



## Disproof of the Riemann Hypothesis

---

Frank Vega

EasyChair preprints are intended for rapid dissemination of research results and are integrated with the rest of EasyChair.

October 11, 2021

# Disproof of the Riemann Hypothesis

Frank Vega

CopSonic, 1471 Route de Saint-Nauphary 82000 Montauban, France

---

## Abstract

We define the function  $\nu(x) = \frac{3 \times \log x + 5}{8 \times \pi \times \sqrt{x} + 1.2 \times \log x + 2} + \frac{\log x}{\log(x + C \times \sqrt{x} \times \log \log x)} - 1$  for some positive constant  $C$  independent of  $x$ . We prove that the Riemann hypothesis is false when there exists some number  $y \geq 13.1$  such that for all  $x \geq y$  the inequality  $\nu(x) \leq 0$  is always satisfied. We know that the function  $\nu(x)$  is monotonically decreasing for all sufficiently large numbers  $x \geq 13.1$ . Hence, it is enough to find a value of  $y \geq 13.1$  such that  $\nu(y) \leq 0$  since for all  $x \geq y$  we would have that  $\nu(x) \leq \nu(y) \leq 0$ . Using the tool *gp* from the project PARI/GP, we note that  $\nu(100!) \approx -2.938735877055718770 E-39 < 0$  for all  $C \geq \frac{1}{1000000!}$ . In this way, we claim that the Riemann hypothesis could be false.

*Keywords:* Riemann hypothesis, Nicolas inequality, Chebyshev function, prime numbers  
*2000 MSC:* 11M26, 11A41, 11A25

---

## 1. Introduction

In mathematics, the Riemann hypothesis is a conjecture that the Riemann zeta function has its zeros only at the negative even integers and complex numbers with real part  $\frac{1}{2}$  [1]. Let  $N_n = 2 \times 3 \times 5 \times 7 \times 11 \times \dots \times p_n$  denotes a primorial number of order  $n$  such that  $p_n$  is the  $n^{\text{th}}$  prime number. Say Nicolas( $p_n$ ) holds provided

$$\prod_{q|N_n} \frac{q}{q-1} > e^\gamma \times \log \log N_n.$$

The constant  $\gamma \approx 0.57721$  is the Euler-Mascheroni constant,  $\log$  is the natural logarithm, and  $q | N_n$  means the prime number  $q$  divides to  $N_n$ . The importance of this property is:

**Theorem 1.1.** [2], [3]. Nicolas( $p_n$ ) holds for all prime numbers  $p_n > 2$  if and only if the Riemann hypothesis is true.

In mathematics, the Chebyshev function  $\theta(x)$  is given by

$$\theta(x) = \sum_{p \leq x} \log p$$

where  $p \leq x$  means all the prime numbers  $p$  that are less than or equal to  $x$ . We know this property for this function:

**Theorem 1.2.** [4]. *There are infinitely many values of  $x$  such that*

$$\theta(x) > x + C \times \sqrt{x} \times \log \log \log x$$

*for some positive constant  $C$  independent of  $x$ .*

We also know that

**Theorem 1.3.** [5]. *If the Riemann hypothesis holds, then*

$$\left( \frac{e^{-\gamma}}{\log x} \times \prod_{q \leq x} \frac{q}{q-1} - 1 \right) < \frac{3 \times \log x + 5}{8 \times \pi \times \sqrt{x}}$$

*for all numbers  $x \geq 13.1$ .*

Let's define  $H = \gamma - B$  such that  $B \approx 0.2614972128$  is the Meissel-Mertens constant [6]. We know from the constant  $H$ , the following formula:

**Theorem 1.4.** [7].

$$\sum_q \left( \log\left(\frac{q}{q-1}\right) - \frac{1}{q} \right) = \gamma - B = H.$$

For  $x \geq 2$ , the function  $u(x)$  is defined as follows

$$u(x) = \sum_{q > x} \left( \log\left(\frac{q}{q-1}\right) - \frac{1}{q} \right).$$

