Risk-Aware Safe Optimal Control of Uncertain Linear Systems

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***Abstract*—This paper synthesizes a risk-aware safe optimal controller for partially unknown linear systems under additive Gaussian noises. The risk is assessed through the concept of Conditional Value-at-Risk (CVaR) to account for the extreme low-probability events that occurred in a one-step cost function without being overly conservative. The safety of the CVaR optimization solution is also guaranteed with high probability by imposing a chance constraint. A state-feedback risk-aware controller is first obtained that provides an upper bound to the formulated safe CVaR optimization problem. Then, an online data-driven quadratic programming (QP) optimization problem is devised to simultaneously and safely learn the unknown dynamics and control the system with high probability. As more measurements are collected, the safety constraint is tightened due to increasing the confidence in estimating the dynamic model. In the end, a numerical example is given to elucidate the efficacy of the proposed method.**

I. Introduction

By increasing the deployment of autonomous systems in real-world settings, concerns regarding their safety have gained increasing attention. Formally, the safety is guaranteed if the closed-loop system trajectory remains within an admissible subset of its state space, called safe set [1]. Several tools, such as control barrier functions (CBF) [2], have been developed to certify the safety of the closed-loop systems. However, a reliable dynamic model is typically required in CBF-based approaches which may not be available in real-world settings due to the uncertainties and unforeseen circumstances.

Uncertainties are typically categorized as epistemic uncertainty (i.e, lack of knowledge) and aleatory uncertainties (i.e., randomness). Several risk measures have been leveraged to deal with aleatory uncertainties [3], [4]. As a risk measure, the chance-constrained optimization (CCO) has been used to impose probabilistic safety guarantees [5] in the safe control design. Besides safety, a risk factor can be imposed on the objective function of the optimal control design problem to avoid performance fluctuation. However, guaranteeing an optimal performance with high probability using the CCO formulation of the optimal control is a daunting challenge. Chance constraints have been widely used in stochastic model predictive control (SMPC) [6] in which the performance and robustness are balanced by solving a distributionally robust optimization problem. However, heavy online computations are required to solve SMPC. Currently, there are two methods to handle chance constraints in solving optimal control problems [7]: probabilistic approximation [8], e.g., using Chebyshev bounds, and sampling-based or scenario approaches [9].

Another commonly used risk measure is the Conditional Value-at-Risk (CVaR), defined as the conditional expectation of the loss function exceeding a predefined threshold [10]. CVaR has gained popularity and acceptance in optimal control theory [11] due to being a coherent risk measure. Recently, by applying the worst-case CVaR, a risk-aware optimal controller is developed [12] for three different quadratic cost functions, which are stationary, one-step and infinite-time horizon. However, the safety constraints are ignored in [12]. Also, epistemic uncertainties are not taken into account and the complete knowledge of system dynamic is required in [12]. Epistemic uncertainties are, however, commonplace in most real-work applications for which an accurate system model is not available and must be learned using data. As a result, designing a data-driven risk-aware safe optimal controller is lacking in the literature, which motivates this research.

In this paper, a risk-averse safe optimal control design method is presented for partially unknown linear systems. Unlike [13], the risk is taken into account in the loss function to account for the extreme events that might lead the autonomous systems into the catastrophic conditions. Also, compared to [12], the proposed controller benefits from the chance constraints to guarantee the safety with high probability. Thanks to [13], the proposed method also enjoys safety in simultaneous learning and control.

A. Notation

Throughout the paper $\mathbb{R}^n$ and $\mathbb{R}^{n\times m}$ denote, respectively, the set of $n \times 1$ real column vectors and the set of $n \times m$ real matrices. $x^+ = max(0, x)$ for any $x \in \mathbb{R}$. $\mathbb{S}^n_+$ and $\mathbb{S}^n_{++}$ denote the set of positive semi-definite and positive definite matrices in $\mathbb{R}^{n\times n}$, respectively. We use $\|.\|_F$ to denote the Frobenius norm. $Tr(A)$ denotes the trace of matrix A.

