EasyChair Preprint

№ 15280

# Deloitte and Stanford Consultants Partner to Investigate Quasiparticle Modes at Exotic Quantum Systems

Nuscha Fathianpour, Matthew Stephenson and Laurine Mbou



October 22, 2024

# Deloitte and Stanford Consultants Partner to Investigate Quasiparticle Modes at Exotic Quantum Systems

Nuscha Fathianpour
*Deloitte University (Risk Advisory)*[*]

Matthew James Stephenson
*Stanford University (ID: 06481437)*[†]

Laurine Mbou[‡]
*Deloitte University (Senior Consultant)*

Toy model at a gedanken calculation of low-temperature properties of a string-inspired gauge theory at finite density built at Deloitte day on September 25th 2024.

## I. THE PLAN

Recently, we've been interested at the bosonic quasiparticle excitations when a chemical potential for a particular $U(1)$ charge is turned on. The associated $U(1)$ symmetry acts as $\phi \to e^{i\alpha}\phi, \psi \to e^{i\alpha}\psi$ on one of the complex scalars $\phi$ and one of the fermions $\psi$. $\phi$ is a complex scalar, while $\psi$ is a Dirac fermion. It seems likely that the trickiest technical point in working out the spectrum may be in learning to handle Yukawa terms, which are of a rather exotic form. Let $\chi$ be a *Majorana* fermion. The type of Yukawa interaction we must deal with is, according to Chesler, Vuorinen and Matt (yes, the one at this very paper)

$$g \,\mathrm{Im}\, \phi\, \bar{\psi} P_+ \chi = \frac{ig}{2}\left(\phi \bar{\psi} P_+ \chi - \phi^\dagger \bar{\chi} P_- \psi\right) \tag{1}$$

where $P_+ = \frac{1}{2}(1+\gamma_5)$. This is invariant under $\phi \to e^{i\alpha}\phi, \psi \to e^{i\alpha}\psi$, as advertised. These Yukawa couplings are weird for a variety of reasons. For instance, if $\phi$ picks a vev it appears that $\chi$ and $\psi$ will mix, and I don't know what that means. (It may mean we can't borrow results from cond-mat literature on polarization tensors for the gluons, etc. But it's not like we'd ever want to either. If everyone was an experimentalist it would be terrible for us) We also have to worry about working with Majorana spinors as well as Dirac spinors, and have to understand the Feynman rules associated with that.

Don't worry, at these models we work our way toward the Yukawa step by step even though it's super cringe, computing quasiparticle spectra at a variety of toy models, adding complications step by step. Hopefully by the end of the process, the exotic Yukawa term no longer seems so exotic, and the general ideas of perturbative calculations of quasiparticle spectra are demystified.

The organization is: Sec. II sets out the conventions and reviews some QFT background. Sec. III works out the spectrum of the simplest interesting theory with a Yukawa, which has a single real scalar interacting with a Dirac fermion at finite chemical potential. Finally, in Sec. IV we work with a theory with a single *complex* scalar and two fermions, one with a chemical potential and one without, with condensation driven by a chemical potential for the scalar. By this step we will be experts, and everything will be trivial because we're geniuses.

If we can do all that, handling the Yukawas in the $\mathcal{N} = 2$ will be a piece of cake, but with more indices.

## II. PRELIMINARIES

### A. Conventions

We fix conventions to be those of the One True Field Theory Book for field theory connosieurs, by Tony Zee (second edition). We use the *mostly minus* metric, so that the action of a free scalar is

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 \tag{2}$$

[*] nfathianpour@deloitte.com
[†] matthew.stephenson@stanford.edu
[‡] laurine.mbou@hec.ca

while the action of a free fermion is

$$\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ - m)\psi \tag{3}$$

Our gamma matrix convention is $\gamma^0 = 1 \otimes \sigma_3, \gamma^i = \sigma^i \otimes i\sigma_2$, and $\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 = 1 \otimes \sigma_1$, so that $\gamma^\mu, \gamma^5$ are Hermitian. We also define $\sigma^{\mu\nu} \equiv \frac{i}{2}[\gamma^\mu, \gamma^\nu]$, so that $\sigma^{0i} = i\sigma_1 \otimes \sigma_i, \sigma^{ij} = \epsilon^{ijk} 1 \otimes \sigma_k$. Parity is represented by $P\psi \equiv \gamma^0\psi$, charge conjugation is realized as $\psi \to \psi_c \equiv C\gamma^0\psi^*$, and with our conventions we can choose $C = \gamma_2$, which satisfies $C^{-1}\gamma^\mu C = -(\gamma^\mu)^T$, and time reversal acts as $\psi \to \psi_t \equiv U\psi^*, U = -i\gamma^1\gamma^3 = \sigma_2 \otimes 1$.

The free scalar Feynman propagator for a particle moving with 4-momentum $p^\mu$ is

$$G(p^\mu) = \frac{i}{p^2 - m^2 + i\epsilon} \tag{4}$$

while for a free fermion the Feynman propagator is

$$S(p)_{\alpha\beta} = \frac{i(p\!\!\!/ + m)_{\alpha\beta}}{p^2 - m^2 + i\epsilon} \tag{5}$$

and $p\!\!\!/ = \gamma^\mu p_\mu$, $\{\gamma^\mu, \gamma^\nu\}_{\alpha\beta} = 2\eta^{\mu\nu} 1_{\alpha\beta}$, and $\alpha, \beta$ are spinor indices. For future use, the relation to the position-space propagator is

$$S^F(x-y)_{\alpha\beta} \equiv \langle T\psi(x)_\alpha \bar{\psi}(y)_\beta \rangle = \begin{cases} \langle \psi(x)_\alpha \bar{\psi}(y)_\beta \rangle & \text{for } x^0 > y^0 \\ -\langle \bar{\psi}(y)_\beta \psi(x)_\alpha \rangle & \text{for } x^0 < y^0 \end{cases} = \int \frac{d^4p}{(2\pi)^4} S(p)_{\alpha\beta} e^{-ip\cdot(x-y)} \tag{6}$$

The Feynman rules for fermions are

1. Each line is associated with $S^F$
2. Each interaction vertex coming from $\mathcal{L} = +g\phi\bar{\psi}\psi$ gives the coupling factor $ig$ in the Feynman diagram, with momentum conservation enforced at the vertices via delta functions times $(2\pi)^4$
3. Each fermion loop means we multiply the amplitude by $-1$.

### B. Finite density

At finite density, the scalar propagator is unchanged except that $p_0 \to p_0 - \mu$. However, the fermion propagator changes in a much more subtle way, since we must shift $p_0 \to p_0 - \mu$, and at the same time do the replacement $i\epsilon \to i\epsilon \operatorname{sgn} p_0$. The change is necessary to get a discontinuity in the particle distribution at the fermi surface, see the discussion in Landau and Lifshitz,[**?** ] in their §7 − §10. What follows is a summary of the relevant part of their discussion, translated to relativistic language.

The particle density matrix can be written as

$$\rho(\mathbf{x}, \mathbf{y}) \equiv \langle \bar{\psi}(t, \mathbf{y})\gamma_0\psi(t, \mathbf{x}) \rangle = \langle \bar{\psi}(t, \mathbf{y})_\beta \psi(t, \mathbf{x})_\alpha \gamma_0^{\beta\alpha} \rangle \tag{7}$$

The left-hand side is independent of $t$ since since we're suppose to be in equilibrium. We want to express this in terms of $S^F$, which means writing

$$\rho(\mathbf{x}, \mathbf{y}) = -S^F(t, \mathbf{x}; t+\epsilon, \mathbf{y})_{\alpha\beta}\gamma_0^{\alpha\beta} = -\operatorname{tr} S^F(t, \mathbf{x}; t+\epsilon, \mathbf{y})\gamma_0 \tag{8}$$

where we assume that $\epsilon > 0$ is infinitesimal. We can't just set $\epsilon = 0$ from the start because we want to work in terms of time-ordered Green's functions. We assume translational invariance, which means $\rho$ and $S^F$ only depend on $\mathbf{x}, \mathbf{y}$ through $\mathbf{r} \equiv \mathbf{x} - \mathbf{y}$. The particle momentum distribution can then be identified as

$$\begin{aligned} n(p) &= -\lim_{\epsilon \to 0^+} \int d^3\mathbf{r}\, \rho(t = -\epsilon; \mathbf{r}) e^{-i\mathbf{p}\cdot(\mathbf{r})} = -\lim_{\epsilon \to 0^+} \int \operatorname{tr} S^F(t = -\epsilon, \mathbf{r}) e^{-i\mathbf{p}\cdot\mathbf{r}} \\ &= -\lim_{\epsilon \to 0^+} \int \frac{dk_0}{2\pi} e^{ik_0\epsilon} \operatorname{tr} S(k_0, p)\gamma_0 \end{aligned} \tag{9}$$

To get from the first line to the second, we plugged in the Fourier transform of $S^F$ and evaluated the spatial integral using the resulting Dirac delta functions. Since there is a factor of $e^{ik_0\epsilon}$ with $\epsilon > 0$ in the integrand, we can close

the $k_0$ contour by an (infinite) semicircle in the upper half plane, since in the upper half plane the $e^{-ik_0\epsilon}$ factor has an exponentially falling factor. It does not make sense to close the contour by an infinite semicircle in the lower-half plane, since there the integrand picks up an exponentially diverging factor and would not really exist.

So taking the $k_0$ contour to close in the upper-half-plane, we can evaluate $n(p)$. Define the energy of a free relativistic particle as $E_p = \sqrt{p^2+m^2} > 0$, and set $\mu > 0$ to be specific. With the funky $i\epsilon$ prescription, the poles in the integrand in Eq. (9) are at $(k_0)_\pm = \mu \pm E_{\mathbf{p}} - i\epsilon$ if $\mu < E_p$, or at $(k_0)_\pm = \mu \pm E_{\mathbf{p}} \mp i\epsilon$ if $\mu > E_{\mathbf{p}}$. Closing the contour into the upper-half plane, for $\mu < E_{\mathbf{p}}$, we get zero. But this means $n(p) = 0$ for $\mu < E_{\mathbf{p}} \Rightarrow \mu^2 - \mu^2 < p^2 \Leftrightarrow p > p_F$. Meanwhile for $\mu > E_{\mathbf{p}} \Leftrightarrow p < p_F$ we pick up the pole at $(k_0)_-$. Noting the factor of 4 coming from the trace, I find that the residue at the pole is 2. Hence $n(p) = 2$ for $\mu < E_{\mathbf{p}}$. Summarizing, we have shown that

$$n(p) = \begin{cases} 2 & \text{for } p < p_F \\ 0 & \text{for } p > p_F \end{cases} \tag{10}$$

This is precisely what we wanted to see. (The factor of 2 reflects the fact that the fermions can have either spin up or spin down.)

### C. Regularization

We will need to regularize things, and we will always use dimensional regularization. The relevant formula for a *Euclidean* integral $I$ is

$$I = \int \frac{d^d q}{(2\pi)^d} \frac{(q^2)^m}{(q^2+\Delta)^p} = \frac{\Delta^{\frac{d}{2}-p+m}}{(4\pi)^{d/2}} \frac{\Gamma(p+m-d/2)}{\Gamma(p+m)}. \tag{11}$$

We will usually plug $d = 3$ into the integral above, since we'll do many of the $k_0$ integrals we encounter via contour integration[1].

Also, the Feynman parameter trick we will need is

$$\frac{1}{A^\alpha B^\beta} = (-1+\alpha+\beta) \int_0^1 dx \frac{x^{\alpha-1} x^{\beta-1}}{[Ax + B(1-x)]^{\alpha+\beta}}. \tag{12}$$

but see [**?** ] for potential important subtleties in the case of interest!