We use the following theorems:

**Theorem 1.5.** [8]. *For  $x > -1$ :*

$$\frac{x}{x+1} \leq \log(1+x).$$

**Theorem 1.6.** [9]. *For  $x \geq 1$ :*

$$\log\left(1 + \frac{1}{x}\right) < \frac{1}{x+0.4}.$$

Let's define:

$$\delta(x) = \left( \sum_{q \leq x} \frac{1}{q} - \log \log x - B \right).$$

**Definition 1.7.** *We define another function:*

$$\varpi(x) = \left( \sum_{q \leq x} \frac{1}{q} - \log \log \theta(x) - B \right).$$

Putting all together yields the proof that the inequality  $\varpi(x) > u(x)$  is satisfied for a number  $x \geq 3$  if and only if  $\text{Nicolas}(p)$  holds, where  $p$  is the greatest prime number such that  $p \leq x$ . In this way, we introduce another criterion for the Riemann hypothesis based on the Nicolas criterion. Using this new criterion, we claim that the Riemann hypothesis could be false.

## 2. Results

**Theorem 2.1.** *The inequality  $\varpi(x) > u(x)$  is satisfied for a number  $x \geq 3$  if and only if  $\text{Nicolas}(p)$  holds, where  $p$  is the greatest prime number such that  $p \leq x$ .*

*Proof.* We start from the inequality:

$$\varpi(x) > u(x)$$

which is equivalent to

$$\left( \sum_{q \leq x} \frac{1}{q} - \log \log \theta(x) - B \right) > \sum_{q > x} \left( \log \left( \frac{q}{q-1} \right) - \frac{1}{q} \right).$$

Let's add the following formula to the both sides of the inequality,

$$\sum_{q \leq x} \left( \log \left( \frac{q}{q-1} \right) - \frac{1}{q} \right)$$

and due to the theorem 1.4, we obtain that

$$\sum_{q \leq x} \log \left( \frac{q}{q-1} \right) - \log \log \theta(x) - B > H$$

because of

$$H = \sum_{q \leq x} \left( \log \left( \frac{q}{q-1} \right) - \frac{1}{q} \right) + \sum_{q > x} \left( \log \left( \frac{q}{q-1} \right) - \frac{1}{q} \right)$$

and

$$\sum_{q \leq x} \log \left( \frac{q}{q-1} \right) = \sum_{q \leq x} \frac{1}{q} + \sum_{q \leq x} \left( \log \left( \frac{q}{q-1} \right) - \frac{1}{q} \right).$$

Let's distribute it and remove  $B$  from the both sides:

$$\sum_{q \leq x} \log \left( \frac{q}{q-1} \right) > \gamma + \log \log \theta(x)$$

since  $H = \gamma - B$ . If we apply the exponentiation to the both sides of the inequality, then we have that

$$\prod_{q \leq x} \frac{q}{q-1} > e^\gamma \times \log \theta(x)$$

which means that  $\text{Nicolas}(p)$  holds, where  $p$  is the greatest prime number such that  $p \leq x$ . The same happens in the reverse implication.  $\square$

**Theorem 2.2.** *The Riemann hypothesis is true if and only if the inequality  $\varpi(x) > u(x)$  is satisfied for all numbers  $x \geq 3$ .*

*Proof.* This is a direct consequence of theorems 1.1 and 2.1.  $\square$

**Theorem 2.3.** *If the Riemann hypothesis holds, then*

$$\frac{3 \times \log x + 5}{8 \times \pi \times \sqrt{x} + 1.2 \times \log x + 2} + \frac{\log x}{\log \theta(x)} > 1$$

for all numbers  $x \geq 13.1$ .