All random variables are defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where Ω is the sample space, $\mathcal{F}$ is its associated Borel σ-algebra and $\mathbb{P}$ is the probability measure. That is $\mathbb{P}$ is a non-negative countably additive set function on $\mathcal{F}$ such that $\mathbb{P}(\Omega) = 1$. For a random variable $w : \Omega \longrightarrow \mathbb{R}^n$ defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, with some abuse of notation, the statement $w \in R^q$ is used to state the dimension of the random variable. $\mathbb{E}_{\mathbb{P}}[.]$ denotes the expectation with respect to the distribution $\mathbb{P}$. Finally, $w \sim \mathcal{N}(\mu, \Sigma)$ denotes a multivariate Gaussian random vector with the mean μ and the covariance Σ.

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$\Phi^{-1}(.) \geq 0$ is the inverse of cumulative distribution function of a zero mean unit variance Gaussian random variable.

II. Preliminaries and Problem Statement

Consider the following discrete-time linear control system

$$x_{t+1} = Ax_t + Bu_t + \omega_t, \tag{1}$$

where $x_t \in \mathbb{R}^{n_x}$ and $u_t \in \mathbb{R}^{n_u}$ are the state of the system and control input, respectively. Moreover $\omega_t \in \mathbb{R}^n$ represents the system noise.

Assumption 1. *The noise ω_t is zero mean Gaussian with variance Σ. That is, $\omega_t \sim \mathcal{N}(0, \Sigma)$*

Assumption 2. *There exist some known constants $r > 0$ and $s > 0$ such that $\|A\|_F \leq s$ and $\Sigma \leq rI$.*

Definition 1 (Conditional Value-at-Risk [14])**.** *For a given measurable loss function $L : \mathbb{R}^n \to \mathbb{R}$ as a function of random vector $\zeta \in \mathbb{R}^n$ distributed with probability $\mathbb{P}$, and tolerance $\epsilon \in (0, 1]$, the CVaR of loss function L at level ϵ with respect to the probability distribution $\mathbb{P}$ is defined as*

$$CVaR_{\epsilon}^{\mathbb{P}}(L(\zeta)) = \inf_{\beta\in\mathbb{R}} \left\{ \beta + \frac{1}{\epsilon}\mathbb{E}_{\mathbb{P}}\left[(L(\zeta) - \beta)^+ \right] \right\} \tag{2}$$

Lemma 1 ([12])**.** *Let $L(\zeta) = \zeta^T P\zeta + 2q^T P^{1/2}\zeta + q^T q + r$, with $P \in \mathbb{S}_+^n$, $q \in \mathbb{R}^n$ and $r \in \mathbb{R}$, then the following inequalities hold.*

$$CVaR_{\epsilon}^{\mathbb{P}}(L) \leq \sup_{\mathbb{P}\in\mathcal{P}} CVaR_{\epsilon}^{\mathbb{P}}(L) \leq r + \frac{1}{\epsilon}\left(Tr(\Sigma P) + q^T q\right) \tag{3}$$

where $\mathcal{P}$ is the set of all probability distributions on $\mathbb{R}^n$ that have the same first- and second-order moments as $\mathbb{P}$.

III. Model-based Risk-aware Linear Controller Design

In this section, we first define the risk-aware safe optimal control problem. Then, a suboptimal solution is given for it assuming that the system dynamics are known. Therefore, the provided solution only accounts for aleatory uncertainties and ignores epistemic uncertainties. The subsequent sections will leveraged the solution provided here to provide a data-based solution that accounts for both epistemic and aleatory uncertainties.

