsf{RQSKSL}, \mathsf{MRQSMSL}) \to L_{\mathrm{Q}}(\mathsf{RSKSL}[\cap\mathsf{RKL}], \mathsf{SMSL}[\cap\mathsf{DML}]), \mathsf{Q} \mapsto ((\mathsf{Q} \cap \mathsf{SMSL})[\cap\mathsf{DML}])$ *and* $g[']: L_{\mathrm{Q}}(\mathsf{RSKSL}[\cap\mathsf{RKL}], \mathsf{SMSL}[\cap\mathsf{DML}]) \to L_{\mathrm{Q}}(\mathsf{RQSKSL}, \mathsf{MRQSMSL}) : \mathsf{Q}' \mapsto ((\mathsf{Q}'\{\uplus^{\mathrm{Q}}\mathsf{SL}\}) \uplus^{\mathrm{Q}} \mathsf{RQSKSL})$ *are inverse to one another isomorphisms between* $\mathfrak{L}_{\mathrm{Q}}(\mathsf{RQSKSL}, \mathsf{MRQSMSL})$ *and* $\mathfrak{L}_{\mathrm{Q}}(\mathsf{RSKSL}[\cap \mathsf{RKL}], \mathsf{SMSL}[\cap\mathsf{DML}])$*, in which case for any* $\mathsf{Q} \in L_{\mathrm{Q}}(\mathsf{RQSKSL}, \mathsf{MRQSMSL}) = (L_{\mathrm{Q}}(\mathsf{MRQSMSL}) \setminus L_{\mathrm{Q}}(\mathsf{SMSL}))$ *and* $\mathsf{Q}' \in L_{\mathrm{Q}}(\mathsf{SMSL})$, $(\mathsf{Q} \cap \mathsf{Q}') = (f(\mathsf{Q}) \cap \mathsf{Q}')$ *and* $(\mathsf{Q} \uplus^{\mathrm{Q}} \mathsf{Q}') = (\mathsf{Q} \uplus^{\mathrm{U}} g(\mathsf{Q}'))$*, so* $\{\langle(\mathsf{S} \cap \mathsf{SMSL})[\cap\mathsf{DML}], (1 - \chi^{L_{\mathrm{Q}}(\mathsf{SMSL})}_{L_{\mathrm{Q}}(\mathsf{MRQSMSL})}(\mathsf{S}))[+(1 - \chi^{L_{\mathrm{Q}}(\mathsf{DML})}_{L_{\mathrm{Q}}(\mathsf{MRQSMSL})}(\mathsf{S}))]\rangle \mid \mathsf{S} \in L_{\mathrm{Q}}(\mathsf{MRQSMSL})\}$ *is an embedding of* $\mathfrak{L}_{\mathrm{Q}}(\mathsf{MRQSMSL})$ *into* $\mathfrak{L}_{\mathrm{Q}}(\mathsf{SMSL}[\cap\mathsf{DML}]) \times \mathfrak{D}_{2[+1]}$*, the former having* $(|L_{\mathrm{Q}}(\mathsf{SMSL})| + |L_{\mathrm{Q}}(\mathsf{RKL}, \mathsf{DML})|) = (15+6) = 21$ *elements and Hasse diagram depicted at Figure 4 with large solid circles-nodes [the latter being embedable into the distributive lattice* $(\mathfrak{D}_5 \times \mathfrak{D}_3) \times \mathfrak{D}_3$*].*

5.2.2.3. Quasi-varieties of Morgan-non-idempotent quasi-strong MS lattices.

Lemma 5.13. $(\mathfrak{MS}_{4:1}\times\mathfrak{B}_2)\notin\mathsf{MRMSL}$ *is embedable into any* $\mathfrak{A}\in((\mathsf{NIQSMSL}\cup\mathsf{DML})\setminus\mathsf{MRMSL})$.

