User Equilibrium and System Optimum with Incomplete Information In Traffic Congestion

Qiang Zhang, Shi Qiang Liu and Mahmoud Masoud

June 5, 2019

User Equilibrium and System Optimum with Incomplete Information In Traffic Congestion

Qiang Zhang
School of Economics and Management
Fuzhou University
Fuzhou, China, 350108
joh_n_zhang@163.com

Shi Qiang Liu*
School of Economics and Management
Fuzhou University
Fuzhou, China, 350108
samsqliu@fzu.edu.cn

Mahmoud Masoud*
School of Mathematical Sciences
Queensland University of Technology
Brisbane, Australia, 4001
mahmoud.masoud@qut.edu.au

***Abstract*—By providing more information about traffic network, such as more feasible paths via intelligent navigation systems (INS), users in the network may change their choices of the path from a source to a destination. This paper investigates a traffic congestion model with incomplete information, in which different users have different information about the network. We introduce the notions of *user equilibrium of incomplete information* (UEII) and *system optimum of incomplete information* (SOII). Then, we prove a theorem about the effect of the change of traffic amount on each couple paths in SOII for the model. Finally, based on this theorem and a property of UEII, we reveal a relationship between UEII and SOII on the cost function.**

Keywords—incomplete information, user equilibrium, system optimum, traffic network

I. INTRODUCTION

Recently the rapid development of mobile internet produces profound influence on the way we live. Intelligent Navigation Systems (INS) based on GPS, such as Google Map or Baidu Map, are representative examples. To reduce the traffic congestion and improve the driving experience, INS provides users more information about transportation network, for instance, more feasible paths to destination. However, how does the more information influence stable state of traffic network? This is the central question of this paper and we focus on the traffic congestion model with the factor of additional information.

In traditional research of traffic congestion effects, it is a common behavior that the users in the transportation network are regarded as independent participants in a noncooperative game [1] with the assumption that users act in selfish to minimize individual travel time by choosing their routes consisting of any arc in the network [2]. The sum of all participants' travel time in the equilibrium of aforementioned game, which is known as user equilibrium (UE) [3], is not an status with the minimization of social time cost, which is called system optimum (SO) [4]. Inspired by the inefficient of UE caused by each user ignoring the welfare of whole population [5], many researchers focused on the impact of selfishness. Reference [2] proved that the time consumption of UE is no more than the cost of an optimal routing of twice as much traffic. For more general, the ratio between UE and SO (also called "the price of anarchy" [6]) was studied in a series of papers with more realistic model features, including traffic capacity of arc [7], [8], nonatomic congestion games [9] and more practical latency function of arc [10], [11] and structure of network [12]–[14]. References [15]–[20] focused on the behavior changes of selfish users affected by pricing strategy and references [21]–[23] concentrate on designing algorithm mechanism.

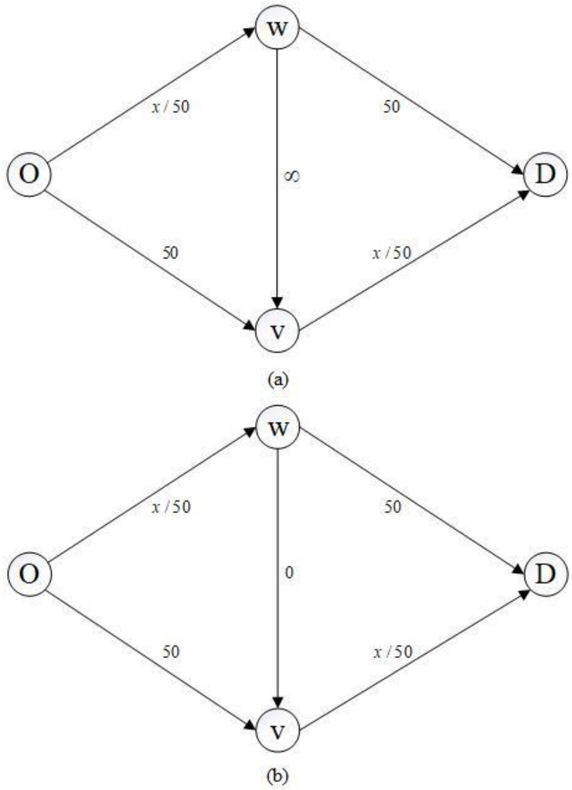

Fig. 1. Example of Braess's paradox

An intuitive measure to improve the performance of network is to decrease the time cost of arcs, but, in UE, there has a counterintuitive and well-known phenomenon called Braess's Paradox (BP) proposed by [24] and [25]. Fig. 1 can be a concise interpretation of the paradox. In Fig. 1, label of each arc is the load-dependent cost function specifying the time to travel along the arc. Suppose that there are 2000 users travel from origin node O to destination node D in the directed network of Fig. 1(a). Base on the assumption of UE, each user minimize their own time cost, the equilibrium result is that half of the 2000 users choose the path $O \to W \to D$, the other half take the path $O \to V \to D$ and all users have the same time cost, 70. Then, suppose that the cost of arc $W \to V$ reduces to 0, and the equilibrium result is that all users choose the path $O \to W \to V \to D$ and every user suffers the increasing time cost, 80. The decreasing of arc cost increase the cost of all users. Encouraged by the finding of the interesting fact, BP, researchers have attempted to explore its significant impact on the designing of the transportation network in congested regions. These previous works mainly contain detecting BP in networks [26]–[30], researching the variants of BP not in the traffic context [31]–[34] and designing the network to reduce the negative impacts from BP [35]–[39].