## III. SINGLE REAL SCALAR WITH A SINGLE FERMION, NO CONDENSATION

Suppose we've a Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu - \frac{1}{2}M^2\phi^2 - \lambda\phi^4 + \bar\psi(i\partial\!\!\!/ - m + \mu\gamma_0)\psi - g\phi\bar\psi\gamma_5\psi\mathcal{L}_{ct} \tag{13}$$

we take the dimensionless couplings $g, \lambda$ to be small $g, \lambda \ll 1$, and $m^2 > 0$, and $\mathcal{L}_{ct}$ is the counter-term Lagrangian. To keep things simple for now, we also assume the hierarchy $\lambda \ll g^2$ [2]. The reason for the $\gamma_5$ in the Yukawa term is to make the action invariant under parity if $\phi$ is a pseudoscalar, so $\phi, \phi^3$ terms cannot be radiatively generated. This should make things simpler later. The chemical potential breaks $\mathcal{C}$ and $\mathcal{T}$, as well as Lorentz invariance, but not rotational or translational invariance in the spatial directions.

We want to compute the one-loop correction to the scalar propagator. To do this we have to resum a chain of bubble diagrams coming from the Yukawa interaction, the prototype of which is shown in Fig. 1. In general, the scalar will have 3-momentum $\mathbf{p}$ and energy $p_0$. If we were interested in a Lorentz-invariant system ($\mu = 0$), we could compute this diagram in the rest frame of the scalar, and set $\mathbf{p} = 0, p_0 = m$ without loss of generality. One way to see this is that Lorentz symmetry constrains the dispersion relation for a scalar to take the form $p_0 = p^2 + m^2_{\text{phys}}$, so

---

[1] Euclidean 'Lorentz' $SO(4)$ symmetry, which is used in particle theory textbooks to do loop integrals after massaging them into $SO(4)$ symmetric forms using Feynman parametrization, is broken by the chemical potential down to rotational $SO(3)$. Hence one has to treat the time and space directions separately; the 'time' integral is usually done via contour integration, while for the space integral you regulate according to taste and the phase of the moon. Dim reg is almost always the most efficient.

[2] In the SUSY system $\lambda \sim g^2$. The reason I suggest neglecting $\phi$ loops for now is that the structure of the $\phi$ loop corrections in the condensed phase with chemical potentials should be quite different than the structure without condensation and Lorentz breaking, and we might as well put off dealing with the complications until we need them.

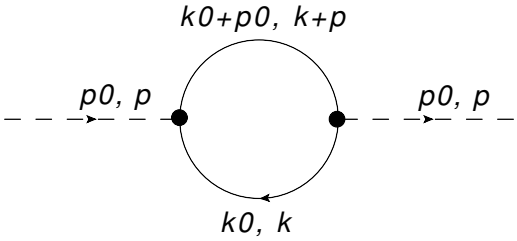


FIG. 1. Yukawa correction to scalar propagator

that the dependence on $p$ is fixed, and the effect of interactions is only to produce shifts in $m_{\rm phys}$ relative to $m_{\rm bare}$, and the calculation with $\mathbf{p}=0, p_0=m$ would give the one-loop shift of $m$.

Once Lorentz invariance is broken, life has more flavor. The interactions can change the functional form of the dispersion relation. The remaining symmetry is rotations and time translations, which imply (a) energy is conserved, and (b) the dependence on $\mathbf{p}$ must be through $p=|\mathbf{p}|$. Because of this, in evaluating the diagram in Fig. 1 we will take the scalar to have $p_\mu=(p_0,\mathbf{p})$. We do *not* impose any relation between $p_0$ and $\mathbf{p}$, since to do that we would need to know the dispersion relation of the scalar field, and that is the thing that we willl be calculating.

The propagator for $\phi$ will take the form

$$G_{\phi,\text{1-loop}}=\frac{i}{p_0^2-p^2-M^2-F(p_0,p)+i\epsilon} \tag{14}$$

where we're trying to compute $F$ at one loop. The quasiparticle dispersion relations are determined by the poles $p_0=\epsilon(p)$ in the propagator and their locations controlled by the form of $F$. Near a pole $p_0=\epsilon(p)$, the propagator takes the form

$$G\to\frac{iZ(p)}{p_0-\epsilon(p)+i\epsilon} \tag{15}$$

and in general the residue $Z\neq 1$. The residue can depend on $p$, but not on $p_0$. (Near the pole any $p_0$'s on top can be replaced by $\epsilon(p)$.) The residue tells us how e.g. canonically-normalized quasiparticle fields fit inside $\phi$, so that in the notation above $\phi(p,p_0)=Z(p)^{1/2}\varphi(p,p_0)$, and $\varphi$ is the quasiparticle field with a kinetic term set by $p_0=\omega_{\mathbf{p}}$. In the case we are studying here, my naive expectation is that for $\mu>m$ and $p\ll k_F=\sqrt{\mu^2-m^2}$, $F(p_0,p\to 0)\to g^2\mu k_F$, with a trivial residue, so that the expectation is that $\omega_{\mathbf{p}}=\sqrt{p^2+M^2+g^2\mu k_F}$.

The one-loop resummed expressions for $G$ can be extracted from the amplitude associated to Fig. 1. To try to get the signs right, let's first 'derive' the tree-level inverse propagator. Viewing e.g. the $-\frac{1}{2}m^2\phi^2\in\mathcal{L}$ term as an 'interaction', note that the Feyman rule associated to this vertex is $-im$. Then an insertion of $m$ at tree level gives the 'amplitude' $-im$. Similarly, $\frac{1}{2}(\partial\phi)^2$ can be associated to a Feyman rule $+ip_\mu p^\mu$. The tree-level amplitude is $i(p^2-m^2)$, which is $-1$ times the inverse propagator as defined by Zee. **I don't understand this $-1$ factor.**

Consider now the one-loop contribution to the amplitude, which we will write as $iF(p_0,p)$ so that $F(p_0,p)$ is easy to pull into the inverse propagator. Using our Feynman rules, we get

$$\begin{aligned}iF(p_0,p)=-1\times(-ig)^2i^2\int\frac{dk_0}{2\pi}\frac{d^3\mathbf{k}}{(2\pi)^3}\Big(&\text{tr}\left\{\left[(\gamma^0(k_0-\mu)-\gamma^i\mathbf{k}_i+m\right]\gamma^5\left[(\gamma^0(p_0+k_0-\mu)-\gamma^i(\mathbf{p}+\mathbf{k})_i+m\right]\gamma^5\right\}\\&\times\frac{1}{[(k_0-\mu)^2-k^2-m^2+i\epsilon\,\text{sgn}(k_0)]\,[(p_0+k_0-\mu)^2-(\mathbf{k}+\mathbf{p})^2-m^2+i\epsilon\,\text{sgn}(p_0+l_0)]}\Big)\end{aligned} \tag{16}$$

Note the funny $i\epsilon$ prescription in the propagators compared to what we are used to when $\mu=0$.

The numerator can be simplified to

$$\text{tr}\left\{\left[(\gamma^0(k_0-\mu)-\gamma^i\mathbf{k}_i+m\right]\gamma^5\left[(\gamma^0(p_0+k_0-\mu)-\gamma^i(\mathbf{k}+\mathbf{p})_i+m\right]\gamma^5\right\} \tag{17}$$

$$=\text{tr}\left\{\left[(-\gamma^0(k_0-\mu)+\gamma^i\mathbf{k}_i+m\right]\left[(\gamma^0(p_0+k_0-\mu)-\gamma^i(\mathbf{k}+\mathbf{p})_i+m\right]\right\} \tag{18}$$

$$=4\left(-(k_0-\mu)(p_0+k_0-\mu)+\mathbf{k}\cdot(\mathbf{k}+\mathbf{p})+m^2\right)\equiv N_5(p_0,\mathbf{p};k_0,\mathbf{k}). \tag{19}$$

Note that different couplings to $\phi$ only change the numerator, so that

$$\phi\bar{\psi}\psi \to 4\left(+(k_0-\mu)(p_0+k_0-\mu) - \mathbf{k}\cdot(\mathbf{k}+\mathbf{p}) + m^2\right) = N(p_0,\mathbf{p};k_0,\mathbf{k}) \tag{20}$$

$$\phi\bar{\psi}\gamma_0\psi \to 4\left(+(k_0-\mu)(p_0+k_0-\mu) + \mathbf{k}\cdot(\mathbf{k}+\mathbf{p}) + m^2\right) = N_0(p_0,\mathbf{p};k_0,\mathbf{k}) \tag{21}$$

$$\phi\bar{\psi}\gamma_5\psi \to 4\left(-(k_0-\mu)(p_0+k_0-\mu) - \mathbf{k}\cdot(\mathbf{k}+\mathbf{p}) + m^2\right) = N_5(p_0,\mathbf{p};k_0,\mathbf{k}) \tag{22}$$

$$\tag{23}$$

This is neat, since for instance working with $N_0$ lets us read off the contribution of a finite density of Dirac fermions to the 00 and $0i$ components of the gauge boson polarization tensor. The reason is that the coupling to a gauge field looks like $A^\mu\bar{\psi}\gamma_\mu\psi$...

Anyway, with the $N_i$ defined above, we decorate our function $F$ with an index '5' to remind ourselves that it encodes only one of several interesting renormalizations of a 'scalar' propagator. Then the integral we have to do is

$$F_5(p_0,p) = 4g^2 i\int\frac{dk_0}{2\pi}\frac{d^3\mathbf{k}}{(2\pi)^3}\Big(\frac{N_5(p_0,\mathbf{p};k_0,\mathbf{k})}{[(k_0-\mu)^2-\mathbf{k}^2-m^2+i\epsilon\,\mathrm{sgn}(k_0)]} \times\frac{1}{[(k_0+p_0-\mu)^2-(\mathbf{p}+\mathbf{k})^2-m^2+i\epsilon\,\mathrm{sgn}(p_0+k_0)]}\Big) \tag{24}$$

As a check on our methods and a warm-up for handling the gauge sector, we will also consider $F_0$ later, defined in the obvious way using $N_0$.

The game now is to evaluate the $k_0$ integral using contour integration, and then do the $d^3\mathbf{k}$ integral using Feynman parameters and dimensional regularization. The analytic structure of the integral is somewhat complicated. Define $k_F = \sqrt{\mu^2-m^2}$ [3].

The poles in the $k_0$ complex plane in the propagators appearing in the loop integral are at $k_\pm = \mu \pm \sqrt{k^2+m^2} \pm i\epsilon\,\mathrm{sgn}(k_\pm)$ and $k'_\pm = \mu - p_0 \pm \sqrt{k^2+m^2} \pm i\epsilon\,\mathrm{sgn}(k'_\pm + p_0)$. Define $E_\mathbf{p} = \sqrt{|\mathbf{p}|^2+m^2}$. Then some short algebra that is basically not worth typing shows that the analytic structure is different in four regions in $\mathbf{k}$ space:

1. $\theta(E_\mathbf{k}-\mu)\theta(E_{\mathbf{k}+\mathbf{p}}-\mu)$: all four poles below the real axis.
2. $\theta(E_\mathbf{k}-\mu)\theta(\mu-E_{\mathbf{k}+\mathbf{p}})$: $l_+, l'_+$ below, $l_-$ below, $l'_-$ above.
3. $\theta(\mu-E_\mathbf{k})\theta(E_{\mathbf{k}+\mathbf{p}}-\mu)$: $l_+, l'_+$ below, $l_-$ above, $l'_-$ below.
4. $\theta(\mu-E_\mathbf{k})\theta(\mu-E_{\mathbf{k}+\mathbf{p}})$: $l_+, l'_+$ below, $l_-, l'_-$ above.

The integrand goes like $1/k_0^2$ at large $k_0$, so we can close the $k_0$ contour by big arcs at $k_0 \to \pm\infty$ as we like. In the first case, we close the contour in the upper half plane, and hence get zero. In the other cases, we can again choose to close the contour in the upper half plane, and pick up one, one, and two poles in the three remaining cases, respectively. Each residue which enters the $d^3\mathbf{k}$ integral must get weighed by the appropriate product of step functions. Note also that we're getting boned by relativity here: in the non-relativistic limit, there would be only two relevant pole contributions, not four[4]. The extras are presumably the anti-particle contributions.

