Periodic Solutions of Degenerate Riemann-Liouville fractional equations

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Abstract. The aim of this work is to study the Solutions of degenerate Riemann-Liouville fractional integro-differential equations $\frac{d}{dt}(\frac{M}{\Gamma(1-\alpha)}\int_{-\infty}^{t}(t-s)^{-\alpha}x(s)ds) = Ax(t) + \int_{-\infty}^{t} a(t-s)x(s)ds + \frac{1}{\Gamma(\beta)}\int_{-\infty}^{t}(t-s)^{\beta-1}x(s)ds + f(t)$. Our approach is based on the R-boundedness of linear operators L^p-multipliers and UMD-spaces.

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1. Introduction

The aim of this paper is to study the existence and of solutions Riemann-Liouville fractional integro-differential equations by using methods of maximal regularity in spaces of vector valued functions.
In this work, we study the existence of periodic solutions for the following Riemann-Liouville fractional integro-differential equations

$$\begin{aligned}\frac{d}{dt}(\frac{M}{\Gamma(1-\alpha)}\int_{-\infty}^{t}(t-s)^{-\alpha}x(s)ds) &= Ax(t) + \int_{-\infty}^{t} a(t-s)x(s)ds \\ &+\frac{1}{\Gamma(\beta)}\int_{-\infty}^{t}(t-s)^{\beta-1}x(s)ds + f(t); \quad 0 \le t \le 2\pi\end{aligned} \tag{1.1}$$

where $\Gamma(.)$ is the Euler gamma function, $\alpha, \beta \in \mathbb{R}^+, 0 \le \beta \le \alpha$, A and M are a linear closed operators on Banach space $(X, \|.\|)$ such that $D(A) \subseteq D(M)$, $f \in L^p([-r_{2\pi}, 0], X)$ for all $p \ge 1$ and $r_{2\pi} := 2\pi N$ (some $N \in \mathbb{N}$), $a \in L^1(\mathbb{R}_+)$, and x_t is an element of $L^p([-r_{2\pi}\ \ , 0],\ \ X)$ which is defined as follows

$$x_t(\theta) = x(t+\theta) \ \text{ for } \ \theta \in [-r_{2\pi},\ \ 0].$$

In [4], Aparicio et al, studied the existence of periodic solution of degenerate integro-differential equations in function spaces described in the following form:

$$(Mu')'(t) - \Lambda u'(t) - \frac{d}{dt}\int_{-\infty}^{t} c(t-s)u(s)ds = \gamma u(t) + Au(t) + \int_{-\infty}^{t} b(t-s)Bu(s)ds + f(t),$$

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and periodic boundary conditions $u(0) = u(2\pi)$, $(Mu')(0) = (Mu')(2\pi)$. Here, A, B, Λ and M are closed linear operators in a Banach space X satisfying the assumption $D(A) \cap D(B) \subset D(\Lambda) \cap D(M)$, $b, c \in L^1(\mathbb{R}_+)$, f is an X-valued function defined on $[0, 2\pi]$, and γ is a constant.
In [21], S.Koumla, Kh.Ezzinbi, R.Bahloul established mild solutions for some partial functional integrodifferential equations with finite delay

$$\frac{d}{dt}x(t) = Ax(t) + \int_0^t B(t-s)x(s)ds + f(t, x_t) + h(t, x_t)$$

where $A : D(A)X \to X$ is the infinitesimal generator of a C_0-semigroup $(T(t))_{t\geq 0}$ on a Banach space X, for $t \geq 0, B(t)$ is a closed linear operator with domain $D(B) \supset D(A)$.

