On Solé and Planat Criterion for the Riemann Hypothesis

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Abstract

There are several statements equivalent to the famous Riemann hypothesis. In 2011, Solé and Planat stated that the Riemann hypothesis is true if and only if the inequality $\zeta(2) \cdot \prod_{q \leq q_k} (1 + \frac{1}{q}) > e^{\gamma} \cdot \log \theta(q_k)$ holds for all prime numbers $q_k > 3$, where $\theta(x)$ is the Chebyshev function, $\gamma \approx 0.57721$ is the Euler-Mascheroni constant, $\zeta(x)$ is the Riemann zeta function and log is the natural logarithm. In this note, using Solé and Planat criterion, we prove that the Riemann hypothesis is true.

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1 Introduction

The Riemann hypothesis is the assertion that all non-trivial zeros have real part $\frac{1}{2}$. It is considered by many to be the most important unsolved problem in pure mathematics. It was proposed by Bernhard Riemann (1859). The Riemann hypothesis belongs to the Hilbert's eighth problem on David Hilbert's list of twenty-three unsolved problems. This is one of the Clay Mathematics Institute's Millennium Prize Problems. In mathematics, the Chebyshev function $\theta(x)$ is given by

$$\theta(x) = \sum_{q \leq x} \log q$$

with the sum extending over all prime numbers q that are less than or equal to x, where log is the natural logarithm. Let's state a property for this function:

Proposition 1.1. *[6, pp. 1]. We have*

$$x \sim \theta(x), \quad (x \to \infty).$$

Leonhard Euler studied the following value of the Riemann zeta function (1734).

Proposition 1.2. *It is known that[1, (1) pp. 1070]:*

$$\zeta(2) = \prod_{k=1}^{\infty} \frac{q_k^2}{q_k^2 - 1} = \frac{\pi^2}{6},$$

where q_k is the kth prime number (We also use the notation q_n to denote the nth prime number).

Franz Mertens obtained some important results about the constants B and H (1874). We define $H = \gamma - B$ such that $B \approx 0.2614972128$ is the Meissel-Mertens constant and $\gamma \approx 0.57721$ is the Euler-Mascheroni constant [4, (17.) pp. 54].

Proposition 1.3. *We have [2, Lemma 2.1 (1) pp. 359]:*

$$\sum_{k=1}^{\infty} \left(\log(\frac{q_k}{q_k - 1}) - \frac{1}{q_k} \right) = \gamma - B = H,$$

where log *is the natural logarithm.*

In mathematics, $\Psi(n) = n \cdot \prod_{q \mid n} \left(1 + \frac{1}{q}\right)$ is called the Dedekind Ψ function, where $q \mid n$ means the prime q divides n. We say that $\mathsf{Dedekind}(q_n)$ holds provided that

$$\prod_{q \leq q_n} \left(1 + \frac{1}{q}\right) > \frac{e^{\gamma}}{\zeta(2)} \cdot \log \theta(q_n).$$

The constant $\gamma \approx 0.57721$ is the Euler-Mascheroni constant and $\zeta(x)$ is the Riemann zeta function. Next, we have Solé and Planat Theorem:

Proposition 1.4. $\mathsf{Dedekind}(q_n)$ *holds for all prime numbers $q_n > 3$ if and only if the Riemann hypothesis is true [7, Theorem 4.2 pp. 5].*

There are several unconditional results from the Riemann hypothesis.

Proposition 1.5. *Unconditionally on Riemann hypothesis, there are infinitely many prime numbers q_n such that* $\mathsf{Dedekind}(q_n)$ *holds [7, Theorem 4.1 pp. 5].*

The following property is based on natural logarithms:

Proposition 1.6. *[3, Theorem 1.1 (13) pp. 3]. For $x \geq 1$:*

$$\log\left(1 + \frac{1}{x}\right) > \frac{1}{x + 0.5}.$$

Putting all together yields a proof for the Riemann hypothesis using the Chebyshev function.

2 What if the Riemann hypothesis were false?

Several analogues of the Riemann hypothesis have already been proved. Many authors expect (or at least hope) that it is true. However, there are some implications in case of the Riemann hypothesis might be false.