Problem 1. *Consider the system* (1)*. Find the optimal controller u_t^* that minimizes the CVaR of one-step cost function* (4a) *such that the time-varying individual safety chance constraint* (4b) *is satisfied.*

$$\min_{u_t\in\mathbb{R}^{n_u}} CVaR_{\epsilon}^{\mathbb{P}_\omega}[x_{t+1}^T Qx_{t+1} + u_t^T Ru_t] \tag{4a}$$

$$s.t.\ \mathbb{P}_\omega\left(G_t x_{t+1} \leq h_t\right) \geq 1 - \eta \tag{4b}$$

where $\mathbb{P}_\omega = \mathcal{N}(\mathbf{0}, \Sigma)$, $Q \in \mathbb{S}_+^{n_x}$, $R \in \mathbb{S}_{++}^{n_u}$, $0 \leq \eta \leq 0.5$, G_t is a row vector, and h_t is a scalar.

Remark 1. *In Problem 1, the risk in the objective function is measured by CVaR to account for the extreme effects of the Gaussian disturbance on the objective function. Note that the guarantees are not intended in a deterministic sense of optimizing against the worst-case realization and safety satisfaction against all possible closed-loop system responses, as the robust optimization does. Instead, optimality and safety guarantees are intended in probabilistic senses of satisfaction of safety constraints against all but a small fraction of the system responses accounting for extreme low-probability events that occur in a one-step cost function. While the robust optimization can be highly conservative since its focus is on a special ill or rare situations, and even lead to infeasibility when the support of the random variable is infinite, the formulated risk-averse safe optimal control problem seeks to minimize not only the mean but also the variance of the performance or cost function while satisfying safety with high probability, leading to a more predictable outcome (i.e., less variance).*

Theorem 1. *Let Assumption 1 hold. Then, the following state-feedback $\hat{u}_t(x_t)$ gives an upper bound to the solution of Problem 1.*

$$\hat{u}_t = -\left(\epsilon R + B^T QB\right)^{-1}\left(B^T QAx_t + \frac{\epsilon}{2}\lambda^T B^T G_t^T\right) \tag{5}$$

where $\lambda \geq 0$ is a Lagrange multiplier defined as

$$\lambda = A_1 x_t - B_1 \tag{6}$$

with

$$A_1 = 2\frac{G_t\left(A - B\left(\epsilon R + B^T QB\right)^{-1} B^T QA\right)}{\epsilon G_t B\left(\epsilon R + B^T QB\right)^{-1} B^T G_t^T},$$

$$B_1 = \frac{2\left(h_t - \Phi^{-1}(1-\eta)\sqrt{G_t^T \Sigma G_t}\right)}{\epsilon G_t B\left(\epsilon R + B^T QB\right)^{-1} B^T G_t^T},$$

if $G_t x_{t+1} + \Phi^{-1}(1-\eta)\sqrt{G_t^T \Sigma G_t} > h_t$, otherwise $\lambda = 0$.

Proof. Instead of minimizing the objective function (4a), the upper-bound of (4a) is obtained according to Lemma 1 as

$$f(u_t) + \frac{1}{\epsilon}Tr(\Sigma_\omega Q) \tag{7}$$

where $f(u_t) = \frac{1}{\epsilon}(Ax_t + Bu_t)^T Q(Ax_t + Bu_t) + u_t^T Ru_t$.

Regarding the safety constraint (4b), one can convert (4b) into the following convex constraint as [5], [15]

$$h_t - Gx_{t+1} \geq \Phi^{-1}(1-\eta)\sqrt{G_t^T \Sigma G_t} \tag{8}$$

Since the objective function and constraint are both convex, we can use the KKT conditions as

$$\frac{2}{\epsilon}B^T Q(Ax_t + Bu_t) + 2Ru_t + \lambda^T B^T G_t^T = 0 \tag{9}$$

$$\lambda^T\left(G_t(Ax_t + Bu_t) + \Phi^{-1}(1-\eta)\sqrt{G_t^T \Sigma G_t} - h_t\right) = 0 \tag{10}$$