This work is organized as follows : In Section 2 we collect some preliminary results and definitions. In section 3, we study the existence and uniqueness of strong L^p-solution of the Eq. (1.1) solely in terms of a property of R-boundedness for the sequence of operators $(ik)^\alpha((ik)^\alpha M - A - \tilde{a}(ik) - (ik)^{-\beta}I)^{-1}$. We optain that the following assertion are equivalent in UMD space :

(1): $((ik)^\alpha M - A - \tilde{a}(ik) - (ik)^{-\beta}I)$ is invertible and $\{((ik)^\alpha((ik)^\alpha M - A - \tilde{a}(ik) - (ik)^{-\beta}I)^{-1}, k \in \mathbb{Z}\}$ is R-bounded.
(2): For every $f \in L^p(\mathbb{T}; X)$ there exist a unique function $u \in H^{\alpha,p}(\mathbb{T}; X)$ such that $u \in D(A)$ and equation (1.1) holds for a.e $t \in [0, 2\pi]$.

2. Preliminaries

In this section, we collect some results and definitions that will be used in the sequel. Let X be a complex Banach space. We denote as usual by $L^1(0, 2\pi, X)$ the space of Bochner integrable functions with values in X. For a function $f \in L^1(0, 2\pi; X)$, we denote by $\hat{f}(k), k \in \mathbb{Z}$ the kth Fourier coefficient of f:

$$\hat{f}(k) = \frac{1}{2\pi}\int_0^{2\pi} e_{-k}(t)f(t)dt,$$

where $e_k(t) = e^{ikt}, t \in \mathbb{R}$.

Let $u \in L^1(0, 2\pi; X)$. We denote again by u its periodic extension to $\mathbb{R}$. Let $a \in L^1(\mathbb{R}_+)$. We consider the the function

$$F(t) = \int_{-\infty}^t a(t-s)u(s)ds, \quad t \in \mathbb{R}.$$

Since

$$F(t) = \int_{-\infty}^t a(t-s)u(s)ds = \int_0^\infty a(s)u(t-s)ds, \tag{2.1}$$

we have $\|F\|_{L^1} \leq \|a\|_1\|u\|_{L^1} = \|a\|_{L^1(\mathbb{R}_+)}\|u\|_{L^1(0,2\pi;X)}$ and F is periodic of period $T = 2\pi$ as u. Now using Fubini's theorem and (2.1) we obtain, for

$k \in \mathbb{Z}$, that

$$\hat{F}(k) = \tilde{a}(ik)\hat{u}(k), k \in \mathbb{Z} \tag{2.2}$$

where $\tilde{a}(\lambda) = \int_0^\infty e^{-\lambda t} a(t)dt$ denotes the Laplace transform of a. This identity plays a crucial role in the paper.

Let X, Y be Banach spaces. We denote by $\mathcal{L}(X,Y)$ the set of all bounded linear operators from X to Y. When $X = Y$, we write simply $\mathcal{L}(X)$.

Proposition 2.1 ([2, Fejer's Theorem]). *Let $f \in L^p(0, 2\pi; X))$, then one has*

$$f = \lim_{n\to\infty} \frac{1}{n+1} \sum_{m=0}^{n} \sum_{k=-m}^{m} e_k \hat{f}(k)$$

with convergence in $L^p(0, 2\pi; Y)$.

R-boundedness-UMD space,L^p-multiplier and Riemann-Liouville fractional integral. We shall frequently identify the spaces of (vector or operator-valued) functions defined on $[0, 2\pi]$ to their periodic extensions to $\mathbb{R}$.

For $j \in \mathbb{N}$, denote by r_j the j-th Rademacher function on $[0, 1]$, i.e. $r_j(t) = sgn(\sin(2^j \pi t))$. For $x \in X$ we denote by $r_j \otimes x$ the vector valued function $t \to r_j(t)x$.

The important concept of R-bounded for a given family of bounded linear operators is defined as follows.