Proof. Take any $a,b,c\in A$ such that $\neg^{\mathfrak{A}}a\leqslant^{\mathfrak{A}}a$, $(a\wedge^{\mathfrak{A}}\neg^{\mathfrak{A}}b)\leqslant^{\mathfrak{A}}(\neg^{\mathfrak{A}}a\vee^{\mathfrak{A}}b)$ but $(\neg^{\mathfrak{A}}b\wedge^{\mathfrak{A}}\neg^{\mathfrak{A}}\neg^{\mathfrak{A}}c)\nleqslant^{\mathfrak{A}}(b\vee^{\mathfrak{A}}c)$, in which case, by $\mathcal{MN}_{i,0}$ with $i\in 2$ and $\mathcal{DM}_1$, we have $((d|e)\|f)\triangleq(\neg^{\mathfrak{A}}\neg^{\mathfrak{A}}(a|b)\|(c\vee^{\mathfrak{A}}\neg^{\mathfrak{A}}c\vee^{\mathfrak{A}}d))=\|\geqslant^{\mathfrak{A}}(\neg^{\mathfrak{A}}\neg^{\mathfrak{A}}(d|e)\|\neg^{\mathfrak{A}}(f/d))(\geqslant\|\leqslant)^{\mathfrak{A}}((\neg^{\mathfrak{A}}d|e)\|f)$, while, by $\mathcal{DM}_j$ with $j\in 2$, we get $(d\wedge^{\mathfrak{A}}\neg^{\mathfrak{A}}e)\leqslant^{\mathfrak{A}}(\neg^{\mathfrak{A}}d\vee^{\mathfrak{A}}e)$, whereas, by (2.6), (5.2) and Theorem 5.3, there are some $\mathfrak{C}\in\{\mathfrak{MS}_{4:1}\times\mathfrak{B}_2,\mathfrak{DM}_4\}$ and some $h\in\hom(\mathfrak{A},\mathfrak{C})$ such that $(\neg^{\mathfrak{C}}h(b)\wedge^{\mathfrak{C}}\neg^{\mathfrak{C}}\neg^{\mathfrak{C}}h(c))\nleqslant^{\mathfrak{C}}(h(b)\vee^{\mathfrak{C}}h(c))$, and so $\mathfrak{C}=(\mathfrak{MS}_{4:1}\times\mathfrak{B}_2)$ and $h((a\|d)|(b\|e)|f)=\langle\langle 0,0|0|1,1|0|1\rangle,3\times\{1\}\rangle$, for $\neg^{\mathfrak{C}}h(a)\leqslant^{\mathfrak{C}}h(a)$ and $(h(a)\wedge^{\mathfrak{C}}\neg^{\mathfrak{C}}h(b))\leqslant^{\mathfrak{C}}(\neg^{\mathfrak{C}}h(a)\vee^{\mathfrak{C}}h(b))$. In that case, using $\mathcal{MN}_{k,0}$ and $\mathcal{DM}_k$ with $k\in 2$, it is routine checking that the mapping $g:(MS_{4:1}\times B_2)\to A$, given by:

$$\begin{aligned}
g(\langle\langle 0,0,0|1\rangle,3\times\{1\}\rangle) &\triangleq ((d\wedge^{\mathfrak{A}}(e\vee^{\mathfrak{A}}(e|\neg^{\mathfrak{A}}d)))\vee^{\mathfrak{A}}\neg^{\mathfrak{A}}f),\\
g(\langle\langle 1|0,1|0,1\rangle,3\times 1\rangle) &\triangleq \neg^{\mathfrak{A}}g(\langle\langle 0,0,0|1\rangle,3\times\{1\}\rangle),\\
g(\langle 3\times\{l\},3\times\{l\}\rangle) &\triangleq (g(\langle 3\times 1,3\times\{1\}\rangle)\diamond^{\mathfrak{A}}_{l}g(\langle 3\times\{1\},3\times 1\rangle)),\\
g(\langle\langle 0,1,1\rangle,3\times\{0|1\}\rangle) &\triangleq (((\neg^{1|0})^{\mathfrak{A}}(d\wedge^{\mathfrak{A}}e)\vee^{\mathfrak{A}}\neg^{\mathfrak{A}}(d\wedge^{\mathfrak{A}}e))\wedge^{\mathfrak{A}}f),
\end{aligned}$$

where $l\in 2$, is a homomorphism from $\mathfrak{MS}_{4:1}\times\mathfrak{B}_2$ to $\mathfrak{A}$ such that $(g\circ h)=\Delta_{MS_{4:1}\times B_2}$, and so it is injective. Finally, $(\mathfrak{MS}_{4:1}\times\mathfrak{B}_2)\not\models\mathcal{MR}[x_n/\langle\langle 0,\max(0,n-1),\max(1-n,n-1)\rangle,3\times\{1\}\rangle]_{n\in 3}$. □