*Proof.* Under the assumption that the Riemann hypothesis is true, then we would have

$$\prod_{q \leq x} \frac{q}{q-1} < e^\gamma \times \log x \times \left(1 + \frac{3 \times \log x + 5}{8 \times \pi \times \sqrt{x}}\right)$$

after of distributing the terms based on the theorem 1.3 for all numbers  $x \geq 13.1$ . If we apply the logarithm to the both sides of the previous inequality, then we obtain that

$$\sum_{q \leq x} \log\left(\frac{q}{q-1}\right) < \gamma + \log \log x + \log\left(1 + \frac{3 \times \log x + 5}{8 \times \pi \times \sqrt{x}}\right).$$

That would be equivalent to

$$\sum_{q \leq x} \frac{1}{q} + \sum_{q \leq x} \left(\log\left(\frac{q}{q-1}\right) - \frac{1}{q}\right) < \gamma + \log \log x + \frac{3 \times \log x + 5}{8 \times \pi \times \sqrt{x} + 1.2 \times \log x + 2}$$

where we know that

$$\begin{aligned} \log\left(1 + \frac{3 \times \log x + 5}{8 \times \pi \times \sqrt{x}}\right) &< \frac{1}{\frac{8 \times \pi \times \sqrt{x}}{3 \times \log x + 5} + 0.4} \\ &= \frac{3 \times \log x + 5}{8 \times \pi \times \sqrt{x} + 0.4 \times (3 \times \log x + 5)} \\ &= \frac{3 \times \log x + 5}{8 \times \pi \times \sqrt{x} + 1.2 \times \log x + 2} \end{aligned}$$

according to theorem 1.6 since  $\frac{8 \times \pi \times \sqrt{x}}{3 \times \log x + 5} \geq 1$  for all numbers  $x \geq 13.1$ . We use the theorem 1.4 to show that

$$\sum_{q \leq x} \left(\log\left(\frac{q}{q-1}\right) - \frac{1}{q}\right) = H - u(x)$$

and  $\gamma = H + B$ . So,

$$H - u(x) < H + B + \log \log x - \sum_{q \leq x} \frac{1}{q} + \frac{3 \times \log x + 5}{8 \times \pi \times \sqrt{x} + 1.2 \times \log x + 2}$$

which is the same as

$$H - u(x) < H - \delta(x) + \frac{3 \times \log x + 5}{8 \times \pi \times \sqrt{x} + 1.2 \times \log x + 2}.$$

We eliminate the value of  $H$  and thus,

$$-u(x) < -\delta(x) + \frac{3 \times \log x + 5}{8 \times \pi \times \sqrt{x} + 1.2 \times \log x + 2}$$

which is equal to

$$u(x) + \frac{3 \times \log x + 5}{8 \times \pi \times \sqrt{x} + 1.2 \times \log x + 2} > \delta(x).$$

We know from the theorem 2.1 that  $\varpi(x) > u(x)$  for all numbers  $x \geq 13.1$  and therefore,

$$\varpi(x) + \frac{3 \times \log x + 5}{8 \times \pi \times \sqrt{x} + 1.2 \times \log x + 2} > \delta(x).$$

Hence,

$$\frac{3 \times \log x + 5}{8 \times \pi \times \sqrt{x} + 1.2 \times \log x + 2} > \log \log \theta(x) - \log \log x.$$

Suppose that  $\theta(x) = \epsilon \times x$  for some constant  $\epsilon > 1$ . Then,

$$\begin{aligned} \log \log \theta(x) - \log \log x &= \log \log(\epsilon \times x) - \log \log x \\ &= \log(\log x + \log \epsilon) - \log \log x \\ &= \log\left(\log x \times \left(1 + \frac{\log \epsilon}{\log x}\right)\right) - \log \log x \\ &= \log \log x + \log\left(1 + \frac{\log \epsilon}{\log x}\right) - \log \log x \\ &= \log\left(1 + \frac{\log \epsilon}{\log x}\right). \end{aligned}$$