Definition 2.2. A family $\mathbf{T} \subset \mathcal{L}(X,Y)$ is called R-bounded if there exists $c_q \geq 0$ such that

$$\|\sum_{j=1}^{n} r_j \otimes T_j x_j\|_{L^q(0,1;X)} \leq c_q \|\sum_{j=1}^{n} r_j \otimes x_j\|_{L^q(0,1;X)} \tag{2.3}$$

for all $T_1, \dots, T_n \in \mathbf{T}, x_1, \dots, x_n \in X$ and $n \in \mathbb{N}$, where $1 \leq q < \infty$. We denote by $R_q(\mathbf{T})$ the smallest constant c_q such that (2.3) holds.

Remark 2.3. Several useful properties of R-bounded families can be found in the monograph of Denk-Hieber-Prüss [14, Section 3], see also [1, 2, 12, 25, 22]. We collect some of them here for later use.

(a) Any finite subset of $\mathcal{L}(X)$ is is R-bounded.
(b) If $\mathbf{S} \subset \mathbf{T} \subset \mathcal{L}(X)$ and $\mathbf{T}$ is R-bounded, then $\mathbf{S}$ is R-bounded and $R_p(\mathbf{S}) \leq R_p(\mathbf{T})$.
(c) Let $\mathbf{S}, \mathbf{T} \subset \mathcal{L}(X)$ be R-bounded sets. Then $\mathbf{S} \cdot \mathbf{T} := \{S \cdot T : S \in \mathbf{S}, T \in \mathbf{T}\}$ is R-bounded and

$$R_p(\mathbf{S} \cdot \mathbf{T}) \leq R_p(\mathbf{S}) \cdot R_p(\mathbf{T}).$$

(d) Let $\mathbf{S}, \mathbf{T} \subset \mathcal{L}(X)$ be R-bounded sets. Then $\mathbf{S} + \mathbf{T} := \{S + T : S \in \mathbf{S}, T \in \mathbf{T}\}$ is R- bounded and

$$R_p(\mathbf{S} + \mathbf{T}) \leq R_p(\mathbf{S}) + R_p(\mathbf{T}).$$

(e) If $\mathbf{T} \subset \mathcal{L}(X)$ is R- bounded, then $\mathbf{T} \cup \{0\}$ is R-bounded and $R_p(\mathbf{T} \cup \{0\}) = R_p(\mathbf{T})$.

(f) If $\mathbf{S}, \mathbf{T} \subset \mathcal{L}(X)$ are R- bounded, then $\mathbf{T} \cup \mathbf{S}$ is R-bounded and

$$R_p(\mathbf{T} \cup \mathbf{S}) \leq R_p(\mathbf{S}) + R_p(\mathbf{T}).$$

(g) Also, each subset $M \subset \mathcal{L}(X)$ of the form $M = \{\lambda I : \lambda \in \Omega\}$ is R-bounded whenever $\Omega \subset \mathbb{C}$ is bounded (I denotes the identity operator on X).

The proofs of (a), (e), (f), and (g) rely on Kahane's contraction principle.

We sketch a proof of (f). Since we assume that $\mathbf{S}, \mathbf{T} \subset \mathcal{L}(X)$ are R-bounded, it follows from (e) (which is a consequence of Kahane's contraction principle) that $\mathbf{S} \cup \{0\}$ and $\mathbf{T} \cup \{0\}$ are R-bounded. We now observe that $\mathbf{S} \cup \mathbf{T} \subset \mathbf{S} \cup \{0\} + \mathbf{T} \cup \{0\}$. Then using (d) and (b) we conclude that $\mathbf{S} \cup \mathbf{T}$ is R-bounded.

We make the following general observation which will be valid throughout the paper, notably in Section 4. Whenever we wish to establish R-boundedness of a family of operators $(M_k)_{k \in \mathbb{Z}}$, if at some point we make an exception such as $(k \neq 0)$, $(k \notin \{-1, 0\})$ and so on, then later we recover the property for the entire family using items (a), (c) and (f) of the foregoing remark. The corresponding observation for boundedness is clear.