This, by Theorems 4.7, 5.3, Corollaries 5.12, 5.6, Lemma 5.4, (5.2) and [9], immediately yields:

Corollary 5.14. *Let* $\mathsf{P}\subseteq(\mathsf{NIQSMSL}\cup\mathsf{DML})$ *be a pre-variety and* $(\mathsf{K}\subseteq)\mathsf{P}'\triangleq(\mathsf{P}\cap\mathsf{DML})$. *Suppose* $\mathsf{P}\nsubseteq\mathsf{MRQSMSL}$ *(and* P' *is the pre-variety generated by* K*). Then,* P *is the pre-variety generated by* $\mathsf{P}'\cup\mathsf{NIQSKSL}$ *(in which case it is the one generated by* $\mathsf{K}\cup\{\mathfrak{MS}_{4:1}\times\mathfrak{B}_2\}$*),* P' *being a finitely-generated quasi-variety, and so being* $\mathsf{P}=(\mathsf{P}'\uplus^{\mathrm{Q}}\mathsf{NIQSKSL})$. *In particular,* $f[']:L_{\mathrm{Q}}(\mathsf{NIQSKSL},\mathsf{NIQSMSL}\cup\mathsf{DML})\to L_{\mathrm{Q}}(\mathsf{NIMRQSKSL}[\cap\mathsf{NIKL}],\mathsf{MRQSMSL}[\cap\mathsf{DML}]),\mathsf{Q}\mapsto((\mathsf{Q}\cap\mathsf{MRQSMSL})[\cap\mathsf{DML}])$ *and* $g[']:L_{\mathrm{Q}}(\mathsf{NIMRQSKSL}[\cap\mathsf{NIKL}],\mathsf{MRQSMSL}[\cap\mathsf{DML}])\to L_{\mathrm{Q}}(\mathsf{NIQSKSL},\mathsf{NIQSMSL}\cup\mathsf{DML}):\mathsf{Q}'\mapsto((\mathsf{Q}'\{\uplus^{\mathrm{Q}}(\mathsf{RQSKSL}\uplus^{\mathrm{Q}}\mathsf{SL})\})\uplus^{\mathrm{Q}}\mathsf{NIQSKSL})$ *are inverse to one another isomorphisms between* $\mathfrak{L}_{\mathrm{Q}}(\mathsf{NIQSKSL},\mathsf{NIQSMSL}\cup\mathsf{DML})$ *and* $\mathfrak{L}_{\mathrm{Q}}(\mathsf{NIMRQSKSL}[\cap\mathsf{NIKL}],\mathsf{MRQSMSL}[\cap\mathsf{DML}])$*, in which case for any* $\mathsf{Q}\in L_{\mathrm{Q}}(\mathsf{NIQSKSL},\mathsf{NIQSMSL}\cup\mathsf{DML})=(L_{\mathrm{Q}}(\mathsf{NIQSMSL}\cup\mathsf{DML})\setminus L_{\mathrm{Q}}(\mathsf{MRQSMSL}))$ *and* $\mathsf{Q}'\in L_{\mathrm{Q}}(\mathsf{MRQSMSL})$, $(\mathsf{Q}\cap\mathsf{Q}')=(f(\mathsf{Q})\cap\mathsf{Q}')$ *and* $(\mathsf{Q}\uplus^{\mathrm{Q}}\mathsf{Q}')=(\mathsf{Q}\uplus^{\mathrm{U}}g(\mathsf{Q}'))$*, so* $\{\langle(\mathsf{S}\cap\mathsf{MRQSMSL})[\cap\mathsf{DML}],(1-\chi^{L_{\mathrm{Q}}(\mathsf{MRQSMSL})}_{L_{\mathrm{Q}}(\mathsf{NIQSMSL}\cup\mathsf{DML})}(\mathsf{S}))[+(1-\chi^{L_{\mathrm{Q}}(\mathsf{DML})}_{L_{\mathrm{Q}}(\mathsf{NIQSMSL}\cup\mathsf{DML})}(\mathsf{S}))+(1-\chi^{L_{\mathrm{Q}}(\mathsf{SMSL})}_{L_{\mathrm{Q}}(\mathsf{NIQSMSL}\cup\mathsf{DML})}(\mathsf{S}))]\rangle\mid\mathsf{S}\in L_{\mathrm{Q}}(\mathsf{NIQSMSL}\cup\mathsf{DML})\}$ *is an embedding of* $\mathfrak{L}_{\mathrm{Q}}(\mathsf{NIQSMSL}\cup\mathsf{DML})$ *into* $\mathfrak{L}_{\mathrm{Q}}(\mathsf{MRQSMSL}[\cap\mathsf{DML}])\times\mathfrak{D}_{2[+2]}$*, the former having* $(|L_{\mathrm{Q}}(\mathsf{MRSMSL})|+|L_{\mathrm{Q}}(\mathsf{NIKL},\mathsf{DML})|)=(21+5)=26$ *elements and Hasse diagram depicted at Figure 4 with large circles-nodes [the latter being embedable into the distributive lattice* $(\mathfrak{D}_5\times\mathfrak{D}_3)\times\mathfrak{D}_4$*].*