Lemma 2.1. *If the Riemann hypothesis is false, then there are infinitely many prime numbers q_k for which* $\mathsf{Dedekind}(q_k)$ *fails (i.e.* $\mathsf{Dedekind}(q_k)$ *does not hold).*

Proof. The Riemann hypothesis is false, if there exists some natural number $x_0 \geq 5$ such that $g(x_0) > 1$ or equivalent $\log g(x_0) > 0$:

$$g(x) = \frac{e^{\gamma}}{\zeta(2)} \cdot \log \theta(x) \cdot \prod_{q \leq x} \left(1 + \frac{1}{q}\right)^{-1}.$$

We know the bound [7, Theorem 4.2 pp. 5]:

$$\log g(x) \geq \log f(x) - \frac{2}{x}$$

where f was introduced in the Nicolas paper [5, Theorem 3 pp. 376]:

$$f(x) = e^{\gamma} \cdot \log \theta(x) \cdot \prod_{q \leq x} \left(1 - \frac{1}{q}\right).$$

When the Riemann hypothesis is false, then there exists a real number $b < \frac{1}{2}$ for which there are infinitely many natural numbers x such that $\log f(x) = \Omega_{+}(x^{-b})$ [5, Theorem 3 (c) pp. 376]. According to the Hardy and Littlewood definition, this would mean that

$$\exists k > 0, \forall y_0 \in \mathbb{N}, \exists y \in \mathbb{N}\ (y > y_0)\colon\ \log f(y) \geq k \cdot y^{-b}.$$

That inequality is equivalent to $\log f(y) \geq \left(k \cdot y^{-b} \cdot \sqrt{y}\right) \cdot \frac{1}{\sqrt{y}}$, but we note that

$$\lim_{y \to \infty} \left(k \cdot y^{-b} \cdot \sqrt{y}\right) = \infty$$

for every possible positive value of k when $b < \frac{1}{2}$. In this way, this implies that

$$\forall y_0 \in \mathbb{N}, \exists y \in \mathbb{N}\ (y > y_0)\colon\ \log f(y) \geq \frac{1}{\sqrt{y}}.$$

Hence, if the Riemann hypothesis is false, then there are infinitely many natural numbers x such that $\log f(x) \geq \frac{1}{\sqrt{x}}$. Since $\frac{2}{x} = o(\frac{1}{\sqrt{x}})$, then it would be infinitely many natural numbers x_0 such that $\log g(x_0) > 0$. In addition, if $\log g(x_0) > 0$ for some natural number $x_0 \geq 5$, then $\log g(x_0) = \log g(q_k)$ where q_k is the greatest prime number such that $q_k \leq x_0$. Actually,

$$\prod_{q \leq x_0} \left(1 + \frac{1}{q}\right)^{-1} = \prod_{q \leq q_k} \left(1 + \frac{1}{q}\right)^{-1}$$

and

$$\theta(x_0) = \theta(q_k)$$

according to the definition of the Chebyshev function. □

3 Central Lemma

Lemma 3.1.

$$\sum_{k=1}^{\infty} \left(\frac{1}{q_k} - \log(1 + \frac{1}{q_k})\right) = \log(\zeta(2)) - H.$$

Proof. We obtain that

$$\begin{aligned}
\log(\zeta(2)) - H &= \log(\prod_{k=1}^{\infty} \frac{q_k^2}{q_k^2 - 1}) - H \\
&= \sum_{k=1}^{\infty} \left(\log(\frac{q_k^2}{(q_k^2 - 1)})\right) - H \\
&= \sum_{k=1}^{\infty} \left(\log(\frac{q_k^2}{(q_k - 1) \cdot (q_k + 1)})\right) - H \\
&= \sum_{k=1}^{\infty} \left(\log(\frac{q_k}{q_k - 1}) + \log(\frac{q_k}{q_k + 1})\right) - H \\
&= \sum_{k=1}^{\infty} \left(\log(\frac{q_k}{q_k - 1}) - \log(\frac{q_k + 1}{q_k})\right) - H \\
&= \sum_{k=1}^{\infty} \left(\log(\frac{q_k}{q_k - 1}) - \log(1 + \frac{1}{q_k})\right) - \sum_{k=1}^{\infty} \left(\log(\frac{q_k}{q_k - 1}) - \frac{1}{q_k}\right) \\
&= \sum_{k=1}^{\infty} \left(\log(\frac{q_k}{q_k - 1}) - \log(1 + \frac{1}{q_k}) - \log(\frac{q_k}{q_k - 1}) + \frac{1}{q_k}\right) \\
&= \sum_{k=1}^{\infty} \left(\frac{1}{q_k} - \log(1 + \frac{1}{q_k})\right)
\end{aligned}$$