Definition 2.4. Let $\varepsilon \in]0, 1[$ and $1 < p < \infty$. Define the operator H_ε by: for all $f \in L^p(\mathbb{R}; X)$

$$(H_\varepsilon f)(t) := \frac{1}{\pi} \int_{\varepsilon < |s| < \frac{1}{\epsilon}} \frac{f(t-s)}{s} ds$$

if $\lim\limits_{\varepsilon \to 0} H_\varepsilon f := Hf$ exists in $L^p(\mathbb{R}; X)$ Then Hf is called the Hilbert transform of f on $L^p(\mathbb{R}, X)$.

Definition 2.5. A Banach space X is said to be UMD space if the Hilbert transform is bounded on $L^p(\mathbb{R}; \; X)$ for all $1 < p < \infty$.

Definition 2.6. For $1 \leq p < \infty$, a sequence $\{M_k\}_{k \in \mathbb{Z}} \subset \mathbf{B}(X, Y)$ is said to be an L^p-multiplier if for each $f \in L^p(\mathbb{T}, X)$, there exists $u \in L^p(\mathbb{T}, Y)$ such that $\hat{u}(k) = M_k \hat{f}(k)$ for all $k \in \mathbb{Z}$.

Proposition 2.7. Let X be a Banach space and $\{M_k\}_{k \in \mathbb{Z}}$ be an L^p-multiplier, where $1 \leq p < \infty$. Then the set $\{M_k\}_{k \in \mathbb{Z}}$ is R-bounded.

Theorem 2.8. (Marcinkiewicz operator-valud multiplier Theorem). Let X, Y be UMD spaces and $\{M_k\}_{k \in \mathbb{Z}} \subset B(X, Y)$. If the sets $\{M_k\}_{k \in \mathbb{Z}}$ and $\{k(M_{k+1} - M_k)\}_{k \in \mathbb{Z}}$ are R-bounded, then $\{M_k\}_{k \in \mathbb{Z}}$ is an L^p-multiplier for $1 < p < \infty$.

Definition 2.9. The Riemann-Liouville fractional integral operator of order $\alpha > 0$ is defined by

$$\mathcal{I}^{\alpha}_{-\infty} f(t) = \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{t} (t-s)^{\alpha - 1} f(s) ds$$

where $\Gamma(\alpha)=\int_0^{+\infty}e^{-t}t^{\alpha-1}dt$, is the Euler gamma function.

Definition 2.10. The Riemann-Liouville fractional integral derivative operator of order $\alpha>0$ is defined by

$$\mathcal{D}^{\alpha}_{-\infty}f(t)=\frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}(\int_{-\infty}^{t}(t-s)^{-\alpha}f(s)ds)$$

Those familiar with the Fourier transform know that the Fourier transform of a derivative can be expressed by the following:

$$\widehat{\frac{dx}{dt}}(k)=ik\hat{x}(k),\forall k\in\mathbb{Z}$$

and more generally,

$$\widehat{\frac{d^nx}{dt^n}}(k)=(ik)^n\hat{x}(k),\forall k\in\mathbb{Z}$$

A similar identity holds for anti-derivatives

$$\widehat{\mathcal{I}^{s}_{-\infty}f}(k)=(ik)^{-s}\hat{x}(k),\forall k\in\mathbb{Z}$$

$$\widehat{\mathcal{D}^{s}_{-\infty}f}(k)=(ik)^{s}\hat{x}(k),\forall k\in\mathbb{Z}$$

Remark 2.11. *If we set* $u(x)=e^{ikx}$ *for* $k\in\mathbb{Z}$ *we have*

$$1)\mathcal{D}^{\alpha}_{-\infty}u(t)=(ik)^{\alpha}e^{ikx}$$
$$2)\mathcal{I}^{\alpha}_{-\infty}u(t)=(ik)^{-\alpha}e^{ikx}.$$