The residue at $k_-$ is

$$\mathrm{Res}\,(k_-)[p_0,\mathbf{p};\mathbf{k}]_5 = \frac{N_5(p_0,\mathbf{p};k_-,\mathbf{k})}{(k_- - k_+)(k_- - k'_+)(k_- - k'_+)} \tag{25}$$

while the residue at $k'_-$

$$\mathrm{Res}\,(k'_-)[p_0,\mathbf{p};\mathbf{k}]_5 = \frac{N_5(p_0,\mathbf{p};k'_-,\mathbf{k})}{(k'_- - k_+)(k'_- - k_-)(k'_- - k'_+)} \tag{26}$$

and hence the integral in $F$ becomes

$$F_5(p_0,p) = -4g^2\int\frac{d^3\mathbf{k}}{(2\pi)^3}\left[\theta(E_\mathbf{k}-\mu)\theta(\mu-E_{\mathbf{k}+\mathbf{p}})\,\mathrm{Res}\,(k'_-)_5 + \theta(\mu-E_\mathbf{k})\theta(E_{\mathbf{k}+\mathbf{p}}-\mu)\,\mathrm{Res}\,(k_-)_5 + \theta(\mu-E_\mathbf{k})\theta(\mu-E_{\mathbf{k}+\mathbf{p}})\left\{\mathrm{Res}\,(k'_-)_5 + \mathrm{Res}\,(k_-)_5\right\}\right], \tag{27}$$

---

[3] Note that the non-relativistic relation between the 'non-relativistic' chemical potential $\tilde{\mu}$ and $\tilde{k}_F$ fall out from this relativistic formula upon writing $\mu = m + \tilde{\mu}$ and expanding in $\tilde{\mu}/m \ll 1$, yielding $\tilde{k}_F = \sqrt{2m\tilde{\mu}}$

[4] See Fetter and Walecka, p 158, Eq 12.29 for the non-relativistic analogue of our Eq. (16).

with the $2\pi i$ from the contour integrals canceling against the original factor of $i$ to give the overall sign above.

Next, change variables in the terms in the integral involving $\mathrm{Res}\,(k'_-)$ to $\mathbf{k}'$, with the relation $\mathbf{k}' = -\mathbf{k} - \mathbf{p}$. Since $E_{\mathbf{p}}$ has the property that $E_{\mathbf{p}} = E_{-\mathbf{p}}$, one can see that $E_{\mathbf{p}+\mathbf{k}} = E_{\mathbf{k}'}$, $E_{\mathbf{k}} = E_{\mathbf{p}+\mathbf{k}'}$. This tells us that $\theta(E_{\mathbf{k}} - \mu)\theta(\mu - E_{\mathbf{k}+\mathbf{p}})$ turns into $\theta(\mu - E_{\mathbf{k}'})\theta(E_{\mathbf{k}'+\mathbf{p}} - \mu)$, and $\mathrm{Res}\,(k'_-)(p_0, \mathbf{p}; \mathbf{k})$ changes as $\mathrm{Res}\,(k'_-)(p_0, \mathbf{p}; \mathbf{k}) = \mathrm{Res}\,(k'_-)(-p_0, \mathbf{p}; \mathbf{k}')$. Dropping the primes on $\mathbf{k}$ from now on, this motivates defining

$$\begin{aligned} f_5(p_0, \mathbf{p}; \mathbf{k}) &= \mathrm{Res}\,(k_-)[p_0, \mathbf{p}; \mathbf{k}]_5 + \mathrm{Res}\,(k_-)[-p_0, \mathbf{p}; \mathbf{k}]_5 \qquad (28) \\ &= -\frac{-kp\cos(\theta)\left(2d + p^2\right) + \left(kp\cos(\theta) + 2\left(k^2 + m^2\right)\right)p_0^2}{\sqrt{k^2+m^2}\left(\left(2kp\cos(\theta) + p^2\right)^2 - 2\left(2\left(kp\cos(\theta) + k^2 + m^2\right) + p^2\right)p_0^2 + p_0^4\right)} \\ f_0(p_0, \mathbf{p}; \mathbf{k}) &= \mathrm{Res}\,(k_-)[p_0, \mathbf{p}; \mathbf{k}]_0 + \mathrm{Res}\,(k_-)[-p_0, \mathbf{p}; \mathbf{k}]_0 \qquad (29) \\ &= -\frac{-2k^2p^2\cos^2(\theta) - 2\left(k^2 + m^2\right)p^2 - kp\cos(\theta)\left(4k^2 + 4m^2 + p^2 - p_0^2\right)}{\sqrt{k^2+m^2}\left(\left(2kp\cos(\theta) + p^2\right)^2 - 2\left(2\left(kp\cos(\theta) + k^2 + m^2\right) + p^2\right)p_0^2 + p_0^4\right)} \end{aligned}$$

which are the new integrands in $F_5$ and $F_0$ respectively. In writing the above I dropped all imaginary parts, since for now we're only trying to get the real part of $F_{5,0}$. Finally, we observe that

$$\theta(\mu - E_{\mathbf{k}})\theta(E_{\mathbf{k}+\mathbf{p}} - \mu) + \theta(\mu - E_{\mathbf{k}})\theta(\mu - E_{\mathbf{k}+\mathbf{p}}) = \theta(\mu - E_{\mathbf{k}}). \qquad (30)$$

Hence we can rewrite the integral defining $F(p_0, p)$ in the simpler form

$$F_{5,0}(p_0, p) = -4g^2 \int \frac{d^3\mathbf{k}}{(2\pi)^3}\theta(\mu - E_{\mathbf{k}})\left[f_{5,0}(p_0, \mathbf{p}; \mathbf{k})\right]. \qquad (31)$$

The step function is non-zero so long as we have the conditions $\mu > m$ and $k < k_F$. Recall that $f(p_0, \mathbf{p}; \mathbf{k})$ has imaginary parts; so does the function we are looking for. These imaginary parts encode the possibility of the scalars losing momentum and energy to the particle-hole pairs. The real and imaginary parts of $F(p_0, p)$ can be extracted using the principal value identity (which is meant to hold under integrals)

$$\lim_{c\to 0^+}\frac{1}{\omega \pm ic} = \mathcal{P}\frac{1}{\omega} \mp i\pi\epsilon(\omega) \qquad (32)$$

which we would use on $f_{5,0}$ with the proper imaginary parts restored.

The game now turns to actually doing the integral. In such a limit, it seems likely we will reproduce the results of Fetter and Walecka, p 158 and on by considering $F_0$. This is useful, as it serves on a check on our methods, and also in principle it gives us the polarization tensors which we will need anyway.

To validate our methods by checking against Fetter and Walecka, consider $F_0$, which can be written as

$$F_0(p_0, p) = \frac{-4g^2}{(2\pi)^2}\int_0^{k_F} dk k^2 \int_{-1}^{1} d[\cos(\theta)]\, f_0(p_0, \mathbf{p}; \mathbf{k}) \qquad (33)$$

Fetter and Walecka work in the non-relativistic limit, where $m$ is by definition the high scale. Expanding the integrand in powers of $1/m$, and then in powers of $p$, we get

$$\begin{aligned} F_0(p_0, p \to 0) &= \frac{4g^2}{(2\pi)^2}\int_0^{k_F} dk k^2 \int_{-1}^{1} d[\cos(\theta)]\left\{\frac{4kp\cos(\theta)}{mp_0^2} + \frac{p^2}{mp_0^2}\right\} \qquad (34) \\ &= -\frac{g^2k_F^3}{3m\pi^2}\frac{p^2}{p_0^2} \qquad (35) \end{aligned}$$

On the other, translated to our notation, Fetter and Walecka get (their Eq. 12.47b, page 162)

$$F_{FW}(p_0, p \to 0) = \frac{k_F^3}{3m\pi^2}\frac{p^2}{p_0^2} \qquad (36)$$

These results agree.

Let's check one more limit against Fetter and Walecka, where one sets $p_0 = 0$. (Static limit.) Expanding in $1/m$ with $p_0 = 0$, I get

$$F_0(p_0 = 0, p) = \frac{4g^2}{(2\pi)^2} \int_0^{k_F} dk\, k^2 \int_{-1}^{1} d[\cos(\theta)]\, \frac{2m}{p(2k\cos(\theta) + p)}$$
$$= -\frac{k_F m}{2\pi^2} + \frac{k_F^2 m \log\left[\frac{1-\frac{k_F}{2p}}{1+\frac{k_F}{2p}}\right]}{2p\pi^2} - \frac{mp \log\left[\frac{1-\frac{k_F}{2p}}{1+\frac{k_F}{2p}}\right]}{8\pi^2} \tag{37}$$

while in the same limit Fetter and Walecka get their Eq. 12.46b

$$F_0(p_0 = 0, p) = -\frac{k_F m}{2\pi^2} + \frac{k_F^2 m \log\left[\frac{1-\frac{k_F}{2p}}{1+\frac{k_F}{2p}}\right]}{2p\pi^2} - \frac{mp \log\left[\frac{1-\frac{2k_F}{p}}{1+\frac{k_F}{2p}}\right]}{8\pi^2}. \tag{38}$$

It's the same answer.

For future use, let's also compute $F_0$ in the ultra-relativistic limit of $m \to 0$. Then some interesting limits of $F$ are

$$F_0(p_0 = 0, p) = \frac{p\left(-8k_F^2 + p^2\left(\log\left[1 - \frac{4k_F^2}{p^2}\right]\right)\right) + k_F\left(-4k_F^2 + 3p^2\right)\log\left[1 - \frac{4k_F}{2k_F + p}\right]}{12p\pi^2} \tag{39}$$

$$F_0(p_0, p \to 0) = \frac{k_F^2}{3\pi^2}\frac{p^2}{p_0^2}\left(1 + \frac{p_0^2}{4k_F^2}\log\left[1 - \frac{4k_F^2}{p_0^2}\right]\right) \tag{40}$$

$$F_0(p_0 = xp, p \to 0) = -\frac{k_F^2}{\pi^2}\left(1 - \frac{1}{2}x\log\left[\frac{1+x}{1-x}\right]\right) \tag{41}$$

Note that the third limiting form in the ultra-relativistic gas is actually the same as in the non-relativistic gas, while the second form is the same if $k_0/k_F \ll 1$, modulo the need to do the substitution $m \to k_F$. This should not be an accident: presumably both of these forms are somehow determined by properties of low-energy particle-hole pairs, and their properties are essentially the essentially same regardless of the dispersion relation of the fermions. I am puzzled that the first limiting form seems to look nothing like the one in the non-relativistic limit. However, I have checked that the form we get for $F_0(p_0 = 0, p)$ agrees with the one calculated by Kapusta and Toimela[**?** ].

Some very quick remarks on the uses and abuses of these limiting forms. If our scalar field is secretly $A_0$, then the first limit should give relativistic Friedel oscillations after a couple of steps (see [**?** ]), while the second limit gives the Debye mass for photons in the ultra-relativistic limit. The third limit should yield zero sound if $\phi$ has a mass term, but for an electron gas this ought not happen due to the long-range nature of the Coulomb interaction in this limit. (This is the simplest way I know to see that in the electron gas interacting via E&M is not really a Fermi liquid.)

## IV. SINGLE COMPLEX SCALAR WITH TWO FERMIONS, WITH CONDENSATION

### A. Dirac fields

The Lagrangian is

$$\mathcal{L} = |D_\mu\phi|^2 - m\phi^\dagger\phi - \frac{\lambda}{4}(\phi^\dagger\phi)^2 + \bar{\psi}(i\not{D} - m)\psi + \bar{\chi}(i\not{\partial})\chi \tag{42}$$
$$+ \frac{ig}{2}\phi\bar{\psi}\,P_+\chi - \frac{ig}{2}\phi\bar{\chi}\,P_-\psi \tag{43}$$

with $[g] = 0$, $D_\nu\phi = \partial_\nu\phi - \mu\delta_{\nu,0}\phi$, $D_\nu\psi = \partial_\nu\psi - \mu\delta_{\nu,0}\psi$, and $\chi$ should be Majorana. However, to keep things simple, let us first pretend that $\chi$ is a *Dirac* fermion field. The normalization of $\lambda$ is chosen so that the associated Feynman rule is $-i\lambda$ **Check this** . In the target theory, $\lambda \sim g^2$. With the Yukawa interactions written above, the theory is invariant under the $U(1)$ global symmetry $\phi \to e^{i\alpha}\phi, \psi \to e^{i\alpha}\psi$, and $\mu$ is the chemical potential for the associated conserved current. The masses of $\psi$ and $\phi$ are chosen to be identical to emulate the system of interest as closely as possible.