As a result, $\hat{u}_t(x_t)$ in (5) can be obtained from (9). By substituting (5) into (10), one has

$$G_t Ax_t - G_t B\left(\epsilon R + B^T QB\right)^{-1}\left(B^T QAx_t + \frac{\epsilon}{2}\lambda^T B^T G_t^T\right) \tag{11}$$

$$- h_t + \Phi^{-1}(1-\eta)\sqrt{G_t^T \Sigma G_t} = 0$$

From the above equation, λ can be found as (6). $\square$

IV. Safe CVaR Control design with unknown dynamics

If A is unknown, the following procedures are made to simultaneously estimate the unknown parameter A and design the control input u_t for the safe CVaR problem.

Let $\theta_i = A_i^T \in \mathbb{R}^{n_x}$, where A_i is the i-th row of matrix A. Also, $(x_i)_t$, and $(\omega_i)_t$ denote i-th entries of vectors x_t and ω_t. The system (1) can be rewritten in terms of state-input data sets as follows

$$(X_i)_t = Z_t\theta_i + (W_i)_t,\ i = 1,\dots,n_x \tag{12}$$

where $(X_i)_t = \begin{bmatrix}(x_i)_1\\ \vdots \\ (x_i)_t\end{bmatrix} - \begin{bmatrix}u_0^T\\ \vdots \\ u_{t-1}^T\end{bmatrix} B_i^T$, $(W_i)_t = \begin{bmatrix}(\omega_i)_0\\ \vdots \\ (\omega_i)_{t-1}\end{bmatrix}$, and $Z_t = \begin{bmatrix}x_0^T\\ \vdots \\ x_{t-1}^T\end{bmatrix}$.

By using the regularized least square method, one has

$$(\hat{\theta}_i)_t = \left(Z_t^T Z_t + \lambda_L I\right)^{-1} Z_t^T (X_i)_t \tag{13}$$

where $(\hat{\theta}_i)_t$ is the estimate of θ_i, $i = 1,\dots,n_x$, and $\lambda_L \geq 0$. Therefore, the estimation of unknown dynamic A is given as

$$\hat{A}_t = \begin{bmatrix}(\hat{\theta}_1^T)_t\\ \vdots \\ (\hat{\theta}_{n_x}^T)_t\end{bmatrix} \tag{14}$$

System (1) is rewritten as

$$x_{t+1} = \hat{A}_t x_t + Bu_t + \nu_t + \omega_t, \tag{15}$$

where $\nu_t = \begin{bmatrix}(\theta_1^T)_t - (\hat{\theta}_1^T)_t\\ \vdots \\ (\theta_{n_x}^T)_t - (\hat{\theta}_{n_x}^T)_t\end{bmatrix} x_t$.

Lemma 2 ([13]). *Let $0 < \delta \leq 1$, $V_t = Z_t^T Z_t + \lambda_L I$, $\beta_t(\delta) = r\sqrt{2\log\left(\det V_t^{1/2}/(\lambda_L^{n_x/2}\delta)\right)} + \lambda_L^{1/2}s$, and $\zeta_t = \|V_t^{-1/2}x_t\|_2$. Under Assumption 2, the following relation is true.*

$$\mathbb{P}_\omega\left\{\left(\omega_t^T\omega_t \leq \frac{2rn_x}{\delta}\right) \wedge \left(\|\nu_t\|_2 \leq \zeta_t n_x \beta_t\left(\frac{\delta}{2n_x}\right)\right)\right\} \geq 1-\delta \tag{16}$$

Problem 2. *Consider the system* (1) *with unknown dynamics A. Find an online data-driven suboptimal control input $\bar{u}_t$ with respect to the CVaR one-step cost function* (4a) *such that the safety chance constraint* (4b) *is satisfied with the probability of at least $1-\delta$.*