3. Periodic solutions in UMD space

For $a\in L^1(\mathbb{R}_+)$, we denote by $a*x$ the function

$$(a*x)(t):=\int_{-\infty}^{t}a(t-s)x(s)ds$$

with this notation we may rewrite Eq. (1.1) in the following was:

$$\mathcal{D}^{\alpha}_{-\infty}Mx(t)=Ax(t)+(a*x)(t)+\mathcal{I}^{\beta}_{-\infty}x(t)+f(t)\ for\ t\in\mathbb{R}. \tag{3.1}$$

we have $\widehat{a*x}(k)=\tilde{a}(ik)\hat{x}(k)$. We define

$$\Delta_k=((ik)^{\alpha}M-A-\tilde{a}(ik)I-(ik)^{-\beta}I)$$

and

$$\sigma_{\mathbb{Z}}(\Delta)=\{k\in\mathbb{Z}:\Delta_k\ is\ not\ bijective\}$$

the periodic vector-valued space is defined by

$$H^{\alpha,p}(\mathbb{T};X)=\{u\in L^p(\mathbb{T},X):\exists v\in L^p(\mathbb{T},X),\hat{v}(k)=(ik)^{\alpha}M\hat{u}(k)\text{ for all }\ k\in\mathbb{Z}\}$$

Definition 3.1. For $1 \leq p < \infty$, we say that a sequence $\{M_k\}_{k\in\mathbb{Z}} \subset \mathbf{B}(X,Y)$ is an $(L^p, H^{1,p})$-multiplier, if for each $f \in L^p(\mathbb{T}, X)$ there exists $u \in H^{1,p}(\mathbb{T}, Y)$ such that

$$\hat{u}(k) = M_k \hat{f}(k) \quad \text{for all} \quad k \in \mathbb{Z}.$$

Lemma 3.2. *Let $1 \leq p < \infty$ and $(M_k)_{k\in\mathbb{Z}} \subset \mathbf{B}(X)$ ($\mathbf{B}(X)$ is the set of all bounded linear operators from X to X). Then the following assertions are equivalent:*
(i) $(M_k)_{k\in\mathbb{Z}}$ is an $(L^p, H^{\alpha,p})$-multiplier.
(ii) $((ik)^\alpha M_k)_{k\in\mathbb{Z}}$ is an (L^p, L^p)-multiplier.

We begin by establishing our concept of strong solution for Eq. (3.1)

Definition 3.3. Let $f \in L^p(\mathbb{T}; X)$. A function $x \in H^{\alpha,p}(\mathbb{T}; X)$ is said to be a 2π-periodic strong L^p-solution of Eq.(3.1) if $x(t) \in D(A)$ for all $t \geq 0$ and Eq. (3.1) holds almost every where.

Proposition 3.4. *Let A be a closed linear operator defined on an UMD space X. Suppose that*
$\sigma_{\mathbb{Z}}(\Delta) = \phi$ .Then the following assertions are equivalent :

- **(i):** $\big((ik)^\alpha((ik)^\alpha M - A - \tilde{a}(ik)I - (ik)^{-\beta}I)^{-1}\big)_{k\in\mathbb{Z}}$ *is an L^p-multiplier for $1 < p < \infty$*
- **(ii):** $\big((ik)^\alpha((ik)^\alpha M - A - \tilde{a}(ik)I - (ik)^{-\beta}I)^{-1}\big)_{k\in\mathbb{Z}}$ *is R-bounded.*