In addition, we know that

$$\log\left(1 + \frac{\log \epsilon}{\log x}\right) \geq \frac{\log \epsilon}{\log \theta(x)}$$

using the theorem 1.5 since  $\frac{\log \epsilon}{\log x} > -1$  when  $\epsilon > 1$ . Certainly, we will have that

$$\log\left(1 + \frac{\log \epsilon}{\log x}\right) \geq \frac{\frac{\log \epsilon}{\log x}}{\frac{\log \epsilon}{\log x} + 1} = \frac{\log \epsilon}{\log \epsilon + \log x} = \frac{\log \epsilon}{\log \theta(x)}.$$

Thus,

$$\frac{3 \times \log x + 5}{8 \times \pi \times \sqrt{x} + 1.2 \times \log x + 2} > \frac{\log \epsilon}{\log \theta(x)}.$$

If we add the following value of  $\frac{\log x}{\log \theta(x)}$  to the both sides of the inequality, then

$$\frac{3 \times \log x + 5}{8 \times \pi \times \sqrt{x} + 1.2 \times \log x + 2} + \frac{\log x}{\log \theta(x)} > \frac{\log \epsilon}{\log \theta(x)} + \frac{\log x}{\log \theta(x)} = \frac{\log \epsilon + \log x}{\log \theta(x)} = \frac{\log \theta(x)}{\log \theta(x)} = 1.$$

We know this inequality is satisfied when  $0 < \epsilon \leq 1$  since we would obtain that  $\frac{\log x}{\log \theta(x)} \geq 1$ . Therefore, the proof is done.  $\square$

**Theorem 2.4.** *If there exists some number  $y \geq 13.1$  such that for all  $x \geq y$  the inequality  $\frac{3 \times \log x + 5}{8 \times \pi \times \sqrt{x} + 1.2 \times \log x + 2} + \frac{\log x}{\log(x + C \times \sqrt{x} \times \log \log x)} \leq 1$  is satisfied for some positive constant  $C$  independent of  $x$ , then the Riemann hypothesis should be false.*

*Proof.* From the theorem 1.2, we know that there are infinitely many values of  $x$  such that

$$\theta(x) > x + C \times \sqrt{x} \times \log \log \log x$$

for some positive constant  $C$  independent of  $x$ . That would be equivalent to

$$\log \theta(x) > \log(x + C \times \sqrt{x} \times \log \log \log x)$$

and so,

$$\frac{1}{\log \theta(x)} < \frac{1}{\log(x + C \times \sqrt{x} \times \log \log \log x)}$$

for all numbers  $x \geq 13.1$ . Hence,

$$\frac{\log x}{\log \theta(x)} < \frac{\log x}{\log(x + C \times \sqrt{x} \times \log \log \log x)}.$$

If the Riemann hypothesis holds, then

$$\frac{3 \times \log x + 5}{8 \times \pi \times \sqrt{x} + 1.2 \times \log x + 2} + \frac{\log x}{\log(x + C \times \sqrt{x} \times \log \log \log x)} > 1$$

for those values of  $x$  that complies with

$$\theta(x) > x + C \times \sqrt{x} \times \log \log \log x$$

due to the theorem 2.3. By contraposition, if there exists some number  $y \geq 13.1$  such that for all  $x \geq y$  the inequality

$$\frac{3 \times \log x + 5}{8 \times \pi \times \sqrt{x} + 1.2 \times \log x + 2} + \frac{\log x}{\log(x + C \times \sqrt{x} \times \log \log \log x)} \leq 1$$

is satisfied for some positive constant  $C$  independent of  $x$ , then the Riemann hypothesis should be false, because of there are infinitely many values of  $x$  which satisfy the inequality in the theorem 1.2 and comply with  $x \geq y$  no matter how big could be  $y$ .  $\square$

**Definition 2.5.** Let's define the function  $\nu(x) = \frac{3 \times \log x + 5}{8 \times \pi \times \sqrt{x} + 1.2 \times \log x + 2} + \frac{\log x}{\log(x + C \times \sqrt{x} \times \log \log \log x)} - 1$  for some positive constant  $C$  independent of  $x$ .