To solve Problem 2, by replacing (15) in (8), the chance constraint (4b) is converted to

$$h_t - G_t(\hat{A}_t x_t + Bu_t + \nu_t + \omega_t) \geq \Phi^{-1}(1-\eta)\sqrt{G_t^T\Sigma G_t} \tag{17}$$

where $\omega_t^T\omega_t \leq \frac{2rn_x}{\delta}$ and $\|\nu_t\|_2 \leq \zeta_t n_x\beta_t\left(\frac{\delta}{2n_x}\right)$ are held with the probability of at least $1-\delta$ according to Lemma 2. Thanks to [13, Theorem 3], a computationally-friendly reformulation to the constraint (17) is given as

$$h_t - G(\hat{A}_t x_t + Bu_t) \geq \Phi^{-1}(1-\eta)\sqrt{G_t^T\Sigma G_t} + e_{t+1} \tag{18}$$

where $e_{t+1} = \left(\zeta_t n_x \beta_t(\delta/(2n_x)) + \sqrt{2rn_x/\delta}\right)\|G_{t+1}\|_2$ is a constraint-tightening term.

Remark 2. *The constraint-tightening term e_{t+1} consists of two parts. The first term is related to the estimation error, and the second one is caused by the disturbance. Under the persistence of excitation (PE) condition, by collecting more data sets, the estimation error tends to zero [13], i.e., $\lim_{t\to\infty}\zeta_t n_x\beta_t(\delta/(2n_x)) = 0$. In future works, the PE condition can be relaxed thanks to the concept of concurrent learning [16].*

Theorem 2. *Under Assumptions 1-2, the control input $\bar{u}_t$ obtained from the following data-driven QP optimization problem* (19) *solves Problem 2.*

$$\bar{u}_t = \underset{u_t\in\mathbb{R}^{n_u}}{\arg\min}\ u_t^T E_t u_t + F_t u_t \tag{19a}$$

$$s.t.\ G_t(\hat{A}_t x_t + Bu_t) + \Phi^{-1}(1-\eta)\sqrt{G_t^T\Sigma G_t} \leq h_t - e_{t+1} \tag{19b}$$

where $E_t = B^TQB + R + \frac{2}{\epsilon}C_t B^TQQ^TB$ and $F_t = \frac{4}{\epsilon}C_t x_t^T\hat{A}_t^TQQ^TB$ with $C_t = \zeta_t^2 n_x^2\beta_t^2(\delta/(2n_x))$.

Remark 3. *In the optimization problem* (19)*, we have the concept of safety in not only the control but also in online learning, similar to [13]. However, unlike [13], the concept of risk is given by using CVaR in this research to quantify the importance of the extreme events in the cost function.*

Proof. By applying Lemma 1, and using (7), one can find the upper bound of (4a) as follows

$$\min_{u_t\in\mathbb{R}^{n_u}}\left(\tfrac{1}{\epsilon}\left(\hat{A}_t x_t + Bu_t + \nu_t\right)^T Q\left(\hat{A}_t x_t + Bu_t + \nu_t\right) + u_t^T R u_t\right) \tag{20}$$

where $\nu_t \in \mathbb{A}$, and $\mathbb{A} = \{\nu_t | \nu_t^T\nu_t \leq C_t\}$ according to Lemma 2. To deal with the time-varying estimation error ν_t which belongs to the closed-set $\mathbb{A}$, we consider the worst-case scenario. Therefore, the following min-max optimization problem is given as

$$\min_{u_t\in\mathbb{R}^{n_u}}\max_{\nu_t\in\mathbb{A}}\left(\tfrac{1}{\epsilon}\left(\hat{A}_t x_t + Bu_t + \nu_t\right)^T Q\left(\hat{A}_t x_t + Bu_t + \nu_t\right) + u_t^T R u_t\right) \tag{21a}$$

$$s.t.\ G_t(\hat{A}_t x_t + Bu_t) + \Phi^{-1}(1-\eta)\sqrt{G_t^T\Sigma G_t} \leq h_t - e_{t+1} \tag{21b}$$