Proof. (i) $\Rightarrow$ (ii) As a consequence of Proposition (2.7)
(ii) $\Rightarrow$ (i) Let $a_{s,k} = (ik)^{-s}, s \in \mathbb{R}, k \neq 0$
Define $M_k = (ik)^\alpha(C_k - A)^{-1}$, where $C_k := (ik)^\alpha M - \tilde{a}(ik)I - (ik)^{-\beta}I$. By Theorem (2.8) it is sufficient to prove that the set $\{k(M_{k+1} - M_k)\}_{k\in\mathbb{Z}}$ is R-bounded. Since

$$\begin{aligned}
&k\,[M_{k+1} - M_k]\\
&= k\left[(i(k+1))^\alpha(C_{k+1} - A)^{-1} - (ik)^\alpha(C_k - A)^{-1}\right]\\
&= k(C_{k+1} - A)^{-1}\left[(i(k+1))^\alpha(C_k - A) - (ik)^\alpha(C_{k+1} - A)\right](C_k - A)^{-1}\\
&= kM_{k+1}\left[a_{\alpha,k}(C_k - A) - a_{\alpha,k+1}(C_{k+1} - A)\right]M_k\\
&= kM_{k+1}\left[a_{\alpha,k}C_k - a_{\alpha,k+1}C_{k+1} + (a_{\alpha,k+1} - a_{\alpha,k})A\right]M_k\\
&= ka_{\alpha,k}M_{k+1}C_kM_k - ka_{\alpha,k+1}M_{k+1}C_{k+1}M_k + k(a_{\alpha,k+1} - a_{\alpha,k})M_{k+1}AM_k\\
&= ka_{\alpha,k}M_{k+1}C_kM_k - ka_{\alpha,k+1}M_{k+1}C_{k+1}M_k\\
&+ k(\frac{a_{\alpha,k+1} - a_{\alpha,k}}{a_{\alpha,k}})M_{k+1}(a_{\alpha,k}M_kC_k - I).
\end{aligned}$$

Observe that for $\alpha > 0$ we have that $|(i(k+1))^\alpha - (ik)^\alpha|$ can be estimated by $(ik)^{\alpha-1}$ uniformly in k according to the definition of $|(ik)^\alpha|$ and the mean value theorem. This implies that $\frac{k(a_{\alpha,k+1} - a_{\alpha,k})}{a_{\alpha,k}}$ is bounded sequence. Since $ka_{\alpha,k}$ also is bounded for $\alpha > 0$. Since products and sums of R-bounded sequences is R-bounded [23, Remark 2.2]. Then the proof is complete. $\square$

Lemma 3.5. *Let $1 \leq p < \infty$. Suppose that $\sigma_{\mathbb{Z}}(\Delta) = \phi$ and that for every $f \in L^p(\mathbb{T}; X)$ there exists a 2π-periodic strong L^p-solution x of Eq. (3.1). Then x is the unique 2π-periodic strong L^p-solution.*

Proof. Suppose that x_1 and x_2 two strong L^p-solution of Eq. (3.1) then $x = x_1 - x_2$ is a strong L^p-solution of Eq. (3.1) corresponding to $f = 0$. Taking Fourier transform in (3.1), we obtain that

$$(ik)^\alpha M\hat{x}(k) = A\hat{x}(k) + \tilde{a}(ik)\hat{x}(k) + (ik)^{-\beta}\hat{x}(k), k \in \mathbb{Z}.$$

Then

$$((ik)^\alpha M - A - \tilde{a}(ik)I - (ik)^{-\beta}I)\hat{x}(k) = 0$$

It follows that $\hat{x}(k) = 0$ for every $k \in \mathbb{Z}$ and therefore $x = 0$. Then $x_1 = x_2$. □

Theorem 3.6. *Let X be a Banach space. Suppose that for every $f \in L^p(\mathbb{T}; X)$ there exists a unique strong solution of Eq. (3.1) for $1 \leq p < \infty$. Then*

(1) *for every $k \in \mathbb{Z}$ the operator $\Delta_k = ((ik)^\alpha M - A - \tilde{a}(ik)I - (ik)^{-\beta}I)$ has bounded inverse*
(2) *$\left\{(ik)^\alpha M\Delta_k^{-1}\right\}_{k\in\mathbb{Z}}$ is R-bounded.*

Before to give the proof of Theorem 3.6, we need the following Lemma.