The foregoing works are based on the assumption that users in transportation network are familiar with all arcs. But, in most cases, due to the complexity of the transportation network, each user cannot be aware of all nodes and arcs. On the other hand, users have various ways to obtain information about the network, especially from the INS. Therefore, different users will have different knowledge of network so that they choose a selfish path based on their own subset of arcs. Reference [40] studied congestion games with player-specific latency functions with a player-specific constant which can be considered as another way to model the incomplete information about the nodes and arcs. Under the consideration of multiple types participants differed by their accessible arcs, Reference [41] extended the conception of UE, proved the existence and uniqueness of it and extended the concept BP to informational Braess's paradox (IBP) in which more information could degrade the network performance. Reference [42] proved that there exists more appropriate configuration of information which can improve the performance of network with uncertain congestion. Reference [43] showed that a suitable private information disclosure mechanisms can improve the overall efficiency in the transportation network. Reference [44] considered a class of Bayesian congestion game in which there are two Traveler Information Systems providing information with two different accuracy and showed that the heterogeneity of information is benefit to the overall efficiency.

In this paper, we further research the model proposed by [41] and our main results are as follows:

- Proving the characteristic that, in the status of SOII, the marginal benefit of decreasing amount of flow one path is no more than the marginal cost of increasing amount of flow on any another path.
- Showing that the marginal factor of cost function neglected by users is the reason of the difference between user equilibrium of incomplete information (UEII) and system optimum of incomplete information (SOII) in the traffic congestion model.

The paper is organized as follows. In Section 2, we introduce the formal definition of the network model. In Section 3, we first state the definition of UEII and provide a numerical example to illustrate the notion of UEII. Then we prove a useful property of SOII and give the relationship between UEII and SOII. Finally, Section 4 contains concluding remarks.

II. MODEL

In this section, we describe the network model and introduce the necessary notation for the analysis.

We consider a directed network $G=(N,A)$ with node set N, arc set A and an origin-destination pair $\{O,D\}$. Each arc $a\in A$ connects two nodes $\{n_i,n_j\}$, which called the start node and end node of arc a respectively. A path $P\in G$ is a series of arcs $a_1,\dots a_n$ to connect O and D, where the end node of a_i is the start node of a_j. We denote the set of all paths by $\mathcal{P}$. To model the incomplete information of arcs, we assume that there are K types of users traveling on G and each type of users set, denoted by T_i, only know an arc subset $A_i\subseteq A$ $(i=1,\dots,k)$.

The obligatory notations for further research as follow.

d_i	total demand flow of all users in T_i
$d_{(1:K)}$	vector of d_i, $i\in\{1,\dots,K\}$
$A_{(1:K)}$	vector of A_i, $i\in\{1,\dots,K\}$
$\mathcal{P}_i$	set of paths composed of arcs in A_i
f_i^P	amount of flow of users in T_i on path P
f^P	total amount of flow on path P, $f^P=\sum_{i=1}^K f_i^P$
f^a	total amount of flow on arc a, $f^a=\sum_{P\in\mathcal{P}:\,a\in P} f^P$
f_i	flow of type i, which is a vector of f_i^P, $P\in\mathcal{P}_i$
$f_{(1:K)}$	flow, which is a vector of f_i, $i\in\{1,\dots,K\}$
$c^a(\cdot)$	load-dependent cost function of each arc $a\in A$, which is nonnegative, nondecreasing and differentiable
$(c^a(\cdot))'$	derivative of $c^a(\cdot)$
$c^P\big(f_{(1:K)}\big)$	cost of path P incurred by flow $f_{(1:K)}$, $c^P\big(f_{(1:K)}\big)=\sum_{a\in P}c^a(f^a)$
$l^a(\cdot)$	total cost function of arc $a\in A$, $l^a(x)=c^a(x)\cdot x$
$\big(l^a(\cdot)\big)'$	derivative of $l^a(\cdot)$, $\big(l^a(x)\big)'=c^a(x)'\cdot x+c^a(x)$
$\Big(l^P\big(f_{(1:K)}\big)\Big)'$	sum of $(l^a(\cdot))'$, $a\in P$, incurred by flow $f_{(1:K)}$, $\Big(l^P\big(f_{(1:K)}\big)\Big)'=\sum_{a\in P}\big(l^a(f^a)\big)'$
$\mathcal{C}$	set of all $c_a(\cdot)$, $a\in A$

Base on the above notations, we define $\big(G,A_{(1:K)},d_{(1:K)},\mathcal{C}\big)$ as an instance of the model. In additional, we call a feasible flow of type i is a vector $f_i=(f_i^P\colon P\in\mathcal{P}_i)$ when $\sum_{P\in\mathcal{P}_i}f_i^P=d_i$ and a feasible flow for an instance is a vector $f_{(1:K)}=(f_i\colon i\in\{1,\dots,K\})$ when each $f_i, i\in\{1,\dots,K\}$ is a feasible flow of type i.