First, we solve (21) for ν_t by fixing the control policy u_t. Therefore, one has

$$\max_{\|\nu_t\|_2\leq\sqrt{C_t}} \frac{1}{\epsilon}\left(\nu_t^TQ\nu_t + 2\left(u_t^TB^T + x_t^T\hat{A}_t^T\right)Q\nu_t\right) \tag{22}$$

Algorithm 1 Online Data-driven Safe CVaR Control Problem

1: **Input:** Control matrix B, Risk factor ϵ, Chance constraint probability level η, Confidence level δ, Number of time-steps T, Regularization parameter λ_L, Number of state n_x, Constants r and s.
2: **for** $t = 1$ **to** T **do**
3: Store the state-input data sets in $(X_i)_t$, $i = 1, \dots, n_x$ and Z_t.
4: Find $(\hat{\theta}_i)_t$, $i = 1, \dots, n_x$, by using the regularized least square method (13).
5: Find $\hat{A}_t$ as the estimation of the unknown dynamic A by stacking $(\hat{\theta}_i)_t$ as (14).
6: Solve the data-driven QP optimization problem (19) for $\bar{u}_t$.
7: Apply $\bar{u}_t$ to the system (1), and store the next state x_{t+1}.
8: **end for**

By applying Hölder's inequality, one can rewrite (22) as

$$\frac{1}{\epsilon}\left(\nu_t^T Q \nu_t + 2\left(u_t^T B^T + x_t^T \hat{A}_t^T\right) Q \nu_t\right) \leq \quad (23)$$
$$\frac{\lambda_{max}(Q)}{\epsilon}\nu_t^T\nu_t + 2\left\|\left(u_t^T B^T + x_t^T \hat{A}_t^T\right) Q\right\|_2 \|\nu_t\|_2 \leq$$
$$\frac{\lambda_{max}(Q)}{\epsilon}\nu_t^T\nu_t + 2\left\|\left(u_t^T B^T + x_t^T \hat{A}_t^T\right) Q\right\|_2^2 \nu_t^T\nu_t$$

Therefore, one has

$$\max_{\|\nu_t\|_2 \leq \sqrt{C_t}} \frac{\lambda_{max}(Q)}{\epsilon}\nu_t^T\nu_t + 2\left\|\left(u_t^T B^T + x_t^T \hat{A}_t^T\right) Q\right\|_2^2 \nu_t^T\nu_t \quad (24)$$
$$= \frac{\lambda_{max}(Q)}{\epsilon} C_t + 2\left\|\left(u_t^T B^T + x_t^T \hat{A}_t^T\right) Q\right\|_2^2 C_t$$

Considering (24) along with (21b) results in (19). □

To implement the proposed safe CVaR data-driven controller, Algorithm 1 is given to illustrate the detailed procedure.

V. Numerical Example

Consider the discretized linear model of steering system in autonomous vehicles as [17]

$$A = \begin{bmatrix} 1 & 0.2 \\ 0 & 1 \end{bmatrix}, \ B = \begin{bmatrix} 0.06 \\ 0.20 \end{bmatrix} \quad (25)$$

The performance constants are defined as $R = 1$ and $Q = 10I_{2\times 2}$. The safety constraint is imposed on the heading angle such that $G = \begin{bmatrix} 0 & -1 \end{bmatrix}$ and $h = 0.1$. The initial condition is defined as $x_1 = [1,\ 1]^T$. The covariance of the Gaussian disturbance is $\Sigma = \begin{bmatrix} 0.010 & 0.003 \\ 0.003 & 0.020 \end{bmatrix}$. The system is simulated for $T = 200$ time steps. Figure 1 depicts the evolution of the heading angle versus time-steps T for different safety probability levels $\eta = 0.2$ and $\eta = 0.5$, and a fixed risk level $\epsilon = 0.2$. The chance constraint (4b) is plotted with the red dotted line. The system is expected to satisfy this constraints with at least 80% (left hand figure) and 50% probability (right hand figure). As can be seen from Fig. 1, by decreasing the probability level η, the number of trajectories violated the safe constraints is decreased, which means the probability of violation is decreased.