Lemma 3.7. *if $((ik)^\alpha M - A - \tilde{a}(ik)I - (ik)^{-\beta}I)(x) = 0$ for all $k \in \mathbb{Z}$, then $u(t) = e^{ikt}x$ is a 2π-periodic strong L^p-solution of the following equation*

$$\mathcal{D}^\alpha_{-\infty}(Mu)(t) = Au(t) + (a * u)(t) + \mathcal{I}^\beta_{-\infty}(u)(t).$$

Proof. We have $((ik)^\alpha M - A - \tilde{a}(ik)I - (ik)^{-\beta}I)x = 0$.
Then

$$(ik)^\alpha Mx = Ax + \tilde{a}(ik)x + (ik)^{-\beta}x$$

We have $u(t) = e^{ikt}x$ and by Remark 2.11 (2),

$$\begin{aligned}\mathcal{D}^\alpha_{-\infty}(Mu)(t) &= (ik)^\alpha e^{ikt}Mx = e^{ikt}((ik)^\alpha x)\\ &= e^{ikt}[Ax + \tilde{a}(ik)x + (ik)^{-\beta}x]\\ &= Ae^{ikt}x + \tilde{a}(ik)e^{ikt}x + (ik)^{-\beta}e^{ikt}x]\\ &= Au(t) + (a * u)(t) + \mathcal{I}^\alpha_{-\infty}u(t)\end{aligned}$$

Proof of Theorem 3.6: 1) Let $k \in \mathbb{Z}$ and $y \in X$. Then for $f(t) = e^{ikt}y$, there exists $x \in H^{\alpha,p}(\mathbb{T}; X)$ such that:

$$\mathcal{D}^\alpha_{-\infty}(Mu)(t) = Au(t) + (a * u)(t) + \mathcal{I}^\beta_{-\infty}(u)(t) + f(t)$$

Taking Fourier transform. We have $\widehat{\mathcal{D}^\alpha_{-\infty}M}x(k) = (ik)^\alpha M\hat{x}(k)$ and $\widehat{\mathcal{I}^\beta_{-\infty}}x(k) = (ik)^{-\beta}\hat{x}(k)$
Consequently, we have

$$(ik)^\alpha M\hat{x}(k) = A\hat{x}(k) + \tilde{a}(ik)\hat{x}(k) + (ik)^{-\beta}\hat{x}(k) + \hat{f}(k)$$

$[(ik)^\alpha M - A - \tilde{a}(ik) - (ik)^{-\beta}]\hat{x}(k) = \hat{f}(k) = y \Rightarrow ((ik)^\alpha M - A - \tilde{a}(ik) - (ik)^{-\beta})$ is surjective.
if $((ik)^\alpha M - A - \tilde{a}(ik) - (ik)^{-\beta})(u) = 0$, then by Lemma 3.7, $x(t) = e^{ikt}u$ is a 2π-periodic strong L^p-solution of Eq.(3.1) corresponing to the function $f(t) = 0$ Hence $x(t) = 0$ and $u = 0$ then $((ik)^\alpha M - A - \tilde{a}(ik) - (ik)^{-\beta})$ is injective.
2) Let $f \in L^p(\mathbb{T}; X)$. By hypothesis, there exists a unique $x \in H^{\alpha,p}(\mathbb{T}, X)$ such that the Eq. (3.1) is valid. Taking Fourier transforms, we deduce that

$$\hat{x}(k) = ((ik)^\alpha M - A - \tilde{a}(ik) - (ik)^{-\beta})^{-1}\hat{f}(k) \quad \text{for all } \ k \in \mathbb{Z}.$$

Hence

$$(ik)^\alpha M\hat{x}(k) = (ik)^\alpha M((ik)^\alpha M - A - \tilde{a}(ik) - (ik)^{-\beta})^{-1}\hat{f}(k) \ \text{ for all } \ k \in \mathbb{Z}$$