III. RELATIONSHIP BETWEEN UEII AND SOII

A. User Equilibrium of Incomplete Information

UEII is an equilibrium of the noncooperative game in which each user act in selfish manner to travel along the available minimum-latency path consisted of arcs in A_i. Therefore, each user has no path to choose to further reduce the cost. We next formalize this user equilibrium notion in our model.

Definition 3.1. A feasible $f_{(1:K)}$ for instance $\big(G,A_{(1:K)},d_{(1:K)},\mathcal{C}\big)$ is at user equilibrium of incomplete information(UEII) if for each $i\in\{1,\dots,K\}$, every couple $P_1,P_2\in\mathcal{P}_i$, $\delta\in\big(0,f_i^{P_1}\big]$, and a new feasible $\overline{f_{(1:K)}}$ constructed by $f_{(1:K)}$ in which is replaced the ith element by the following $\overline{f_i}$,

$$\overline{f_i}=\begin{cases}f_i^P-\delta & \text{if } P=P_1\\ f_i^P+\delta & \text{if } P=P_2\\ f_i^P & \text{if } P\notin\{P_1,P_2\}\end{cases},$$

we have $c^{P_1}\big(f_{(1:K)}\big)\le c^{P_2}\big(\overline{f_{(1:K)}}\big)$

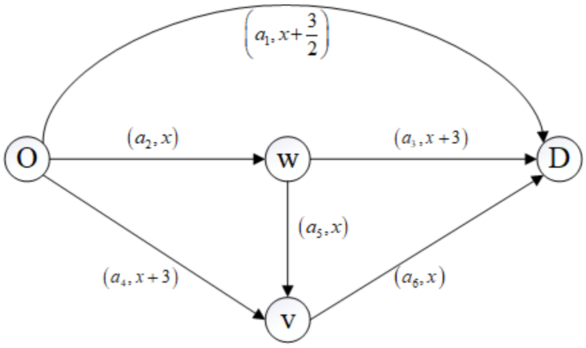

Fig. 1. Example of a network

Because of the nondecreasing and differentiable cost function of each arc, making δ approximate 0 generates the following useful lemma of UEII.

Lemma 3.2. A feasible flow $f_{(1:K)}$ for instance $\left(G, A_{(1:K)}, d_{(1:K)}, \mathcal{C}\right)$ is at user equilibrium of incomplete information (UEII) if and only if for each $i \in \{1, \ldots, K\}$, every couple $P, \overline{P} \in \mathcal{P}_i$ with $f_i^P > 0$, we have $c^P\left(f_{(1:K)}\right) \leq c^{\overline{P}}\left(f_{(1:K)}\right)$.

From Lemma 3.2, if a feasible flow $f_{(1:K)}$ is at UEII, users of each type have the equal cost on their paths consisted of their own arc set. Lemma 3.2 can be deemed to the Wardrop's principle [45] of incomplete information [41].

The following is a numerical example of the UEII in a specific network.

Example 1. Consider the directed network $G = (N, A)$ given in Fig. 1, where each arc a_i labeled by a pair $(a_i,\ c^{a_i}(x))$. We denote all different paths from origin O to destination D as following: $P_1 = a_1$, $P_2 = a_2 \to a_3$, $P_3 = a_2 \to a_5 \to a_6$, and $P_4 = a_4 \to a_6$. Suppose that there are two types of users with respect to arc subset $A_1 = \{a_1\}$, $A_2 = A$ and total demand flow $d_1 = d$, $d_2 = 1 - d$ $(0 \leq d \leq 1)$. Then we have $\mathcal{P}_1 = \{P_1\}$ and $\mathcal{P}_2 = \{P_1, P_2, P_3, P_4\}$.

Suppose the flow of type 1 on $\mathcal{P}_1$ is $f_1^{P_1}$ and the flows of type 2 on $\mathcal{P}_2$ are $f_2^{P_1}, f_2^{P_2}, f_2^{P_3}, f_2^{P_4}$. Then we have the following equations:

$$c^{P_1}\left(f_{(1:2)}\right) = f_1^{P_1} + f_2^{P_1} + 3/2 \qquad (1.1)$$

$$c^{P_2}\left(f_{(1:2)}\right) = \left(f_2^{P_2} + f_2^{P_3}\right) + (f_2^{P_2} + 3) \qquad (1.2)$$

$$c^{P_3}\left(f_{(1:2)}\right) = (f_2^{P_2} + f_2^{P_3}) + f_2^{P_3} + (f_2^{P_3} + f_2^{P_4}) \qquad (1.3)$$

$$c^{P_4}\left(f_{(1:2)}\right) = f_2^{P_4} + 3 \qquad (1.4)$$

From (1.1), (1.2), (1.3), (1.4), we have $c^{P_3}\left(f_{(1:2)}\right) \leq c^{P_2}\left(f_{(1:2)}\right)$ and $c^{P_3}\left(f_{(1:2)}\right) \leq c^{P_4}\left(f_{(1:2)}\right)$. Therefore, according to Lemma 3.2, users of type 2 will not choose the paths, P_2 and P_4. That is we have $f_2^{P_2} = f_2^{P_4} = 0$ and $f_2^{P_1} + f_2^{P_3} = 1 - d$. The critical condition of type 2 to use path P_1 or not, that is $f_2^{P_1}$ equals 0 or not, can be described as the following equation

$$f_1^{P_1} + 3/2 = 3f_2^{P_3},$$

where $f_1^{P_1} = d$ and $f_2^{P_3} = 1 - d$.