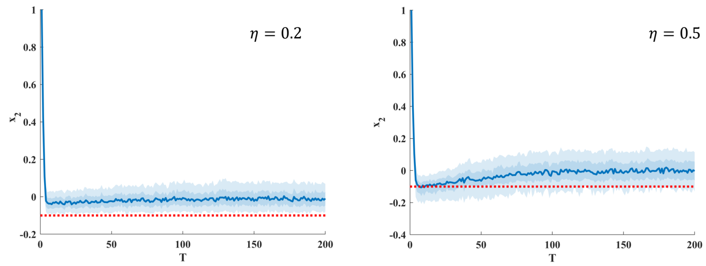

Fig. 1. The evolution of the heading angle for different safety probability levels. Shaded blue areas represent the 50% and 75% confidence bounds for 500 independent experiments. The safety constraint is plotted with the red dotted line.

To illustrate the efficacy of using CVaR in the proposed method, Fig. 2 depicts the histogram of loss function evaluated by 500 independent experiments with a fixed safety probability $\eta = 0.2$. The left-hand side histogram is plotted for risk-neutral case in which the expectation of loss function is considered. On the other hand, the risk is considered in the right-hand side figure to capture the extreme effects of the noise on the loss function. As can be observed from this figure, the risk-neutral case has a long tail compared to the risk-averse one.

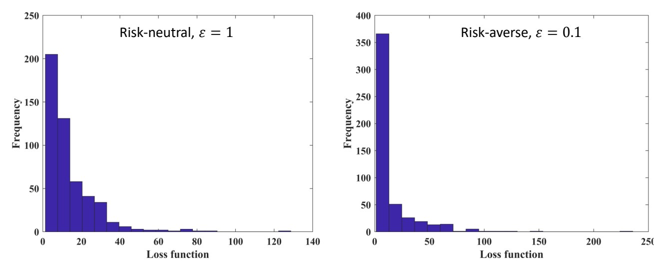

Fig. 2. Histograms of loss function for different risk levels.

In Fig.3, the control input u_t is plotted for different risk levels. As can be observed from this figure, the risk-averse controller has tight confidence bounds compared to the risk-neutral case.

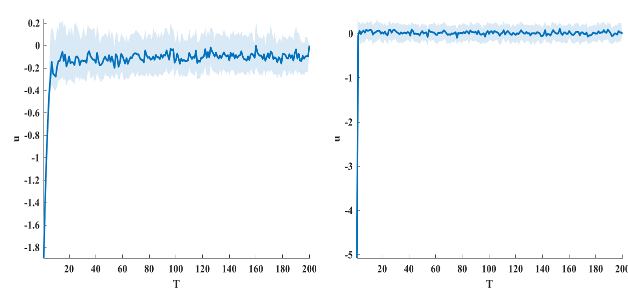

Fig. 3. The evolution of u_t for different risk levels.

VI. Conclusions

This paper is focused on designing a risk-aware safe sub-optimal controller for unknown linear systems under additive Gaussian noises. To measure the risk in the loss function, which is here a one-step quadratic cost function, the concept of CVaR is leveraged to account for the extreme events that might lead the autonomous systems into catastrophic conditions. In the meanwhile, the safety is guaranteed in probability thanks to applying chance constraints. To solve this problem, a state-feedback sub-optimal controller is designed, and we have shown that the safety is guaranteed in probability. Also, a data-driven method is devised by providing a QP optimization problem in which the safety is guaranteed in simultaneous learning and control. Finally, the proposed controller is examined by a numerical example.

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