In the presence of this kind of Yukawa interaction, one may worry that a $\psi$ fermion may be able to decay to $\phi$ particles plus a $\chi$ fermion. This makes the ground state at finite density rather difficult to understand. When $g = 0$

and $\mu > m$, $\psi$ particles will populate the non-interacting vacuum. However, turning on the interactions may cause the $\psi$ density to decrease in favor of an increased density of $\phi$ particles, because adding an extra fermion to the system costs an energy $\sim k_F$, while the boson density is much cheaper to increase. But what really happens is far from obvious, because once $\mu > m$ the $\phi$ field condenses, and the Yukawa terms lead to mixing between $\psi$ and $\chi$ fields, as will be discussed in detail below. To build some intuition it helps to remember that field theory is more clever than we are, so we will temporarily let ourselves be guided by its formalism until we understand the physics better.

To summarize: shut up and calculate.

So suppose that $\mu^2 > m^2$. Then it is reasonable to expect that $\phi$ will pick a vev which at tree level will be $\langle\phi\rangle^2 = v^2 \sim (\mu^2 - m^2)/\lambda$, and it behooves us to rewrite $\phi$ to make this evident. For now, let us pick the cartesian parametrization $\phi = v + h$, where $h$ is a complex field. Now it is clear that the Yukawa terms generate mixing terms of the form $gv\bar{\chi}P_-\psi$ as well the interaction terms like $gh\bar{\chi}P_-\psi$. As the name suggests, the mixing term mixes the two fermion fields at the quadratic level, and to make perturbation theory go, in general one has to move to a new basis which diagonalizes the quadratic action of the $\chi, \psi$ system. (Only once this is done is it sensible to define propagators and the associated machinery.) We should be able to do the diagonalization using a suitable generalization of the Nambu-Gorkov formalism.

To proceed, we go to momentum space, and rewrite the fermion part of the action in a matrix form. To get the signs right below note that in the Fourier transform relating time-position to energy-momentum space the fields go like $\psi(x) \sim \exp(-ik_0 t + ik_i x_i)$, so that $i\partial\!\!\!/ \to p\!\!\!/ = p_0\gamma_0 - p^i\gamma_i$. The action in matrix notation is

$$\mathcal{L}_F = \int \frac{d^4p}{(2\pi)^4} \begin{pmatrix} \bar{\psi}(-p) & \bar{\chi}(-p) \end{pmatrix} \begin{pmatrix} p\!\!\!/ + \mu\gamma_0 - m & \frac{igv}{2}P_+ \\ -\frac{igv}{2}P_- & p\!\!\!/ \end{pmatrix} \begin{pmatrix} \psi(p) \\ \chi(p) \end{pmatrix} \tag{44}$$

$$= \int \frac{d^4p}{(2\pi)^4} \bar{\Psi}(-p) \cdot K_{\text{toy}}(p) \cdot \Psi(p) \tag{45}$$

Trying to work out an explicit relation between $\psi, \chi$ and the fermionic eigenmodes of the condensed system appears to be very complicated, and the necessary linear transformation is likely highly non-local, and hence not especially illuminating. Instead of trying to find the explicit transformation, we play some tricks to get the answers we need without working up too much of a sweat.

Consider integrating out $\Psi$ in Eq. (45). The result is a fermion determinant $\det K_{\text{toy}}$. It turns out that there exists a relation for determinants of block matrices that we can use to solve for the eigenmodes. Assuming that $A, B, C, D$ are block matrices, with $A, B$ invertible and $C, D$ not necessarily square, the formula reads

$$\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det A \det(D - CA^{-1}B) = \det D \det(A - BD^{-1}C), \tag{46}$$

and I have checked numerically that these formulas (obtained from Wikipedia) really work. Using the first form above, we obtain

$$\det K_{\text{toy}}(p) = \det(p\!\!\!/ + \mu\gamma_0 - m) \det\left(p\!\!\!/ - [-igvP_-] \frac{p\!\!\!/ + \mu\gamma_0 + m}{(p_0-\mu)^2 - p^2 - m^2} [igvP_+]\right) \tag{47}$$

$$= \det(p\!\!\!/ + \mu\gamma_0 - m) \det\left(p\!\!\!/ - \frac{p\!\!\!/ + \mu\gamma_0}{(p_0-\mu)^2 - p^2 - m^2} \left[g^2v^2P_+\right]\right) \tag{48}$$

$$= \det(p\!\!\!/ + \mu\gamma_0 - m) \det\left(\frac{p\!\!\!/\left((p_0-\mu)^2 - p^2 - m^2\right) - (p\!\!\!/ + \mu\gamma_0)g^2v^2P_+}{(p_0-\mu)^2 - p^2 - m^2}\right) \tag{49}$$

$$= \det(p\!\!\!/ + \mu\gamma_0 - m) \times \tag{50}$$

$$\det\left[-m - \mu\gamma_0 + p\!\!\!/\left(1 + \frac{g^2v^2}{8\left(\mathbf{p}^2 - p_0^2\right)}\right) - p\!\!\!/\gamma_5\left(\frac{g^2v^2}{8\left(\mathbf{p}^2 - p_0^2\right)}\right)\right] \tag{51}$$

where in the second step we used the identities $P_+^2 = P_+, P_-P_+ = 0$, and in the last step we rewrote the numerator by projecting onto the basis of $4 \times 4$ matrices given by the space of the possible Dirac structures. This looks ugly, but the nice thing is that this manipulation immediately lets us write the Lagrangian for the in-medium fermionic fields, by integrating some fermions back in, replacing the determinants by equivalent fermionic actions. These fermionic actions may be ugly, but physics is hard and no one promised us a field of roses.

The Lagrangian for the in-medium fermionic eigenmodes can be written (in momentum space) as

$$\mathcal{L}_F = \bar{\psi}(\not{D} - m)\psi \tag{52}$$

$$+ \bar{\xi}\left[\not{p} - \mu\gamma_0 - \mathcal{M}_{\rm Dirac}\right]\xi \tag{53}$$

where

$$\mathcal{M}_{\rm Dirac} = m + \frac{\not{p} g^2 v^2 P_-}{4\left(\mathbf{p}^2 - p_0^2\right)} \tag{54}$$

where we recycled the name of the field $\psi$ for obvious reasons, and introduced a new field $\xi$ to represent the mixed mode of the original $\psi$ and $\chi$ fields. The point of doing all this is that now $\psi$ is still a fermion field that feels a chemical potential, and everything we said about such fields above still applies. In particular we can write a Feynman propagator for $\psi$ the same way as usual. For $\xi$ we have to do a bit more work, but in principle writing the Feynman propagator should not be that hard at least in various limits, such as e.g. low momentum.

If you're paying attention, *this should be surprising!* The original chemical potential $\mu$ coupled to a conserved charge $Q = Q_{\rm scalar} + \bar{\psi}\gamma_0\psi$. But in the condensed phase $Q_{\rm scalar}$ no longer annihilates the vacuum by definition. Also, the original fermion number symmetry $\psi \to e^{i\alpha}\psi$ is not a symmetry of the action in the condensed phase. Hence it is not clear what symmetry $\bar{\psi}'\gamma_0\psi'$ corresponds to in terms of the original $\psi, \chi$ fields. A natural guess is $\psi \to e^{i\alpha}\psi, \chi \to e^{-i\alpha}\chi$. An examination of the Majorana system should tell us whether that's really plausible, since such transformations cannot be symmetries of the action in the Majorana theory.

Independently of the subtlety above, we can now easily work out the form of the interaction terms of the in-medium fields, simply by including them in the determinants above. For instance, suppose we write $\phi = (v + h)e^{i\varphi}$, so that $h$ should pick a mass when $v \neq 0$, while $\varphi$ ought to be a Goldstone mode. It is now easy to trace through the above derivation and see that $\varphi$ does not couple to either $\psi$ or $\xi$, while $h$ only couples to $\xi$, and to work out the precise form of the couplings of $h$ to $\xi$. Hence the coupling to the fermion sector via the Yukawa does not mess with the Goldstone theorem, as of course one might have expected.

### B. Majorana fields

Now we take off the Dirac training wheels, and deal with the fact that $\chi$ is Majorana. Write $\psi = \frac{1}{\sqrt{2}}(\eta + i\theta)$, with $\eta, \theta$ Majorana fermions, as discussed in the Appendix. Then

$$\frac{ig}{2}\left(\phi\bar{\psi}\, P_+\chi - \phi^\dagger\bar{\chi}\, P_-\psi\right) = \frac{1}{2} \times \frac{ig}{\sqrt{2}}\left(\phi\bar{\eta}P_+\chi - i\phi\bar{\theta}P_+\chi - \phi^\dagger\bar{\chi}P_-\eta - i\phi^\dagger\bar{\chi}P_-\theta\right). \tag{55}$$

Now assume that $\phi \to \langle\phi\rangle = v$, and rewrite the action in the Majorana basis and in momentum space, yielding

$$\mathcal{L}_F = \frac{1}{2}\int \frac{d^4p}{(2\pi)^4}\begin{pmatrix} \bar{\eta}(-p) & \bar{\theta}(-p) & \bar{\chi}(-p) \end{pmatrix}\begin{pmatrix} \not{p} - m & +i\mu\gamma_0 & \frac{igv}{\sqrt{2}}P_+ \\ -i\mu\gamma_0 & \not{p} - m & \frac{gv}{\sqrt{2}}P_+ \\ \frac{-igv}{\sqrt{2}}P_- & \frac{gv}{\sqrt{2}}P_- & \not{p} \end{pmatrix}\begin{pmatrix} \eta(p) \\ \theta(p) \\ \chi(p) \end{pmatrix} \tag{56}$$

$$= \frac{1}{2}\int \frac{d^4p}{(2\pi)^4}\Psi_M^T(-p)\cdot CK_M(p)\cdot\Psi_M(p). \tag{57}$$

Integrating out $\Psi_M$, we get $\mathrm{Pf}(CK_M)$. This can manipulated into the following:

$$\mathrm{Pf}[CK_M] = \mathrm{Pf}[CK_D]\mathrm{Pf}\left[C\not{p} - \begin{pmatrix} \frac{-igv}{\sqrt{2}}CP_- & \frac{gv}{\sqrt{2}}CP_- \end{pmatrix}\cdot(CK_D)^{-1}\cdot\begin{pmatrix} \frac{igv}{\sqrt{2}}CP_+ \\ \frac{gv}{\sqrt{2}}CP_+ \end{pmatrix}\right] \tag{58}$$

$$= \mathrm{Pf}[CK_D]\mathrm{Pf}\left[C\left\{\not{p} - \begin{pmatrix} \frac{-igv}{\sqrt{2}}P_- & \frac{gv}{\sqrt{2}}P_- \end{pmatrix}\cdot(K_D)^{-1}\cdot\begin{pmatrix} \frac{igv}{\sqrt{2}}P_+ \\ \frac{gv}{\sqrt{2}}P_+ \end{pmatrix}\right\}\right] \tag{59}$$

where

$$K_D = \begin{pmatrix} \not{p} - m & +i\mu\gamma_0 \\ -i\mu\gamma_0 & \not{p} - m \end{pmatrix} \tag{60}$$

is the Majorana-basis massive Dirac operator at finite density.