Hence, the critical point of d is $3/8$ and we have the following result of UEII:

If $d \leq 3/8$, the flow at UEII is $f_{(1:2)} = \left((f_1^{P_1}), (f_2^{P_1}, f_2^{P_2}, f_2^{P_3}, f_2^{P_4})\right)$, in which $f_1^{P_1} = d$, $f_2^{P_1} = 3/8 - d$, $f_2^{P_3} = 5/8$ and $f_2^{P_2} = f_2^{P_4} = 0$. The cost of path P_1 of type 1 is $c^{P_1}\left(f_{(1:K)}\right) = 15/8$, and the cost of paths of type 2 is $c^{P_1}\left(f_{(1:K)}\right) = c^{P_3}\left(f_{(1:K)}\right) = 15/8$. From this result, we can see that type 1 and type 2 have the same equilibrium cost when they share a common path P_1.

If $d > 3/8$, the flow at UEII is $f_{(1:2)} = \left((f_1^{P_1}), (f_2^{P_1}, f_2^{P_2}, f_2^{P_3}, f_2^{P_4})\right)$, in which $f_1^{P_1} = d$, $f_2^{P_3} = 1 - d$ and $f_2^{P_1} = f_2^{P_2} = f_2^{P_4} = 0$. The cost of path P_1 of type 1 is $c^{P_1}\left(f_{(1:K)}\right) = d + 3/2$, and the cost of path P_3 of type 2 is $c^{P_3}\left(f_{(1:K)}\right) = 3(1 - d)$. The result shows that type 1 and type 2 can have the different equilibrium cost.

B. System Optimum of Incomplete Information

SOII is a feasible flow that minimizes total cost of all users in the noncooperative game with incomplete information. According to our definition of total cost and feasible flow, SOII is a solution of the following optimization problem:

$$\begin{aligned}
\min \quad & \sum_{a \in A} l^a(f^a) & \\
\text{s.t.:} \quad & \sum_{P \in \mathcal{P}_i} f_i^P = d_i & \forall i \in \{1, \ldots, k\} \\
& f^a = \sum_{i=1}^{K} \sum_{P \in \mathcal{P}_i : a \in P} f_i^P & \forall a \in A \\
& f_i^P \geq 0 & \forall i \in \{1, \ldots, k\}, \\
& & \forall P \in \mathcal{P} \qquad (*)
\end{aligned}$$

Because of the differentiable cost function of each arc, the optimal solution for problem $(*)$ is exist. In the sense of SOII, making an arbitrarily small flow from one path to another will lead to an increase in the total cost. That is, the marginal benefit of cutting amount of flow on path P_i is no more than the marginal cost of adding amount of flow on another path P_j. We next formalize the characteristic of SOII and give a proof via the Karush-Kuhn-Tucker (KKT) theorem [46].

Lemma 3.3. A flow $f_{(1:K)}$ for instance $\left(G, A_{(1:K)}, d_{(1:K)}, \mathcal{C}\right)$ is at system optimum of incomplete information (SOII) if and only if for each $i \in \{1, \ldots, K\}$, every couple $P, \overline{P} \in \mathcal{P}_i$ with $f_i^P > 0$, we have $\left(l^P\left(f_{(1:K)}\right)\right)' \leq \left(l^{\overline{P}}\left(f_{(1:K)}\right)\right)'$.

Proof: Suppose $(l^P\left(f_{(1:K)}\right))'$ of type i at SOII equals ζ_i. From the conditions of Lemma 3.3, we have the following expression:

$$\left(l^P\left(f_{(1:K)}\right)\right)' = \begin{cases} = \zeta_i & \text{if } f_i^P > 0 \\ \geq \zeta_i & \text{if } f_i^P = 0 \end{cases}.$$

Let η_i^P be:

$$\eta_i^P = \begin{cases} 0 & \text{if } f_i^P > 0 \\ \left(l^P\left(f_{(1:K)}\right)\right)' - \zeta_i & \text{if } f_i^P = 0 \end{cases}.$$

Then we have the following equation:

$$\frac{\partial}{\partial f_i^P}\left(\sum_{a\in A} l^a(f^a) - \sum_{i=1}^{K}\zeta_i\left(\sum_{P\in\mathcal{P}_i} f_i^P - d_i\right) - \sum_{i=1}^{K}\sum_{P\in\mathcal{P}_i}\eta_i^P f_i^P\right) = 0.$$

Therefore, the KKT conditions are satisfied by the flow $f_{(1:K)}$ with parameters ζ_i and η_i^P. In problem (OP), due to the convexity of objective function and the affine property of constrain functions, the KKT conditions is the sufficient condition for optimal solution and the flow $f_{(1:K)}$ for instance $(G, A_{(1:K)}, d_{(1:K)}, \mathcal{C})$ is at SOII.