5.2.2.4. The lattice of quasi-varieties of quasi-strong Morgan-Stone lattices.

Lemma 5.15. $\mathfrak{MS}_{4:1}\in\mathsf{K}\triangleq(\mathsf{IQSMSL}\setminus[\mathsf{N}]\mathsf{DML})$ *is embedable into any* $\mathfrak{A}\in\mathsf{K}$.

Proof. By Lemma 5.8, there are some $a,e\in A$ such that $\neg^{\mathfrak{A}}e=e$ and $c\neq d\neq b$, where $b,c,d\in A$ are as in Lemma 5.8, in which case $b\leqslant^{\mathfrak{A}}(f|g)\triangleq((e\wedge^{\mathfrak{A}}(c|d))\vee^{\mathfrak{A}}b)=(g\wedge^{\mathfrak{A}}(c|d))$, and so, by $\mathcal{DM}_i$ with $i\in 2$, we have $b\neq f\leqslant^{\mathfrak{A}}g=\neg^{\mathfrak{A}}(f|g)\notin\{c,d\}$, for, otherwise, we would get $b=g=d$. Then, by $\mathcal{Q}$, we get $g=(\neg^{\mathfrak{A}}f\wedge^{\mathfrak{A}}\neg^{\mathfrak{A}}\neg^{\mathfrak{A}}f)\leqslant^{\mathfrak{A}}f\leqslant^{\mathfrak{A}}c$, in which case $\{\langle 0,0,0,b\rangle,\langle 0,0,1,g\rangle,\langle 0,1,1,c\rangle,\langle 1,1,1,d\rangle\}\in\hom_{\mathrm{I}}(\mathfrak{MS}_{4:1},\mathfrak{A})$, so Lemmas 4.6, 5.1 and Theorem 4.7 complete the argument. □

This, by Theorems 4.7, 5.3, Corollaries 5.14, 5.6, Lemma 5.4, (5.2) and [9], eventually yields:

Theorem 5.16. *Let* $\mathsf{P} \subseteq \mathsf{QSMSL}$ *be a pre-variety and* $(\mathsf{K} \subseteq)\mathsf{P}' \triangleq (\mathsf{P} \cap \mathsf{DML})$. *Suppose* $\mathsf{P} \nsubseteq (\mathsf{NIQSMSL} \cup \mathsf{DML})$ *(and* P' *is the pre-variety generated by* K*). Then,* P *is the pre-variety generated by* $\mathsf{P}' \cup \mathsf{QSKSL}$ *(in which case it is the one generated by* $\mathsf{K} \cup \{\mathfrak{MS}_{4:1}\}$*),* P' *being a finitely-generated quasi-variety, and so being* $\mathsf{P} = (\mathsf{P}' \uplus^{\mathrm{Q}} \mathsf{QSKSL})$. *In particular,* $f[']: L_{\mathrm{Q}}(\mathsf{QSKSL}, \mathsf{QSMSL}) \to L_{\mathrm{Q}}((\mathsf{NIQSKSL} \cup \mathsf{KL})[\cap \mathsf{KL}], (\mathsf{NIQSMSL} \cup \mathsf{DML})[\cap \mathsf{DML}]), \mathsf{Q} \mapsto ((\mathsf{Q} \cap (\mathsf{NIQSMSL} \cup \mathsf{DML}))[\cap \mathsf{DML}])$ *and* $g[']: L_{\mathrm{Q}}((\mathsf{NIQSKSL} \cup \mathsf{KL})[\cap \mathsf{KL}], (\mathsf{NIQSMSL} \cup \mathsf{DML})[\cap \mathsf{DML}]) \to L_{\mathrm{Q}}(\mathsf{QSKSL}, \mathsf{QSMSL}) : \mathsf{Q}' \mapsto ((\mathsf{Q}'\{\uplus^{\mathrm{Q}}((\mathsf{QSKSL} \uplus^{\mathrm{Q}} \mathsf{RQSKSL}) \uplus^{\mathrm{Q}} \mathsf{SL})\}) \uplus^{\mathrm{Q}} \mathsf{NIQSKSL})$ *are inverse to one another isomorphisms between* $\mathfrak{L}_{\mathrm{Q}}(\mathsf{QSKSL}, \mathsf{QSMSL})$ *and* $\mathfrak{L}_{\mathrm{Q}}((\mathsf{NIQSKSL} \cup \mathsf{KL})[\cap \mathsf{KL}], (\mathsf{NIQSMSL} \cup \mathsf{DML})[\cap \mathsf{DML}])$*, in which case, for any* $\mathsf{Q} \in L_{\mathrm{Q}}(\mathsf{QSKSL}, \mathsf{QSMSL}) = (L_{\mathrm{Q}}(\mathsf{QSMSL}) \setminus L_{\mathrm{Q}}(\mathsf{NIQSMSL} \cup \mathsf{DML}))$ *and* $\mathsf{Q}' \in L_{\mathrm{Q}}(\mathsf{NIQSMSL} \cup \mathsf{DML})$, $(\mathsf{Q} \cap \mathsf{Q}') = (f(\mathsf{Q}) \cap \mathsf{Q}')$ *and* $(\mathsf{Q} \uplus^{\mathrm{Q}} \mathsf{Q}') = (\mathsf{Q} \uplus^{\mathrm{U}} g(\mathsf{Q}'))$*, so* $\{\langle (\mathsf{S} \cap (\mathsf{NIQSMSL} \cup \mathsf{DML}))[\cap \mathsf{DML}], (1 - \chi^{L_{\mathrm{Q}}(\mathsf{NIQSMSL} \cup \mathsf{DML})}_{L_{\mathrm{Q}}(\mathsf{QSMSL})}(\mathsf{S}))[+(1 - \chi^{L_{\mathrm{Q}}(\mathsf{MRQSMSL})}_{L_{\mathrm{Q}}(\mathsf{QSMSL})}(\mathsf{S})) + (1 - \chi^{L_{\mathrm{Q}}(\mathsf{SMSL})}_{L_{\mathrm{Q}}(\mathsf{QSMSL})}(\mathsf{S})) + (1 - \chi^{L_{\mathrm{Q}}(\mathsf{DML})}_{L_{\mathrm{Q}}(\mathsf{QSMSL})}(\mathsf{S}))] \rangle \mid \mathsf{S} \in L_{\mathrm{Q}}(\mathsf{QSMSL})\}$ *is an embedding of* $\mathfrak{L}_{\mathrm{Q}}(\mathsf{QSMSL})$ *into* $\mathfrak{L}_{\mathrm{Q}}((\mathsf{NIQSMSL} \cup \mathsf{DML})[\cap \mathsf{DML}]) \times \mathfrak{D}_{2[+3]}$*, the former having* $(|L_{\mathrm{Q}}(\mathsf{NIQSMSL} \cup \mathsf{DML})| + |L_{\mathrm{Q}}(\mathsf{KL}, \mathsf{DML})|) = (26+3) = 29$ *elements and Hasse diagram depicted at Figure 4 [the latter being embedable into the distributive lattice* $(\mathfrak{D}_5 \times \mathfrak{D}_3) \times \mathfrak{D}_5$*].*

Finally, Theorems 4.7, 5.3, 5.7 and Corollary 5.10 provide finite generating sets of all sub-quasi-varieties of QSMSL.

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