We massage the Pfaffian on the right using Mathematica, with the surprisingly simple result

$$\mathrm{Pf}\left[C\left\{\not{p}\left(1+\frac{g^2v^2}{2[m^2+\mathbf{p}^2-(p_0+\mu)^2]}\right)+\frac{g^2v^2\mu\gamma_0}{2[m^2+\mathbf{p}^2-(p_0+\mu)^2]}+\frac{g^2v^2(\not{p}+\mu\gamma_0)\gamma_5}{2[m^2+\mathbf{p}^2-(p_0+\mu)^2]}\right\}\right] \tag{61}$$

We now integrate the fermions back in, and find that the diagonalized Lagrangian of the fermionic fields after condensation of $\phi$ is given by

$$\mathcal{L}_F^{\mathrm{cond}} = \bar{\psi}(\not{p}-m-\mu\gamma_0)\psi \tag{62}$$

$$+\frac{1}{2}\bar{\xi}\left[\not{p}\left(1+\frac{g^2v^2}{2[m^2+\mathbf{p}^2-(p_0+\mu)^2]}\right)+\frac{g^2v^2\mu\gamma_0}{2[m^2+\mathbf{p}^2-(p_0+\mu)^2]}+\frac{g^2v^2(\not{p}+\mu\gamma_0)\gamma_5}{2[m^2+\mathbf{p}^2-(p_0+\mu)^2]}\right]\xi \tag{63}$$

There is one Dirac fermion with mass $m$ and chemical potential $\mu$, and one Majorana fermion with a super weird kinetic term we'll discuss in more detail at a later iteration of this paper to be discussed in detail at a future Deloitte day.

Interactions of the fermions with the fluctuations of $\phi$ around its vev are very simple to take into account by trading $v \to \phi = v + h$ and keeping track of complex conjugation, with the diagonalized action taking the form

$$\mathcal{L}_F^{\mathrm{cond}} = \bar{\psi}(\not{p}-m-\mu\gamma_0)\psi + \frac{1}{2}\bar{\xi}\left[\not{p}-\mathcal{M}_{p_0,|\mathbf{p}|}P_+\right]\xi \tag{64}$$

where

$$\mathcal{M}_{p_0,|\mathbf{p}|} = \frac{g^2\phi^\dagger\phi(\not{p}+\mu\gamma_0)}{(p_0+\mu)^2-\mathbf{p}^2-m^2} \tag{65}$$

Note that the scalar field only couples to $\xi$ after diagonalization through the term $\mathcal{M}$. Note that when fluctuations around the scalar vev are turned off, $\mathcal{M}$ is a rather complicated-looking momentum-dependent mass term.

As a check on the algebra, I have verified that the spectrum of spinor modes following from Eq. (64) is the same as the one following from a brute-force eigenmode computation applied to Eq. (56).

### C. Color

Let's turn on color. Now $\phi_a, \psi_b$ are $SU(N_c)$ fundamentals with the index $a, b = 1, \ldots, N_c$, while $\chi_{ab} = \sqrt{2}t^A_{ab}\chi_A$ is an adjoint field with $A = 1, \ldots, N_c^2 - 1$ is an adjoint index, and the $t^A$ are the Hermitian generators of $SU(N_c)$ normalized so that $\mathrm{tr}\, t^A t^B = \frac{1}{2}\delta^{AB}$. The normalizations are chosen such that both the gluon kinetic term and the kinetic terms for the $\chi_A$ fields are canonically normalized:

$$\mathcal{L}_F = \bar{\psi}(i\not{D}_{\mathrm{F}}-m-\mu\gamma_0)\psi + \mathrm{tr}\,\bar{\chi}i\not{D}_{\mathrm{adj}}\chi + \frac{1}{2}\mathrm{tr}\,F_{\mu\nu}F^{\mu\nu} + \mathcal{L}_{\mathrm{Yukawa}} \tag{66}$$

$$= \bar{\psi}_a(i\not{D}^{ab}_F-m-\mu\gamma_0)\psi_b + \bar{\chi}_A i\not{D}^{AB}_{\mathrm{adj}}\chi_B + \frac{1}{4}\mathrm{tr}\,F^A_{\mu\nu}(F^A)^{\mu\nu} + \mathcal{L}_{\mathrm{Yukawa}} \tag{67}$$

where $\not{D}_{\mathrm{F}} = \not{\partial} - i\not{A}, \not{D}_{\mathrm{Adj}}\chi = \not{\partial}\chi - i[\not{A}, \chi]$, with $A_\mu = A^A_\mu t^A$. Note that

$$\mathrm{tr}\,\bar{\chi}[\not{A},\chi] = \mathrm{tr}\,\bar{\chi}[\not{A}^A t^A, \sqrt{2}\chi^B t^B] = 2i\mathrm{tr}\,\bar{\chi}^D t^D f^{ABC} t^C \not{A}^A \chi^B \tag{68}$$

$$= -i\bar{\chi}^C f^{ABC}\not{A}^A\chi^B \tag{69}$$

so that $\not{D}^{AB}_{\mathrm{adj}}(p) = \not{p}\delta^{AB} - if^{CBA}\not{A}^C$.

Writing $\phi_a = v_a + h_a$, the interaction term with color indices shows explicitly is

$$\mathcal{L}_{\mathrm{Yukawa}} = \frac{ig}{\sqrt{2}}\left(\bar{\psi}^a P_+ t^A_{ab}\chi_A\phi^b - (\phi^\dagger)_a\bar{\chi}_B(t^B)^{ab}P_-\psi_b\right) \tag{70}$$

Call $\Psi_{\mathcal{A}} = (\eta_a\, \theta_a\, \chi_A)^T$. Then fermion matrix takes the form

$$\mathcal{L}_{\rm Yukawa} = \frac{1}{2}\int \frac{d^4p}{(2\pi)^4}\Psi^T_{\mathcal{B}}(-p)\begin{pmatrix} \delta^{ab}(\not{p}-m) - iA^A_\mu(t^A)^{ab} & +i\mu\gamma_0\delta^{ab} & igP_+t^A_{ab}\phi^b \\ -i\mu\gamma_0\delta^{ab} & \delta^{ab}(\not{p}-m) - iA^A_\mu(t^A)^{ab} & gP_+t^A_{ab}\phi^b \\ -igP_-(t^B)^{ab}(\phi^\dagger)_a & gP_-(t^B)^{ab}(\phi^\dagger)_a & \not{D}^{AB}_{\rm adj}(p) \end{pmatrix}\Psi_{\mathcal{A}}(p) \tag{71}$$

$$= \frac{1}{2}\int \frac{d^4p}{(2\pi)^4}\Psi^T_M(-p)\cdot CK_M\cdot\Psi_M(p). \tag{72}$$

and now $\Psi_M$ is $N^2+N-1$ vector in 'color' space. Tracking through the algebra, the tricky point is how to handle the fact that $K_D$, the Dirac matrix in Majorana space, now depends on $A_\mu$. We are interested in setting up a perturbative calculation, and for this we need to find out the propagators and the lowest-order interaction terms of the fermions with the scalars and gauge fields. The propagators for the fermions are defined with the gauge fields set to zero.

Finally, the result for the diagonalized modes changes to

$$\mathcal{L}^{\rm cond}_F = \bar\psi_a(\not{D}(p) - m - \mu\gamma_0)\psi^a + \frac{1}{2}\bar\xi_A\left[\not{D}^{AB}_{\rm adj}(p) - \mathcal{M}^{AB}_{p_0,|\mathbf{p}|}P_+\right]\xi_B. \tag{73}$$

where

$$\mathcal{M}^{AB}_{p_0,|\mathbf{p}|} = \frac{g^2(\Phi^\dagger)^A\Phi^B(\not{p}+\mu\gamma_0)}{(p_0+\mu)^2 - \mathbf{p}^2 - m^2} \tag{74}$$

with $\Phi^A = t^A\phi$. I have not yet worked out a simple form for the matrix $(\Phi^\dagger)^A\Phi^B$.

## V. MATTHEWS NOTES

### A. Yukawa diagonalization - Majorana style

Let's write the Lagrangian as

$$\mathcal{L} = \mathcal{L}_b + \mathcal{L}_f + \mathcal{L}_{int}, \tag{75}$$

where

$$\mathcal{L}_b = |D_\mu\phi|^2 - m^2\phi^\dagger\phi - \frac{\lambda}{2}(\phi^\dagger\phi)^2, \tag{76}$$

$$\mathcal{L}_f = \bar\psi(i\not{D} - m)\psi + \frac{1}{2}\bar\chi(i\not{\partial})\chi \tag{77}$$

and

$$\mathcal{L}_{Yuk} = \frac{ig}{2}\left(\phi\bar\psi\, P_+\chi - \phi^\dagger\bar\chi\, P_-\psi\right). \tag{78}$$

The system has a global $U(1)$ rotating $\psi$ and $\phi$ simultaneously. We can use the Majorana basis with $\psi = \frac{1}{\sqrt{2}}(\eta + i\theta)$ and $\phi = v + h$ to find

$$\mathcal{L} = \mathcal{L}_b\,[v+h] + \frac{1}{2}\int\frac{d^4p}{(2\pi)^4}\begin{pmatrix}\bar\eta(-p) & \bar\theta(-p) & \bar\chi(-p)\end{pmatrix}\begin{pmatrix} \not{p}-m & +i\mu\gamma_0 & \frac{igv}{\sqrt{2}}P_+ \\ -i\mu\gamma_0 & \not{p}-m & \frac{gv}{\sqrt{2}}P_+ \\ \frac{-igv}{\sqrt{2}}P_- & \frac{gv}{\sqrt{2}}P_- & \not{p}\end{pmatrix}\begin{pmatrix}\eta(p)\\ \theta(p)\\ \chi(p)\end{pmatrix}$$
$$+\int d^4x\frac{ig}{2}\left(h\bar\psi\, P_+\chi - h^\dagger\bar\chi\, P_-\psi\right). \tag{79}$$

Let us just focus on

$$\mathcal{L}_F = \frac{1}{2}\int\frac{d^4p}{(2\pi)^4}\begin{pmatrix}\bar\eta(-p) & \bar\theta(-p) & \bar\chi(-p)\end{pmatrix}\begin{pmatrix} \not{p}-m & +i\mu\gamma_0 & \frac{igv}{\sqrt{2}}P_+ \\ -i\mu\gamma_0 & \not{p}-m & \frac{gv}{\sqrt{2}}P_+ \\ \frac{-igv}{\sqrt{2}}P_- & \frac{gv}{\sqrt{2}}P_- & \not{p}\end{pmatrix}\begin{pmatrix}\eta(p)\\ \theta(p)\\ \chi(p)\end{pmatrix} \tag{80}$$

and write

$$\begin{aligned} K_D &= \begin{pmatrix} \not{p} - m & +i\mu\gamma_0 \\ -i\mu\gamma_0 & \not{p} - m \end{pmatrix}, \\ B &= \begin{pmatrix} \frac{igv}{\sqrt{2}} P_+ \\ \frac{gv}{\sqrt{2}} P_+ \end{pmatrix} \\ C &= \begin{pmatrix} \frac{-igv}{\sqrt{2}} P_- & \frac{gv}{\sqrt{2}} P_- \end{pmatrix} \\ K_M &= \not{p}, \end{aligned} \tag{81}$$

so that

$$M = \begin{pmatrix} K_D & B \\ C & K_M \end{pmatrix} = \begin{pmatrix} K_D & 0 \\ C & \mathbb{1}_1 \end{pmatrix} \cdot \begin{pmatrix} \mathbb{1}_2 & K_D^{-1} B \\ 0 & K_M - C K_D^{-1} B \end{pmatrix} = M_1 \cdot M_2. \tag{82}$$

We now wish to diagonalize $M_1$ and $M_2$ individually, so that

$$\begin{aligned} M &= P P^{-1} M_1 P P^{-1} Q Q^{-1} M_2 Q Q^{-1} \\ &= P \bar{M}_1 P^{-1} Q \bar{M}_2 Q^{-1} \end{aligned} \tag{83}$$

where

$$\begin{aligned} \bar{M}_1 &= P^{-1} M_1 P = \begin{pmatrix} K_D & 0 \\ 0 & \mathbb{1}_1 \end{pmatrix} \\ \bar{M}_2 &= Q^{-1} M_2 Q = \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & K_M - C K_D^{-1} B \end{pmatrix} \end{aligned} \tag{84}$$