Conversely, because of the convexity of objective function and the affine property of constrain functions, KKT conditions are satisfied and for all $i \in \{1, \dots, K\}$, $P \in \mathcal{P}_i$, we have the following expression:

$$\frac{\partial}{\partial f_i^P}\left(\sum_{a\in A} l^a(f^a) - \sum_{i=1}^{K}\zeta_i\left(\sum_{P\in\mathcal{P}_i} f_i^P - d_i\right) - \sum_{i=1}^{K}\sum_{P\in\mathcal{P}_i}\eta_i^P f_i^P\right) = 0, \tag{1}$$

where $\eta_i^P = \begin{cases} = 0 & \text{if } f_i^P > 0 \\ \geq 0 & \text{if } f_i^P = 0 \end{cases}$. Then, we can simplify the (1) to the following:

$$\sum_{a\in A}\frac{\partial f^a}{\partial f_i^P}\left(l^a(f^a)\right)' = \left(l^P\left(f_{(1:K)}\right)\right)' = \begin{cases} = \zeta_i & \text{if } f_i^P > 0 \\ \geq \zeta_i & \text{if } f_i^P = 0 \end{cases}.$$

Therefore, we get the conclusion, $\left(l^P\left(f_{(1:K)}\right)\right)' \leq \left(l^{\overline{P}}\left(f_{(1:K)}\right)\right)'$. □

C. Relationship Between UEII and SOII

We notice Lemma 3.2 and Lemma 3.3 have some formal resemblance. Base on them, we will further reveal the relationship between UEII and SOII of the traffic congestion model. In the following theorem, we define $\mathcal{L}$ as the set of all $\left(l^a(\cdot)\right)'$, $a \in A$.

Theorem 3.5. A feasible flow $f_{(1:K)}$ for instance $(G, A_{(1:K)}, d_{(1:K)}, \mathcal{L})$ is at user equilibrium of incomplete information (UEII) if and only if $f_{(1:K)}$ is at system optimum of incomplete information (SOII) for instance $(G, A_{(1:K)}, d_{(1:K)}, \mathcal{C})$.

Proof: From the Lemma 3.4, a flow $f_{(1:K)}$ is at SOII for instance $(G, A_{(1:K)}, d_{(1:K)}, \mathcal{C})$, if and only if for each $i \in \{1, \dots, K\}$, every couple $P, \overline{P} \in \mathcal{P}_i$ with $f_i^P > 0$, we have $\left(l^P\left(f_{(1:K)}\right)\right)' \leq \left(l^{\overline{P}}\left(f_{(1:K)}\right)\right)'$. On the other hand, from the definition of notations, we have:

$$\left(l^P\left(f_{(1:K)}\right)\right)' \leq \left(l^{\overline{P}}\left(f_{(1:K)}\right)\right)' \Leftrightarrow$$
$$\sum_{a\in P} c^a(f^a)'\cdot f^a + c^a(f^a) \leq \sum_{a\in \overline{P}} c^a(f^a)'\cdot f^a + c^a(f^a) \tag{2}$$

Equation (2) is the definition of UEII of the instance$(G, A_{(1:K)}, d_{(1:K)}, \mathcal{L})$ according to Lemma 3.2. □

From Theorem 3.5, we can see that a flow $f_{(1:K)}$ of SOII will be a flow of UEII when the arc cost function has a new form: $\left(l^a(x)\right)' = c^a(x)'\cdot x + c^a(x)$. The new form consists of the origin form, $c^a(x)$, and a new component, $c^a(x)'\cdot x$, which can be illustrated as the traffic congestion incurred by the marginal increase traffic suffering from the already existing. Therefore, comparing to SOII, the poor performance of the UEII is the ignorance of users with the additional negative effect, $c^a(x)'\cdot x$.

IV. Conclusion

The widely-used INS gives users more information about transportation network and exerts important influences on traffic decision. In this paper, we investigate the traffic congestion model with incomplete information. We first propose the definition of UEII of incomplete information model in which different type users have different arc sets to choose their path. Then, we propose a new theorem of SOII whose form is similar to a necessary and sufficient condition of UEII. Finally, the theorem is applied to reveal the relationship of cost function of UEII and SOII which can account for the subtle difference between them. The proposed methodology will be applied to bioenergy production, biomass supply chain, mining industry, robotics, railway, aviation and healthy industries[47]–[71] [72]–[74].

Acknowledgment

This work was supported by the National Natural Science Foundation of China under Grant nos. 71871064, 71801051 and 61873173.

References

[1] E. Koutsoupias and C. Papadimitriou, "Worst-Case Equilibria," in *STACS 99*, 1999, pp. 404–413.

[2] T. Roughgarden and É. Tardos, "How bad is selfish routing?," *J. ACM*, vol. 49, no. 2, pp. 236–259, 2002.

[3] S. Dafermos and A. Nagurney, "On some traffic equilibrium theory paradoxes," *Transp. Res. Part B Methodol.*, vol. 18, no. 2, pp. 101–110, Apr. 1984.