by Propositions 1.2 and 1.3. □

4 A New Criterion

Theorem 4.1. $\mathsf{Dedekind}(q_n)$ *holds if and only if the inequality*

$$\sum_{k=1}^{\infty} \left(\frac{1}{q_k} - (\chi_{\{x:\ x>q_n\}}(q_k)) \cdot \log(1+\frac{1}{q_k}) \right) > B + \log\log\theta(q_n)$$

is satisfied for the prime number q_n*, where the set* $S = \{x : \ x > q_n\}$ *contains all the real numbers greater than* q_n *and* χ_S *is the characteristic function of the set* S *(This is the function defined by* $\chi_S(x) = 1$ *when* $x \in S$ *and* $\chi_S(x) = 0$ *otherwise).*

Proof. When $\mathsf{Dedekind}(q_n)$ holds, we apply the logarithm to the both sides of the inequality:

$$\log(\zeta(2)) + \sum_{q \leq q_n} \log(1+\frac{1}{q}) > \gamma + \log\log\theta(q_n)$$

$$\log(\zeta(2)) - H + \sum_{q \leq q_n} \log(1+\frac{1}{q}) > B + \log\log\theta(q_n)$$

$$\sum_{k=1}^{\infty} \left(\frac{1}{q_k} - \log(1+\frac{1}{q_k}) \right) + \sum_{q \leq q_n} \log(1+\frac{1}{q}) > B + \log\log\theta(q_n)$$

after of using the Lemma 3.1. Let's distribute the elements of the previous inequality to obtain that

$$\sum_{k=1}^{\infty} \left(\frac{1}{q_k} - (\chi_{\{x:\ x>q_n\}}(q_k)) \cdot \log(1+\frac{1}{q_k}) \right) > B + \log\log\theta(q_n)$$

when $\mathsf{Dedekind}(q_n)$ holds. The same happens in the reverse implication. □

5 The Main Insight

Theorem 5.1. *The Riemann hypothesis is true if the inequality*

$$\theta(q_{n+1}) \geq \theta(q_n)^{1+\frac{1}{q_{n+1}}}$$

is satisfied for all sufficiently large prime numbers q_n*.*

Proof. For large enough prime q_n, if $\mathsf{Dedekind}(q_{n+1})$ holds then

$$\sum_{k=1}^{\infty} \left(\frac{1}{q_k} - (\chi_{\{x:\ x>q_{n+1}\}}(q_k)) \cdot \log(1+\frac{1}{q_k}) \right) > B + \log\log\theta(q_{n+1})$$

by Theorem 4.1. That is equivalent to

$$\sum_{k=1}^{\infty}\left(\frac{1}{q_k} - (\chi_{\{x:\ x>q_n\}}(q_k)) \cdot \log(1+\frac{1}{q_k})\right)$$
$$> B + \log\log\theta(q_{n+1}) - \log(1+\frac{1}{q_{n+1}})$$

after subtracting the value of $\log(1+\frac{1}{q_{n+1}})$ to the both sides. Thus,

$$\sum_{k=1}^{\infty}\left(\frac{1}{q_k} - (\chi_{\{x:\ x>q_n\}}(q_k)) \cdot \log(1+\frac{1}{q_k})\right)$$
$$> B + \log\log\theta(q_n) + \left(\log\log\theta(q_{n+1}) - \log\log\theta(q_n) - \log(1+\frac{1}{q_{n+1}})\right)$$

since $\log\log\theta(q_n) - \log\log\theta(q_n) = 0$. If we obtain that

$$\left(\log\log\theta(q_{n+1}) - \log\log\theta(q_n) - \log(1+\frac{1}{q_{n+1}})\right) \geq 0$$

then

$$\sum_{k=1}^{\infty}\left(\frac{1}{q_k} - (\chi_{\{x:\ x>q_n\}}(q_k)) \cdot \log(1+\frac{1}{q_k})\right) > B + \log\log\theta(q_n)$$

which means that $\mathsf{Dedekind}(q_n)$ holds by Theorem 4.1. Hence, it is enough to guarantee that

$$\left(\log\log\theta(q_{n+1}) - \log\log\theta(q_n) - \log(1+\frac{1}{q_{n+1}})\right) \geq 0$$