We can find the corresponding $P$ and $Q$, which do the job [I think...]

$$\begin{aligned} P &= \begin{pmatrix} \mathbb{1}_1 & 0 & 0 \\ 0 & \mathbb{1}_1 & 0 \\ \frac{-igv}{\sqrt{2}} P_- \left(\not{p} - m + \mu\gamma_0 - \mathbb{1}_1\right)^{-1} & \frac{gv}{\sqrt{2}} P_- \left(\not{p} - m + \mu\gamma_0 - \mathbb{1}_1\right)^{-1} & \mathbb{1}_1 \end{pmatrix} = \begin{pmatrix} \mathbb{1}_1 & 0 & 0 \\ 0 & \mathbb{1}_1 & 0 \\ -iP_1 & P_1 & \mathbb{1}_1 \end{pmatrix} = \\ Q &= \begin{pmatrix} \mathbb{1}_1 & 0 & \frac{igv\left(\not{p} - m - \mu\gamma_0\right) P_+}{\sqrt{2}(\det K_D)\left(K_M - C K_D^{-1} B - 1\right)} \\ 0 & \mathbb{1}_1 & \frac{gv\left(\not{p} - m - \mu\gamma_0\right) P_+}{\sqrt{2}(\det K_D)\left(K_M - C K_D^{-1} B - 1\right)} \\ 0 & 0 & \mathbb{1}_1 \end{pmatrix} = \begin{pmatrix} \mathbb{1}_1 & 0 & iQ_1 \\ 0 & \mathbb{1}_1 & Q_1 \\ 0 & 0 & \mathbb{1}_1 \end{pmatrix} \end{aligned} \tag{85}$$

and very conveniently

$$\begin{aligned} P^{-1} &= \begin{pmatrix} \mathbb{1}_1 & 0 & 0 \\ 0 & \mathbb{1}_1 & 0 \\ iP_1 & -P_1 & \mathbb{1}_1 \end{pmatrix} \\ Q^{-1} &= \begin{pmatrix} \mathbb{1}_1 & 0 & -iQ_1 \\ 0 & \mathbb{1}_1 & -Q_1 \\ 0 & 0 & \mathbb{1}_1 \end{pmatrix}. \end{aligned} \tag{86}$$

Now commute

$$\begin{aligned} M &= P \bar{M}_1 P^{-1} Q \bar{M}_2 Q^{-1} = \\ &= \bar{M}_1 \bar{M}_2 + \left[P, \bar{M}_1\right] P^{-1} \bar{M}_2 + \bar{M}_1 \left[Q, \bar{M}_2\right] Q^{-1} + \left[P, \bar{M}_1\right] P^{-1} \left[Q, \bar{M}_2\right] Q^{-1} \\ &= \bar{M}_1 \bar{M}_2 + \left[P, \bar{M}_1\right] + \bar{M}_1 \left[Q, \bar{M}_2\right] + \left[P, \bar{M}_1\right] \left[Q, \bar{M}_2\right] \end{aligned} \tag{87}$$

This decomposition gives the expected

$$\begin{aligned}
\left[P, \bar{M}_1\right] &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -\frac{igv}{\sqrt{2}} P_- & \frac{gv}{\sqrt{2}} P_- & 0 \end{pmatrix} \\
\bar{M}_1 \left[Q, \bar{M}_2\right] &= \begin{pmatrix} 0 & 0 & \frac{igv}{\sqrt{2}} P_+ \\ 0 & 0 & \frac{gv}{\sqrt{2}} P_+ \\ 0 & 0 & 0 \end{pmatrix} \\
\left[P, \bar{M}_1\right] \left[Q, \bar{M}_2\right] &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & C K_D^{-1} B \end{pmatrix}
\end{aligned} \tag{88}$$

so that the terms add up to the initial matrix block by block. I guess this $0 = 0$ result simply shows from a matrix decomposition point of view why Pfaffians do not factorize as determinants. Everything should have vanished somehow, and I don't see how that's possible, except for the $\bar{M}_1 \bar{M}_2$ term.

### B. Pfaffians

A simple reason why things don't factorize:

$$\mathrm{Pf}(AB) = \sqrt{\det AB} = \sqrt{\det A \det B} \tag{89}$$

But now for $\mathrm{Pf}(AB)$ to be non-zero $AB$ has to be antisymmetric. However, a product of two anti-symmetric matrices cannot be anti-symmetric so $A$ and $B$ are not individually antisymmetric and hence the roots of their individual determinants can not become Pfaffians.

### C. Chemical potentials

After thinking about what will happen to symmetries I think that in the case of simple (non-SUSY) Yukawa, the U(1) will be broken by the purely scalar contribution, even though the Yukawa terms and the Dirac kinetic terms might still have a U(1) rotating $h$ and $\psi$ simultaneously. But as in the usual spontaneous symmetry breaking the $h$ rotation will simply not be a symmetry of the scalar term any more.

In the SUSY case of the $U(1)$ gauge theory, I think that we are actually giving a chemical potential to a different global $U(1)$, which is not the gauge $U(1)$, but a global $U(1)$ rotating only $\phi_\pm, \psi$. Once we break this "baryon" $U(1)$, the gauge group should still remain.

## VI. APPLICATION: SUPER-QED, DARK MATTER, AND DARK WHITE DWARFS STARS

It turns out that a slightly generalization of our toy model, which actually brings it closer to our target theory, is very close to $\mathcal{N} = 1$ super-QED. So before considering the non-abelian theory, we might as well work out the quasiparticle spectrum of $\mathcal{N} = 1$ QED at high density, using what we learned above. The action of $\mathcal{N} = 1$ super QED two charged chiral 'electron' multiplet with charges $\pm 1$ and a common mass $m$ is[**?** **?** ]

$$\mathcal{L}_{\mathrm{sQED}} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\lambda} i \partial\!\!\!/ \lambda \tag{90}$$

$$+ \bar{\psi}(i D\!\!\!/ - m)\psi + |D_\mu \phi_-|^2 + |D_\mu \phi_+|^2 - m|\phi_-|^2 - m|\phi_+|^2 + \mathcal{L}_{\mathrm{int}} \tag{91}$$

$$\mathcal{L}_{\mathrm{int}} = i\sqrt{2} g(\phi_- \bar{\psi} P_- \lambda - \phi_-^\dagger \bar{\lambda} P_+ \psi - \phi_+ \bar{\lambda} P_- \psi + \phi_+^\dagger \bar{\psi} P_+ \lambda) - \frac{g^2}{2}(|\phi_+|^2 - |\phi_-|^2)^2 \tag{92}$$

where $D_\mu = \partial_\mu - igA_\mu$, where $g$ is the charge. The Dirac fermion $\psi$ has charge $q = -1$, the complex scalars $\phi_\pm$ have charges $q = \pm 1$ respectively, and $\lambda$ is a neutral Majorana field, which is simply the photino. Our game would be to turn on a chemical potential for an interesting-seeming $U(1)$ global symmetry of the system which couples to the fermions and at least one of the scalars, and see what happens. The point is to race to get out something publishable rather than getting depressed by the prospects of engaging in year-long grungy calculation which may necessary to handle the system with AdS/CMT relevance.

Aside from pure intellectual curiosity about the system, there is a not entirely laughable motivation for doing this in the physics of dark matter. It is quite conceivable that the dark sector is almost exactly supersymmetric, with the tiny couplings to the visible sector being the source of SUSY breaking. Given the plethora of long and short range interactions in the visible sector, it is also conceivable that the dark sector has some long-range interactions along with short-range ones, so that dark matter may be atomic or molecular rather made of fundamental particles, as most commonly assumed. The phenomenology of this was recently explored in Wacker et al in [**?** ]. A bit earlier Wacker et al[**?** ] and Herzog et al[**?** ] calculated the properties of atoms in super-QED, with unbroken supersymmetry. This all followed lots of non-supersymmetric composite dark matter pheno papers apparently starting in the mid 80s in some work by Shmuel Nussinov. (Small world!)

If the dark matter is initially atomic, with atomic size set by $r_0 = (\alpha M)^{-1}$, where $\alpha = g^2/4\pi$ with $g$ the $U(1)$ dark-sector gauge coupling and $M$ the relevant reduced mass, it may well clump and form molecules, or perhaps even stars. If densities in these stars are such that $n^{1/3} \gg 1/r_0$, the supersymmetric atoms will ionize, and the physics will be well-described by the super-QED Lagrangian, as opposed to the QM actions appropriate for the study of supersymmetric atoms and molecules. From previous experience and e.g. discussions of electron gases in Fetter and Walecka, one expects that the perturbative computations of the properties of such neutral plasmas will be controlled by the small parameter $r_0 n^{1/3}$. On general grounds, the scalars may be expected to bose-condense, with various interesting consequences. In the non-supersymmetric context, such physics was explored in [**?** **?** ]. A study of the high-density region of the $\mathcal{N} = 1$ may provide some insight into the phenomenology of **dark white dwarfs stars**, or whatever a big clump of such dark matter may become; clearly it depends on the equation of state of the plasma, which we'd have to compute. **Dark white dwarf** is such a cool term that it justifies working on this all by itself.

While a neat notion, we can't get too excited about the pheno: to my recollection any significant lumpiness of dark matter is very severely constrained.

### A. Vacuum alignment

The potential for the scalars including the chemical potential contributions is

$$V = \frac{g^2}{2}(|\phi_+|^2 - |\phi_-|^2)^2 + (|m|^2 - \mu^2)(|\phi_+|^2 + |\phi_-|^2) \tag{93}$$

$$\tag{94}$$

In writing this I've assumed that the chemical potential couples to the time component of the conserved current corresponding to the electric charge

$$J^\mu = \bar{\psi}\gamma^\mu\psi + (\phi_-^\dagger i\partial_\mu\phi_- + h.c.) - (\phi_+^\dagger i\partial_\mu\phi_+ + h.c.) \tag{95}$$

Note that the dependence on the mass parameter is via $|m|^2$, as a consequence of supersymmetry. The minima of the potential must simultaneously satisfy $\partial V/\partial\phi_+^\dagger = 0, \partial V/\partial\phi_-^\dagger = 0$, so that the vevs $v_+, v_-$ should satisfy

$$g^2 v_+(|v_+|^2 - |v_-|^2) + (|m|^2 - \mu^2)v_+ = 0 \tag{96}$$

$$g^2 v_-(|v_+|^2 - |v_-|^2) + (|m|^2 - \mu^2)v_- = 0 \tag{97}$$

When $m = \mu = 0$, the vacuum is a continuous manifold $\phi_+ = \phi_- = \varphi$, $\varphi \in \mathbb{C}$, and on the vacuum configurations $V = 0$, which is the condition for a supersymmetric vacuum. If $\mu = 0$ but $m \neq 0$, the only stable extremum is at $v_\pm = 0$, where the potential again vanishes, signaling that SUSY is preserved. When $0 < \mu \leq m$, the vacuum remains at $v_+ = v_- = 0$, and classically it so happens that $V = 0$, although I do not expect this to be true at the quantum level with $\mu \neq 0$. Finally, when $\mu^2 > |m|$, the vacuum is at $v_+ = g^{-1}\sqrt{\mu^2 - |m|^2}e^{i\theta}, v_- = 0$, or at $v_+ = 0, v_- = g^{-1}\sqrt{\mu^2 - |m|^2}e^{i\theta}$, with $\theta$ an arbitrary phase. Vacua where supersymmetry is unbroken have $V = 0$, and indeed this is an if and only if relation and is stable under quantum corrections. In vacua where SUSY is spontaneously broken $V > 0$, but this is no longer an if and only if statement: if $V > 0$, the underlying theory may or may not be supersymmetric. Finally, $V < 0$ is inconsistent with SUSY. The results confirm our expectation that the chemical potential we have in mind breaks SUSY completely in region $\mu > m$, and a one-loop calculation of the vacuum energy would confirm (in a brute-force way) that SUSY is broken for any $\mu \neq 0$.