[4] A. Downs, "The law of peak-hour expressway congestion," *Traffic Q.*, vol. 16, no. 3, 1962.

[5] P. Dubey, "Inefficiency of Nash Equilibria," *Math. Oper. Res.*, vol. 11, no. 1, pp. 1–8, 1986.

[6] C. H. Papadimitriou, "Algorithms, Games, and the Internet," in *Automata, Languages and Programming*, 2001, pp. 1–3.

[7] J. R. Correa, A. S. Schulz, and N. E. Stier-Moses, "Selfish Routing in Capacitated Networks," *Math. Oper. Res.*, vol. 29, no. 4, pp. 961–976, 2004.

[8] M. Mavronicolas and P. Spirakis, "The Price of Selfish Routing," in *Proceedings of the Thirty-third Annual ACM Symposium on Theory of Computing*, 2001, pp. 510–519.

[9] T. Roughgarden and É. Tardos, "Bounding the inefficiency of equilibria in nonatomic congestion games," *Games Econ. Behav.*, vol. 47, no. 2, pp. 389–403, May 2004.

[10] H. Yang, W. Xu, and B. Heydecker, "Bounding the efficiency of road pricing," *Transp. Res. Part E Logist. Transp. Rev.*, vol. 46, no. 1, pp. 90–108, Jan. 2010.

[11] A. Czumaj and B. Vöcking, "Tight Bounds for Worst-case Equilibria," *ACM Trans. Algorithms*, vol. 3, no. 1, p. 4:1--4:17, 2007.

[12] E. Koutsoupias and C. Papadimitriou, "Worst-case equilibria," *Comput. Sci. Rev.*, vol. 3, no. 2, pp. 65–69, 2009.

[13] T. Roughgarden, "The price of anarchy is independent of the network topology," *J. Comput. Syst. Sci.*, vol. 67, no. 2, pp. 341–364, 2003.

[14] R. Colini-Baldeschi, R. Cominetti, and M. Scarsini, "Price of Anarchy for Highly Congested Routing Games in Parallel Networks," *Theory Comput. Syst.*, vol. 63, no. 1, pp. 90–113, 2019.

[15] R. Cole, Y. Dodis, and T. Roughgarden, "How much can taxes help selfish routing?," *J. Comput. Syst. Sci.*, vol. 72, no. 3, pp. 444–467, 2006.

[16] R. Cole, Y. Dodis, and T. Roughgarden, "Pricing Network Edges for Heterogeneous Selfish Users," in *Proceedings of the Thirty-fifth Annual ACM Symposium on Theory of Computing*, 2003, pp. 521–530.

[17] V. Bonifaci, M. Salek, and G. Schäfer, "Efficiency of Restricted Tolls in Non-atomic Network Routing Games," in *Algorithmic Game Theory*, 2011, pp. 302–313.

[18] Y. Hamdouch and S. Lawphongpanich, "Congestion Pricing for Schedule-Based Transit Networks," *Transp. Sci.*, vol. 44, no. 3, pp. 350–366, 2010.

[19] B. Awerbuch, Y. Azar, and A. Epstein, "The Price of Routing Unsplittable Flow," *SIAM J. Comput.*, vol. 42, no. 1, pp. 160–177, 2013.

[20] V. Bilò and C. Vinci, "Dynamic Taxes for Polynomial Congestion Games," in *Proceedings of the 2016 ACM Conference on Economics and Computation*, 2016, pp. 839–856.

[21] J. Feigenbaum, C. H. Papadimitriou, and S. Shenker, "Sharing the cost of multicast transmissions," *J. Comput. Syst. Sci.*, vol. 63, no. 1, pp. 21–41, 2001.

[22] N. Nisan and A. Ronen, "Algorithmic mechanism design," *Games Econ. Behav.*, vol. 35, no. 1–2, pp. 166–196, 2001.

[23] V. Bilò, A. Fanelli, M. Flammini, G. Melideo, and L. Moscardelli, "Designing Fast Converging Cost Sharing Methods for Multicast Transmissions," *Theory Comput. Syst.*, vol. 47, no. 2, pp. 507–530, 2010.

[24] D. Braess, "Über ein Paradoxon aus der Verkehrsplanung," *Unternehmensforschung*, vol. 12, no. 1, pp. 258–268, 1968.

[25] J. D. Murchland, "Braess's paradox of traffic flow," *Transp. Res.*, vol. 4, no. 4, pp. 391–394, 1970.

[26] M. Frank, "The Braess paradox," *Math. Program.*, vol. 20, no. 1, pp. 283–302, 1981.

[27] K.-H. Park, "A Model for Detecting Braess Paradox in General Transportation Networks," *J. Korean Oper. Res. Manag. Sci. Soc.*, vol. 32, no. 4, pp. 19–35, 2007.

[28] K. Park, "Detecting Braess Paradox Based on Stable Dynamics in General Congested Transportation Networks," *Networks Spat. Econ.*, vol. 11, no. 2, pp. 207–232, 2011.

[29] Y. Xia and D. J. Hill, "Dynamic Braess's Paradox in Complex Communication Networks," *IEEE Trans. Circuits Syst. II Express Briefs*, vol. 60, no. 3, pp. 172–176, 2013.