to assure that $\mathsf{Dedekind}(q_n)$ holds for a large enough prime number q_n when $\mathsf{Dedekind}(q_{n+1})$ holds. Since there are infinitely many prime numbers $q_{n+1} > 5$ such that $\mathsf{Dedekind}(q_{n+1})$ holds, then we can guarantee that $\mathsf{Dedekind}(q_n)$ holds as well when

$$\left(\log\log\theta(q_{n+1}) - \log\log\theta(q_n) - \log(1+\frac{1}{q_{n+1}})\right) \geq 0$$

by Proposition 1.5. Furthermore, if the inequality

$$\left(\log\log\theta(q_{n+1}) - \log\log\theta(q_n) - \log(1+\frac{1}{q_{n+1}})\right) \geq 0$$

holds for all pairs (q_n, q_{n+1}) of consecutive large enough primes such that $q_n < q_{n+1}$, then we can confirm that $\mathsf{Dedekind}(q_n)$ always holds for all large enough prime numbers q_n by Theorem 4.1. As result, if the inequality

$$\left(\log\log\theta(q_{n+1}) - \log\log\theta(q_n) - \log(1+\frac{1}{q_{n+1}})\right) \geq 0$$

is satisfied for all sufficiently large prime numbers q_n, then there won't exist infinitely many prime numbers q_k such that $\mathsf{Dedekind}(q_k)$ fails and so, the Riemann hypothesis must be true by Lemma 2.1. Let's distribute the elements of the previous inequality to obtain that

$$\theta(q_{n+1}) \geq \theta(q_n)^{1+\frac{1}{q_{n+1}}}.$$

□

6 The Main Theorem

Theorem 6.1. *The Riemann hypothesis is true.*

Proof. The Riemann hypothesis is true when

$$\theta(q_{n+1}) \geq \theta(q_n)^{1+\frac{1}{q_{n+1}}}$$

is satisfied for all sufficiently large prime numbers q_n because of the Theorem 5.1. That is the same as

$$\left(\frac{\theta(q_{n+1})}{\theta(q_n)}\right)^{q_{n+1}} \geq \theta(q_n).$$

We know that

$$q_{n+1} \cdot \log\left(\frac{\theta(q_{n+1})}{\theta(q_n)}\right) \geq \log\theta(q_n).$$

By definition of the Chebyshev function, we have

$$\log\left(\frac{\theta(q_{n+1})}{\theta(q_n)}\right) = \log\left(1 + \frac{\log q_{n+1}}{\theta(q_n)}\right).$$

In addition, we have

$$\log\left(1 + \frac{\log q_{n+1}}{\theta(q_n)}\right) > \frac{\log q_{n+1}}{\theta(q_n) + 0.5 \cdot \log q_{n+1}} = \frac{\log q_{n+1}}{\theta(q_{n+1}) - 0.5 \cdot \log q_{n+1}}$$

since $\frac{\theta(q_n)}{\log q_{n+1}} \geq 1$ by Proposition 1.6. Hence, it is enough to show that

$$q_{n+1} \cdot \frac{\log q_{n+1}}{\theta(q_{n+1}) - 0.5 \cdot \log q_{n+1}} > \log\theta(q_n)$$

which is

$$q_{n+1} \cdot \log q_{n+1} > (\theta(q_{n+1}) - 0.5 \cdot \log q_{n+1}) \cdot \log\theta(q_n).$$

However, we know that

$$q_{n+1} \sim \theta(q_{n+1}) > (\theta(q_{n+1}) - 0.5 \cdot \log q_{n+1})$$

and

$$\log q_{n+1} \sim \log \theta(q_{n+1}) > \log \theta(q_n)$$

as n tends to infinity by Proposition 1.1. Consequently, the inequality

$$\theta(q_{n+1}) \geq \theta(q_n)^{1+\frac{1}{q_{n+1}}}$$

is satisfied for all sufficiently large prime numbers q_n and therefore, the Riemann hypothesis is true. $\square$

7 Conclusions

Practical uses of the Riemann hypothesis include many propositions that are known to be true under the Riemann hypothesis and some that can be shown to be equivalent to the Riemann hypothesis. Indeed, the Riemann hypothesis is closely related to various mathematical topics such as the distribution of primes, the growth of arithmetic functions, the Lindelöf hypothesis, the Large Prime Gap Conjecture, etc. Certainly, a proof of the Riemann hypothesis could spur considerable advances in many mathematical areas, such as number theory and pure mathematics in general.

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