## VII. $\mathcal{N}=1$ SUSY THEORY - WEYL SPINORS

The Lagrangian is (factor of 1/2 in the Majorana and $\phi_-$, $\phi_+$ replacement in the Yukawa compared to above, to match Herzog, et. al.)

$$\begin{aligned}\mathcal{L}_{\text{sQED}} &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\bar{\lambda}i\not{\partial}\lambda \\ &\quad + \bar{\psi}(i\not{D} - m)\psi + |D_\mu\phi_-|^2 + |D_\mu\phi_+|^2 - m|\phi_-|^2 - m|\phi_+|^2 + \mathcal{L}_{\text{int}} \qquad (98)\\ \mathcal{L}_{\text{int}} &= i\sqrt{2}g(\phi_+\bar{\psi}P_-\lambda - \phi_+^\dagger\bar{\lambda}P_+\psi - \phi_-\bar{\lambda}P_-\psi + \phi_-^\dagger\bar{\psi}P_+\lambda) - \frac{g^2}{2}(|\phi_+|^2 - |\phi_-|^2)^2 \qquad (99)\end{aligned}$$

We found that at tree level, we can set $v_- = 0$ and keep only $v_+$. For now we will only keep $v_+$ (taking it to be real for now), so that

$$i\sqrt{2}gv_+\left(\bar{\psi}P_-\lambda - \bar{\lambda}P_+\psi\right) + i\sqrt{2}g\left(h_+\bar{\psi}P_-\lambda - h_+^\dagger\bar{\lambda}P_+\psi - h_-\bar{\lambda}P_-\psi + h_-^\dagger\bar{\psi}P_+\lambda\right) \qquad (100)$$

We will use the Weyl basis for the Dirac matrices and write the Dirac fermion as two Weyl fermions

$$\psi = \begin{pmatrix}\psi_L\\ \psi_R\end{pmatrix} = P_-\psi + P_+\psi = \psi_L + \psi_R \qquad (101)$$

so that

$$\bar{\psi}(i\not{D} - m)\psi = \psi_L^\dagger\left(i\sigma^\mu\partial_\mu + \mu\right)\psi_L + \psi_R^\dagger\left(i\bar{\sigma}^\mu\partial_\mu + \mu\right)\psi_R - m\psi_L^\dagger\psi_R - m\psi_R^\dagger\psi_L. \qquad (102)$$

Now we write the Majorana fermion $\lambda$ as a Weyl fermion

$$\lambda = \begin{pmatrix}\chi\\ -i\sigma^2\chi^*\end{pmatrix} \qquad (103)$$

so that

$$\frac{i}{2}\bar{\lambda}\gamma^\mu\partial_\mu\lambda = i\chi^\dagger\bar{\sigma}^\mu\partial_\mu\chi \qquad (104)$$

The condensate part of the Yukawa becomes

$$\sqrt{2}igv_+\left(\bar{\psi}P_-\lambda - \bar{\lambda}P_+\psi\right) = \sqrt{2}igv_+\left(\psi_R^\dagger\chi - \chi^\dagger\psi_R\right) \qquad (105)$$

The quadratic fermion part, which we need to diagonalize is then

$$\begin{aligned}&\begin{pmatrix}\psi_L^\dagger(-p) & \psi_R^\dagger(-p) & \chi^\dagger(-p)\end{pmatrix}\begin{pmatrix}\bar{\sigma}^\mu p_\mu & 0 & 0\\ 0 & \sigma^\mu p_\mu & 0\\ 0 & 0 & \bar{\sigma}^\mu p_\mu\end{pmatrix}\begin{pmatrix}\psi_L(p)\\ \psi_R(p)\\ \chi(p)\end{pmatrix} + \\ &+\begin{pmatrix}\psi_L^\dagger(-p) & \psi_R^\dagger(-p) & \chi^\dagger(-p)\end{pmatrix}\begin{pmatrix}\mu & -m & 0\\ -m & \mu & \sqrt{2}igv_+\\ 0 & -\sqrt{2}igv_+ & 0\end{pmatrix}\begin{pmatrix}\psi_L(p)\\ \psi_R(p)\\ \chi(p)\end{pmatrix} \qquad (106)\end{aligned}$$

Now this is actually doable! You can type a general $v_+$ into Mathematica, find the determinant and solve the cubic. For some reason it refuses to evaluate the eigenvalues directly. But no matter... If we actually pick the vev

$$v_+ = \frac{\sqrt{\mu^2 - m^2}}{g}, \qquad (107)$$

we find the eigenvalues

$$\left\{2\mu, -\sqrt{\mu^2 - m^2}, \sqrt{\mu^2 - m^2}\right\} \qquad (108)$$

and the eigenvector matrix

$$P=\begin{pmatrix} -\frac{\sqrt{2}im}{\sqrt{\mu^2-m^2}} & \frac{i\left(\sqrt{\mu^2-m^2}-\mu\right)}{\sqrt{2}m} & \frac{i\left(\sqrt{\mu^2-m^2}+\mu\right)}{\sqrt{2}m} \\ \frac{\sqrt{2}i\mu}{\sqrt{\mu^2-m^2}} & -\frac{i}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ 1 & 1 & 1 \end{pmatrix} \tag{109}$$

so that $P^{-1}MP=D_1$. But things go wrong as usual!
The momentum matrix isn't actually diagonal. It has $\sigma$ and $\bar{\sigma}$. Let's try to go to the Weyl basis where all the spinor have the same handedness.

Let us introduce $\psi_R=-i\sigma^2\theta^*$, so that after integrating by parts, using $\eta^T M\chi=-\chi^T M^T\eta$ identity for two spinors and going back to momentum space we find

$$\psi_R^\dagger\sigma^\mu p_\mu\psi_R=\theta^\dagger\bar{\sigma}^\mu p_\mu\theta \tag{110}$$

and

$$\mu\left(-i\sigma^2\theta^*\right)^\dagger\left(-i\sigma^2\theta^*\right)=-\mu\theta^\dagger\theta \tag{111}$$

so that

$$\begin{aligned}\mathcal{L}(p)\supset&\begin{pmatrix}\psi_L^\dagger(-p) & \theta^\dagger(-p) & \chi^\dagger(-p)\end{pmatrix}\begin{pmatrix}\bar{\sigma}^\mu p_\mu & 0 & 0\\ 0 & \bar{\sigma}^\mu p_\mu & 0\\ 0 & 0 & \bar{\sigma}^\mu p_\mu\end{pmatrix}\begin{pmatrix}\psi_L(p)\\ \theta(p)\\ \chi(p)\end{pmatrix}+\\ &+\begin{pmatrix}\psi_L^\dagger(-p) & \theta^\dagger(-p) & \chi^\dagger(-p)\end{pmatrix}\begin{pmatrix}-\mu & 0 & 0\\ 0 & \mu & \sqrt{2}igv_+\\ 0 & -\sqrt{2}igv_+ & 0\end{pmatrix}\begin{pmatrix}\psi_L(p)\\ \theta(p)\\ \chi(p)\end{pmatrix}+\\ &+\begin{pmatrix}\psi_L^\dagger(-p) & \psi_R^\dagger(-p)\end{pmatrix}\begin{pmatrix}0 & -m\\ -m & 0\end{pmatrix}\begin{pmatrix}\psi_L(p)\\ \psi_R(p)\end{pmatrix}\end{aligned} \tag{112}$$

Now we can diagonalize to get the eigenvalues

$$\left\{-\mu,\frac{1}{2}\left(\mu-\sqrt{\mu^2+8g^2v_+^2}\right),\frac{1}{2}\left(\mu+\sqrt{\mu^2+8g^2v_+^2}\right)\right\} \tag{113}$$

and a projector matrix

$$P=\begin{pmatrix}1 & 0 & 0\\ 0 & \frac{i\left(\mu-\sqrt{\mu^2+8g^2v_+^2}\right)}{2\sqrt{2}gv_+} & \frac{i\left(\mu+\sqrt{\mu^2+8g^2v_+^2}\right)}{2\sqrt{2}gv_+}\\ 0 & 1 & 1\end{pmatrix} \tag{114}$$

with its inverse

$$P^{-1}=\begin{pmatrix}1 & 0 & 0\\ 0 & \frac{i\sqrt{2}gv_+}{\sqrt{\mu^2+8g^2v_+^2}} & \frac{1}{2}\left(1+\frac{\mu}{\sqrt{\mu^2+8g^2v_+^2}}\right)\\ 0 & -\frac{i\sqrt{2}gv_+}{\sqrt{\mu^2+8g^2v_+^2}} & \frac{1}{2}\left(1-\frac{\mu}{\sqrt{\mu^2+8g^2v_+^2}}\right)\end{pmatrix} \tag{115}$$

The new basis is then

$$\begin{pmatrix}\psi_L\\ \psi_2\\ \psi_3\end{pmatrix}=P^{-1}\begin{pmatrix}\psi_L\\ \theta\\ \chi\end{pmatrix}=\begin{pmatrix}\psi_L\\ \left(\frac{\sqrt{2}igv_+}{\sqrt{\mu^2+8g^2v_+^2}}\right)\theta+\left(\frac{\mu+\sqrt{\mu^2+8g^2v_+^2}}{2\sqrt{\mu^2+8g^2v_+^2}}\right)\chi\\ -\left(\frac{\sqrt{2}igv_+}{\sqrt{\mu^2+8g^2v_+^2}}\right)\theta-\left(\frac{\mu-\sqrt{\mu^2+8g^2v_+^2}}{2\sqrt{\mu^2+8g^2v_+^2}}\right)\chi\end{pmatrix} \tag{116}$$

and, of course,

$$P\begin{pmatrix}\psi_L\\ \psi_2\\ \psi_3\end{pmatrix}=\begin{pmatrix}\psi_L\\ \theta\\ \chi\end{pmatrix}$$
$$PB=A \tag{117}$$

by definition. Let's write schematically the propagator terms as

$$\begin{aligned}A^\dagger D_0 A+A^\dagger M A &= A^\dagger P P^{-1} D_0 P P^{-1} A + A^\dagger P P^{-1} M P P^{-1} A = \\ &= A^\dagger P D_0 P^{-1} A + A^\dagger P D_1 P^{-1} A = A^\dagger P D_0 B + A^\dagger P D_1 B\end{aligned} \tag{118}$$

Now because $P$ is in general not anti-Hermitian we introduce $R=P^\dagger P$, so that we find

$$A^\dagger D_0 A+A^\dagger M A = B^\dagger (R D_0 + R D_1) B \tag{119}$$

where

$$R=\begin{pmatrix}1 & 0 & 0\\ 0 & 2-\frac{2\mu}{\mu+\sqrt{\mu^2+8g^2v_+^2}} & 0\\ 0 & 0 & 2+\frac{\mu\left(\mu+\sqrt{\mu^2+8g^2v_+^2}\right)}{4g^2v_+^2}\end{pmatrix} \tag{120}$$

The fermionic Langragian mixing the two bases becomes

$$\begin{aligned}\mathcal{L}\supset\ &\psi_L^\dagger(-p)\left(\bar\sigma^\mu p_\mu-\mu\right)\psi_L(p)+\psi_2^\dagger(-p)\left[2\bar\sigma^\mu p_\mu+\left(\mu-\sqrt{\mu^2+8g^2v_+^2}\right)\right]\left[\frac{\sqrt{\mu^2+8g^2v_+^2}}{\mu+\sqrt{\mu^2+8g^2v_+^2}}\right]\psi_2(p)\\ &+\psi_3^\dagger(-p)\left[2\bar\sigma^\mu p_\mu+\left(\mu+\sqrt{\mu^2+8g^2v_+^2}\right)\right]\left[1+\frac{\mu\left(\mu+\sqrt{\mu^2+8g^2v_+^2}\right)}{8g^2v_+^2}\right]\psi_3(p)\\ &-m\psi_L^\dagger(-p)\psi_R(p)-m\psi_R^\dagger(-p)\psi_L(p)+\mathcal{L}_{Yuk}[\psi_L,\psi_R,\chi]\end{aligned} \tag{121}$$

## VIII. APPENDIX: MAJORANA FERMIONS *VS.* DIRAC FERMIONS

In the target theory $\chi$ is a Majorana field. Let's use the Majorana representation of the gamma matrices, in which all $\gamma_\mu$ are pure imaginary, and the charge conjugation matrix $C$ can be set to be unity. Then the charge conjugate of a spinor $\psi$ is just $\psi_c=\psi^*$. A Majorana spinor field satisfies the relation $\psi^c=\pm\psi$, that is, it is an eigenstate of the charge conjugation operator.