[30] S. A. Bagloee, A. Ceder, M. Tavana, and C. Bozic, "A heuristic methodology to tackle the Braess Paradox detecting problem tailored for real road networks," *Transp. A Transp. Sci.*, vol. 10, no. 5, pp. 437–456, 2014.

[31] D. Acemoglu, R. Johari, and A. Ozdaglar, "Partially optimal routing," *IEEE J. Sel. areas Commun.*, vol. 25, no. 6, pp. 1148–1160, 2007.

[32] D. Acemoglu and A. Ozdaglar, "Competition and efficiency in congested markets," *Math. Oper. Res.*, vol. 32, no. 1, pp. 1–31, 2007.

[33] I. Ashlagi, D. Monderer, and M. Tennenholtz, "Two-terminal routing games with unknown active players," *Artif. Intell.*, vol. 173, no. 15, pp. 1441–1455, 2009.

[34] Y. Li and A. S. Morse, "The power allocation game on a network: a paradox," *IEEE/CAA J. Autom. Sin.*, vol. 5, no. 4, pp. 771–776, 2018.

[35] T. Roughgarden, "Designing networks for selfish users is hard," in *Proceedings 42nd IEEE Symposium on Foundations of Computer Science*, 2001, pp. 472–481.

[36] T. Roughgarden, "On the severity of Braess's Paradox: Designing networks for selfish users is hard," *J. Comput. Syst. Sci.*, vol. 72, no. 5, pp. 922–953, 2006.

[37] K. Sohn, "Multi-objective optimization of a road diet network design," *Transp. Res. part A policy Pract.*, vol. 45, no. 6, pp. 499–511, 2011.

[38] D. Fotakis, A. C. Kaporis, and P. G. Spirakis, "Efficient methods for selfish network design," *Theor. Comput. Sci.*, vol. 448, pp. 9–20, 2012.

[39] S. A. Bagloee, M. Sarvi, and M. Patriksson, "A Hybrid Branch-and-Bound and Benders Decomposition Algorithm for the Network Design Problem," *Comput. Civ. Infrastruct. Eng.*, vol. 32, no. 4, pp. 319–343, 2017.

[40] M. Mavronicolas, I. Milchtaich, B. Monien, and K. Tiemann, "Congestion Games with Player-Specific Constants," in *Mathematical Foundations of Computer Science 2007*, 2007, pp. 633–644.

[41] D. Acemoglu, A. Makhdoumi, A. Malekian, and A. Ozdaglar, "Informational Braess' Paradox: The Effect of Information on Traffic Congestion," *Oper. Res.*, vol. 66, no. 4, pp. 893–917, 2018.

[42] S. Das, E. Kamenica, and R. Mirka, "Reducing congestion through information design," in *2017 55th Annual Allerton Conference on Communication, Control, and Computing (Allerton)*, 2017, pp. 1279–1284.

[43] H. Tavafoghi and D. Teneketzis, "Informational incentives for congestion games," in *2017 55th Annual Allerton Conference on Communication, Control, and Computing (Allerton)*, 2017, pp. 1285–1292.

[44] M. Wu, J. Liu, and S. Amin, "Informational aspects in a class of bayesian congestion games," in *2017 American Control Conference (ACC)*, 2017, pp. 3650–3657.

[45] R. Steinberg and R. E. Stone, "The prevalence of paradoxes in transportation equilibrium problems," *Transp. Sci.*, vol. 22, no. 4, pp. 231–241, 1988.

[46] D. P. Bertsekas, *Nonlinear programming*. Athena Scientific, 2016.

[47] P. Yan, S. Q. Liu, C.-H. Yang, and M. Masoud, "A comparative study on three graph-based constructive algorithms for multi-stage scheduling with blocking," *J. Ind. Manag. Optim.*, vol. 15, no. 1, pp. 221–233, 2019.

[48] A. Moradi Afrapoli, M. Tabesh, and H. Askari-Nasab, "A multiple objective transportation problem approach to dynamic truck dispatching in surface mines," *Eur. J. Oper. Res.*, vol. 276, pp. 331–342, 2019.

[49] M. Masoud, E. Kozan, G. Kent, and S. Q. Liu, "A new constraint programming approach for optimising a coal rail system," *Optim. Lett.*, vol. 11, no. 4, pp. 725–738, 2017.

[50] S. Q. Liu, E. Kozan, M. Masoud, Y. Zhang, and F. T. S. Chan, "Job shop scheduling with a combination of four buffering constraints," *Int. J. Prod. Res.*, 2017.

[51] S. Q. Liu and E. Kozan, "A hybrid metaheuristic algorithm to optimise a real-world robotic cell," *Comput. Oper. Res.*, vol. 84, pp. 188–194, 2017.

[52] E. Kozan and S. Q. Liu, "A new open-pit multi-stage mine production timetabling model for drilling, blasting and excavating operations," *Min. Technol.*, vol. 125, no. 1, pp. 47–53, 2017.

[53] E. Kozan and S. Q. Liu, "An operational-level multi-stage mine production timetabling model for optimally synchronising drilling, blasting and excavating operations," *Int. J. Mining, Reclam. Environ.*, vol. 31, no. 7, pp. 457–474, 2017.