Since $x \in H^{\alpha,p}(\mathbb{T}; X)$, then there exists $v \in L^p(\mathbb{T}; X)$ such that

$$\hat{v}(k) = (ik)^\alpha M\hat{x}(k) = (ik)^\alpha M((ik)^\alpha M - A - \tilde{a}(ik) - (ik)^{-\beta})^{-1}\hat{f}(k).$$

Then $\left\{(ik)^\alpha M\Delta_k^{-1}\right\}_{k\in\mathbb{Z}}$ is an L^p-multiplier and $\left\{(ik)^\alpha M\Delta_k^{-1}\right\}_{k\in\mathbb{Z}}$ is *R*-bounded. □

4. Main result

Our main result in this work is to establish that the converse of Theorem 3.6, are true, provided X is an UMD space.

Theorem 4.1. *Let X be an UMD space and $A : D(A) \subset X \to X$ be an closed linear operator. Then the following assertions are equivalent for $1 < p < \infty$.*

(1): *for every $f \in L^p(\mathbb{T}; X)$ there exists a unique 2π-periodic strong L^p-solution of Eq.* (3.1).
(2): *$\sigma_{\mathbb{Z}}(\Delta) = \phi$ and $\left\{(ik)^\alpha M\Delta_k^{-1}\right\}_{k\in\mathbb{Z}}$ is R-bounded.*

Lemma 4.2. [2]. *Let $f, g \in L^p(\mathbb{T}; X)$. If $\hat{f}(k) \in D(A)$ and $A\hat{f}(k) = \hat{g}(k)$ for all $k \in \mathbb{Z}$ Then*

$$f(t) \in D(A) \ \text{ and } \ Af(t) = g(t) \ \text{ for all } \ \ t \in [0, 2\pi].$$

Proof. 1) ⇒ 2) see Theorem 3.6
1) ⇐ 2) Let $f \in L^p(\mathbb{T}; X)$. Define

$$\Delta_k = ((ik)^\alpha M - A - \tilde{a}(ik)I - (ik)^{-\beta}I)$$

By Lemma 3.2, the family $\left\{(ik)^\alpha M\Delta_k^{-1}\right\}_{k\in\mathbb{Z}}$ is an L^p-multiplier it is equivalent to the family $\left\{\Delta_k^{-1}\right\}_{k\in\mathbb{Z}}$ is an L^p-multiplier that maps $L^p(\mathbb{T}; X)$ into $H^{\alpha,p}(\mathbb{T}; X)$,
namely there exists $x \in H^{1,p}(\mathbb{T}, X)$ such that

$$\hat{x}(k) = \Delta_k^{-1}\hat{f}(k) = ((ik)^\alpha M - A - \tilde{a}(ik)I - (ik)^{-\beta}I)^{-1}\hat{f}(k) \tag{4.1}$$

In particular, $x \in L^p(\mathbb{T};X)$ and there exists $v \in L^p(\mathbb{T};X)$ such that $\hat{v}(k) = (ik)^\alpha M\hat{x}(k)$

$$\widehat{\mathcal{D}^{\alpha}_{-\infty}M}x(k) := \hat{v}(k) = (ik)^\alpha M\hat{x}(k) \tag{4.2}$$

Using now (4.1) and (4.2) we have:

$$\widehat{\mathcal{D}^{\alpha}_{-\infty}M}x(k) = (ik)^\alpha M\hat{x}(k) = A\hat{x}(k) + \widehat{a * x}(k) + \widehat{\mathcal{I}^{\beta}_{-\infty}}x(k) + \hat{f}(k) \quad \text{for all } \ k \in \mathbb{Z}.$$

Since A is closed, then $x(t) \in D(A)$ [Lemma 4.2]
and from the uniqueness theorem of Fourier coefficients, that Eq. (3.1) is valid . □

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