Let $\psi$ be a Dirac fermion field, and consider the Dirac ass term $\bar\psi\psi=\psi^\dagger\gamma_0\psi$. The mass term, which is Hermitian, is number, not a matrix, and hence $(\bar\psi\psi)^T=\bar\psi\psi$. On the other hand, note that $\overline{(\psi^c)}=\psi^T\gamma_0$. Hence $\overline{(\psi^c)}\psi^c=\psi^T\gamma_0\psi^*=+(\psi^\dagger\gamma_0\psi)^T$, where the third equality follows from $\gamma_0^T=-\gamma^0$ and $(\psi^\dagger\psi)^T=-\psi^\dagger\psi$. This shows that $\overline{(\psi^c)}\psi^c=\bar\psi\psi$. On the hand, $\overline{(\psi^c)}\gamma_0\psi^c=\psi^T\psi^*=-(\psi^\dagger\psi)^*=-\psi^\dagger\psi=-\bar\psi\gamma_0\psi$, where the second equality comes from the fact that $\psi$ is Grassmann, while third equality follows from the reality of $\psi^\dagger\psi$, which is a charge density and hence most certainly real. These observations allow us to write

$$m\bar\psi\psi=\tfrac{1}{2}m\left(\bar\psi\psi+\overline{(\psi^c)}\psi^c\right) \tag{122}$$
$$\mu\bar\psi\gamma_0\psi=\frac{\mu}{2}\left(\bar\psi\gamma_0\psi-\overline{(\psi^c)}\gamma_0\psi^c\right). \tag{123}$$

To track what happens for the kinetic term, we have to be a bit more careful. Consider the $p_0$ piece of the kinetic term, $p_0\bar\psi(-p_0)\gamma_0\psi(p_0)$. Now $\psi(p_0)^c=\psi^*(-p_0)$, but then $\overline{[\psi(p_0)^c]}=\psi^T(p_0)\gamma_0$, and $\overline{[\psi(p_0)^c]}\gamma_0\psi^c(-p_0)=\psi^T(p_0)\psi^*(-p_0)=+(\psi^\dagger(p_0)\gamma_0\psi(-p_0))^*$. The derivative involved in the kinetic term sits between the two fields and acts to the right. Hence if $\bar\psi(-p_0)\gamma_0\psi(p_0)$ is multiplied by $p_0$, then $\overline{[\psi(p_0)^c]}\gamma_0\psi^c(-p_0)$ is multiplied by $-p_0$, and we obtain

$$\bar\psi\not{p}\psi=\tfrac{1}{2}\left(\bar\psi\not{p}\psi+\overline{(\psi^c)}\not{p}\psi^c\right). \tag{124}$$

We now define two Majorana spinor fields $\psi_M^1, \psi_M^2$ as

$$\psi_M^1 = \frac{1}{\sqrt{2}}\left(\psi + \psi^c\right) \tag{125}$$

$$\psi_M^2 = \frac{1}{\sqrt{2}}\left(\psi - \psi^c\right). \tag{126}$$

Note that $\psi_M^{1,2}$ are even and odd under $C$ respectively, and

$$\psi = \frac{1}{\sqrt{2}}\left(\psi_M^1 + \psi_M^2\right), \quad \psi^c = \frac{1}{\sqrt{2}}\left(\psi_M^1 - \psi_M^2\right) \tag{127}$$

$$\bar{\psi} = \frac{1}{\sqrt{2}}\left(\bar{\psi}_M^1 + \bar{\psi}_M^2\right), \quad \overline{(\psi^c)} = \frac{1}{\sqrt{2}}\left(\bar{\psi}_M^1 - \bar{\psi}_M^2\right). \tag{128}$$

Then multiplying terms we have

$$\bar{\psi}\psi = \frac{1}{2}\left(\bar{\psi}_M^1\psi_M^1 + \bar{\psi}_M^2\psi_M^2 + \bar{\psi}_M^2\psi_M^1 + \bar{\psi}_M^1\psi_M^2\right) \tag{129}$$

$$\overline{(\psi^c)}\psi^c = \frac{1}{2}\left(\bar{\psi}_M^1\psi_M^1 + \bar{\psi}_M^2\psi_M^2 - \bar{\psi}_M^2\psi_M^1 - \bar{\psi}_M^1\psi_M^2\right), \tag{130}$$

with appropriate insertions of gamma matrices in the other relevant cases. Putting everything together, we see that

$$\bar{\psi}\psi = \tfrac{1}{2}\left(\bar{\psi}_M^2\psi_M^1 + \bar{\psi}_M^1\psi_M^2\right) \tag{131}$$

$$\bar{\psi}\not{p}\psi = \tfrac{1}{2}\left(\bar{\psi}_M^1\not{p}\psi_M^1 + \bar{\psi}_M^2\not{p}\psi_M^2\right) \tag{132}$$

$$\mu\bar{\psi}\gamma_0\psi = \frac{\mu}{2}\left(\bar{\psi}_M^1\gamma_0\psi_M^2 + \bar{\psi}_M^2\gamma_0\psi_M^1\right). \tag{133}$$

Finally, it is convenient to adopt the convention that all Majorana fields are even under charge conjugation, in which case we must shift $\psi_M^2 \to i\psi_M^2$. The only place where this makes a visual difference is in the expression for the chemical potential term, which becomes

$$\mu\bar{\psi}\gamma_0\psi = \frac{i\mu}{2}\left(\bar{\psi}_M^1\gamma_0\psi_M^2 - \bar{\psi}_M^2\gamma_0\psi_M^1\right). \tag{134}$$

All of this makes sense. Two decoupled Majorana fermions with identical mass terms map to a Dirac fermion, just as the old folklore proclaims, and the chemical potential for the $U(1)$ phase symmetry of the Dirac Lagragian becomes a chemical potential for the $SO(2)$ symmetry rotating the two degenerate Majorana fermions. In terms of a doublet Majorana field $\Psi_M$, the action is just

$$\mathcal{L} = \bar{\psi}(\not{p} - m - \mu\gamma_0)\psi = \frac{1}{2}\bar{\Psi}_M\left(\not{p} - m - \mu\gamma_0\tau_2\right)\Psi_M \tag{135}$$

where $\tau_2$ is the Hermitian flavor generator (numerically just the Pauli matrix $\sigma_2$), which generates the $SO(2)$ flavor group by exponentiation $\exp(i\tau_2\theta) = SO(2)$.

## IX. APPENDIX: PFAFFIANS

Consider a Majorana fermion $\chi$. By definition, $\chi = C\gamma_0\bar{\chi}^T$. Hence $\bar{\chi} = \chi^T C$. This has a perhaps surprising implication for the path integral over fermions. For Dirac fermions $\psi$, the Euclidean path integral is defined by integrating over $\psi$ and $\bar{\psi}$ independently, and that gives the determinant:

$$\int d\psi d\bar{\psi}\, e^{-\bar{\psi}K\psi} = \det(K). \tag{136}$$

However, for Majorana fermions it would be inconsistent to treat $\chi$ and $\bar{\chi}$ independently, even in Euclidean space, thanks to the relation $\bar{\chi} = \chi^T C$ Indeed, the overall factor of $1/2$ in the Majorana fermion action in 4-component notation is hint that a standard functional determinant cannot be the result of integrating over Majorana fermions.

The 1/2 factor suggests that the right answer should be something like $\det(K)^{1/2}$. This is actually correct up to a *non-trivial* phase. It turns out that

$$\int d\chi \, e^{-\frac{1}{2}\chi^T C K \chi} = \mathrm{Pf}(CK). \tag{137}$$

The Pfaffian Pf$M$ of a matrix $M$ is defined for even-dimensional matrices, and is only non-zero for antisymmetric matrices. It obeys the identity $\mathrm{Pf}(M)^2 = \det(M)$. The fact that there is a non-trivial phase difference between roots of determinants and Pfaffians is (I think) not important in perturbation theory, but can be important non-perturbatively.

I have checked numerically that the determinant identity in Eq. (46) holds for Pfaffians.

## X. APPENDIX: NAMBU-GOTOING EXERCISE

An outline of an important example

1. Consider the non-relativistic field theory described by the Lagrangian

$$\mathcal{L} = \psi^\dagger \left( i\partial_0 + \frac{\nabla^2}{2M} + \mu \right) \psi - \frac{g}{2M} (\psi^\dagger \psi)^2 \tag{138}$$

   where $\psi$ is a complex scalar field, and $g$ has mass dimension $-1$. Fact you should not bother showing unless you're a glutton for punishment: in the $\bar{M}S$ scheme, one relates $g$ to the scattering length $a$ via $g = 4\pi a$. This describes a gas of spinless bosons with a repulsive contact interaction in 'second quantized formalism', if you're talking to cond-mat people.

2. Show that a perturbative treatment of this theory makes sense in the dilute-gas limit where $n^{1/3}/a \gg 1$, and $n$ is the particle density at tree-level.

3. Assume that $\mu < 0$, and find the vev of $\psi$ at tree level.

4. Expand $\psi$ around its vev as $\psi = v + \chi$, where $\chi$ is a *complex* field. (Working with a radial parametrization is also instructive, but won't show the lesson I want to show at the moment.)

5. Convince yourself that to work out the lowest-order quasiparticle spectrum it suffices to look at the terms which are at most quadratic on $\chi$.

6. Observe the appearance of terms like $\chi\chi$ and $\chi^\dagger\chi^\dagger$ in the quadratic action. These are weird: they represent $\chi$ modes entering and leaving the condensate in pairs. Obviously such a process better be a virtual one...

7. Diagonalize the action, which amounts to finding the propagator and hence the spectrum. Hint: To do this go to momentum space, and write the action as a $2 \times 2$ matrix with the help of the new 'column vector' field $(\chi(p)\, \chi^\dagger(-p))^T$. This is the 'Nambu-Gor'kov spinor', except it's a boson.

8. The quasiparticle you now see is the Nambu-Goldstone boson. Bonus: where is the Higgs mode?

9. Super-bonus you should not do unless you have a good reason: from here to getting the first correction to the energy density of a Bose gas as a function of density, as done in e.g. Landau-Lifshitz Stat Phys Vol II, page 101, is roughly a page of calculation. Marvel at the beauty and power of modern QFT technology as presented by your favorite consultants from Deloitte and Stanford, compared to the hoops that Landau and Lifshitz jump through. QFT technology is great and we are great.

---


[1] J. Maldacena, "The Large N Limit of Superconformal Field Theories and Supergravity," Adv. Theor. Math. Phys. **2**, 231–252 (1998).

[2] J. Polchinski, *String Theory*, Cambridge University Press, Cambridge (2005).

[3] E. Witten, "Anti-de Sitter Space and Holography," Adv. Theor. Math. Phys. **2**, 253–291 (1998).

[4] O. Aharony, S. S. Gubser, J. Maldacena, H. Ooguri and Y. Oz, "Large N Field Theories, String Theory and Gravity," Phys. Rept. **323**, 183–386 (2000).

[5] I. R. Klebanov and M. J. Strassler, "Supergravity and a Confining Gauge Theory: Duality Cascades and $\chi$SB-Resolution of Naked Singularities," JHEP **2000**, 052 (2000).
[6] S. A. Hartnoll, C. P. Herzog and G. T. Horowitz, "Building a Holographic Superconductor," Phys. Rev. Lett. **101**, 031601 (2008).
[7] S. Sachdev, "Strange Metals and the AdS/CFT Correspondence," J. Stat. Mech. **2011**, P11022 (2011).
[8] M. J. Stephenson, "Identification of Universal Features in the Conductivity of Classes of Two-Dimensional QFTs Using the AdS/CFT Correspondence," Fund. J. Mod. Phys. **20**, 25–35 (2023).