[54] W. Allihaibi, M. Masoud, M. Cholette, J. Burke, A. Karim, and S. Liu, "Optimising the service of emergency department in a hospital," in *22nd International Congress on Modelling and Simulation*, 2017.

[55] A. Mousavi, E. Kozan, and S. Q. Liu, "Open-pit block sequencing optimization: A mathematical model and solution technique," *Eng. Optim.*, vol. 48, no. 11, pp. 1932–1950, 2016.

[56] A. Mousavi, E. Kozan, and S. Q. Liu, "Comparative analysis of three metaheuristics for short-term open pit block sequencing," *J. Heuristics*, vol. 22, no. 3, pp. 301–329, 2016.

[57] M. Masoud, E. Kozan, G. Kent, and S. Q. Liu, "An integrated approach to optimise sugarcane rail operations," *Comput. Ind. Eng.*, vol. 98, pp. 211–220, 2016.

[58] M. Masoud, G. Kent, E. Kozan, and S. Q. Liu, "A new multi-objective model to optimize rail transport scheduler," *J. Transp. Technol.*, vol. 6, pp. 86–98, 2016.

[59] R. Matindi, P. Hobson, M. Masoud, G. Kent, and S. Q. Liu, "Developing a versatile simulation, scheduling and economic model framework for bioenergy production systems," *Int. J. Ind. Eng. Comput.*, vol. 10, no. 1, 2019.

[60] A. Mousavi, E. Kozan, and S. Q. Liu, "Chapter 5: Integrated approach to optimize open-pit mine block sequencing," in *Industrial Engineering Non-Traditional Applications in International Settings*, B. Bidanda, I. Sabuncuoglu, and B. Y. Kara, Eds. CRC Press, USA, 2014, pp. 83–98.

[61] S. Q. Liu and E. Kozan, "Scheduling trains with priorities: a no-wait blocking parallel-machine job-shop scheduling model," *Transp. Sci.*, vol. 45, no. 2, pp. 175–198, 2011.

[62] E. Kozan and S. Q. Liu, "Operations Research for Mining: A Classification and Literature Review," *ASOR Bull.*, vol. 30, no. 1, pp. 2–23, 2011.

[63] S. Q. Liu and E. Kozan, "Scheduling trains as a blocking parallel-machine job shop scheduling problem," *Comput. Oper. Res.*, vol. 36, pp. 2840–2852, 2009.

[64] S. Q. Liu and E. Kozan, "An Interactive Planning and Scheduling Framework for Optimising Pits-to-Crushers Operations," *Ind. Eng. Manag. Syst.*, vol. 11, no. 1, pp. 94–102, 2012.

[65] S. Q. Liu and E. Kozan, "Integration of mathematical models for ore mining industry," *Int. J. Syst. Sci. Oper. Logist.*, vol. 6, pp. 55–68, 2019.

[66] W. Li *et al.*, "Innovation-driven industrial green development: The moderating role of regional factors," *J. Clean. Prod.*, vol. 222, pp. 344–354, 2019.

[67] P. Yan, S. Q. Liu, T. Sun, and K. Ma, "A dynamic scheduling approach for optimizing the material handling operations in a robotic cell," *Comput. Oper. Res.*, vol. 99, pp. 166–177, 2018.

[68] P. Yan, A. Che, E. Levner, and S. Q. Liu, "A heuristic for inserting randomly arriving jobs into an existing hoist schedule," *IEEE Trans. Autom. Sci. Eng.*, vol. 15, no. 3, pp. 1423–1430, 2018.

[69] R. Matindi, M. Masoud, P. Hobson, G. Kent, and S. Q. Liu, "Harvesting and transport operations to optimise biomass supply chain and industrial biorefinery processes," *Int. J. Ind. Eng. Comput.*, vol. 9, no. 3, pp. 265–288, 2018.

[70] S. Q. Liu, E. Kozan, M. Masoud, Y. Zhang, and F. T. S. Chan, "Job shop scheduling with a combination of four buffering constraints," *Int. J. Prod. Res.*, vol. 56, no. 9, pp. 3274–3293, 2018.

[71] P. Yan, A. Che, E. Levner, and S. Q. Liu, "A heuristic for inserting randomly arriving jobs into an existing hoist schedule," *IEEE Trans. Autom. Sci. Eng.*, pp. 1–8, 2017.

[72] M. Samà, A. D'Ariano, P. D'Ariano, and D. Pacciarelli, "Scheduling models for optimal aircraft traffic control at busy airports: Tardiness, priorities, equity and violations considerations," *Omega (United Kingdom)*, vol. 67, pp. 81–98, 2017.

[73] A. D'Ariano, D. Pacciarelli, and M. Pistelli, "Real-Time Scheduling of Aircraft Arrivals and Departures in a Terminal Maneuvering Area," *Networks*, vol. 65, no. 3, pp. 212–227, 2015.

[74] D. Bertsimas, G. Lulli, and A. Odoni, "An Integer Optimization Approach to Large-Scale Air Traffic Flow Management," *Oper. Res.*, vol. 59, no. 1, pp. 211–227, 2011.