**Theorem 2.6.** The Riemann hypothesis could be false.

*Proof.* From the theorem 2.4, we know that the Riemann hypothesis is false when there exists some number  $y \geq 13.1$  such that for all  $x \geq y$  the inequality  $\nu(x) \leq 0$  is always satisfied. We know that the function  $\nu(x)$  is monotonically decreasing for all sufficiently large numbers  $x \geq 13.1$ . Let  $\nu'(x)$  be the derivative of  $\nu(x)$ . We can check the value of  $\nu'(x)$  from this web site <https://www.wolframalpha.com/input> and see that  $\nu'(x)$  is lesser than zero for all sufficiently large numbers  $x \geq 13.1$ . Indeed, a function  $\nu(x)$  of a real variable  $x$  is monotonically decreasing in some interval if the derivative of  $\nu(x)$  is lesser than zero and the function  $\nu(x)$  is continuous over that interval [10]. In this way, it is enough to find a value of  $y \geq 13.1$  such that  $\nu(y) \leq 0$  since for all  $x \geq y$  we would have that  $\nu(x) \leq \nu(y) \leq 0$ . We can check that  $\nu(100!) \approx -2.938735877055718770 E-39 < 0$  for all  $C \geq \frac{1}{1000000!}$  using the tool *gp* from the web site <https://pari.math.u-bordeaux.fr>. Consequently, we claim that the Riemann hypothesis could be false.  $\square$

## Appendix

We use the following input:

$$(3*\log(x)+5)/(8*\pi*\sqrt{x}+1.2*\log(x)+2)+\log(x)/\log(x+C*\sqrt{x}*\log(\log(x)))-1$$

from the web site <https://www.wolframalpha.com/input>. Besides, we use the following input into a single command line:

$$(3*\log(100!)+5)/(8*3.14*\sqrt{100!}+1.2*\log(100!)+2) \\ +\log(100!)/\log(100!+(1/1000000!)*\sqrt{100!}*\log(\log(100!)))-1$$

using the tool *gp* from the project <https://pari.math.u-bordeaux.fr>.

## Acknowledgments

I thank Richard J. Lipton and Craig Helfgott for helpful comments.

## References

- [1] P. B. Borwein, S. Choi, B. Rooney, A. Weirathmueller, *The Riemann Hypothesis: A Resource for the Afficionado and Virtuoso Alike*, Vol. 27, Springer Science & Business Media, 2008.
- [2] J.-L. Nicolas, Petites valeurs de la fonction d’Euler et hypothese de Riemann, *Séminaire de Théorie des nombres DPP*, Paris 82 (1981) 207–218.
- [3] J.-L. Nicolas, Petites valeurs de la fonction d’Euler, *Journal of number theory* 17 (3) (1983) 375–388. doi:10.1016/0022-314X(83)90055-0.
- [4] A. E. Ingham, *The Distribution of Prime Numbers*, no. 30, Cambridge University Press, 1990.
- [5] J. B. Rosser, L. Schoenfeld, Sharper Bounds for the Chebyshev Functions  $\theta(x)$  and  $\psi(x)$ , *Mathematics of computation* (1975) 243–269 doi:10.1090/S0025-5718-1975-0457373-7.
- [6] F. Mertens, Ein Beitrag zur analytischen Zahlentheorie., *J. reine angew. Math.* 1874 (78) (1874) 46–62. doi:10.1515/crll.1874.78.46. URL <https://doi.org/10.1515/crll.1874.78.46>
- [7] Y. Choie, N. Lichiardopol, P. Moree, P. Solé, On Robin’s criterion for the Riemann hypothesis, *Journal de Théorie des Nombres de Bordeaux* 19 (2) (2007) 357–372. doi:10.5802/jtnb.591.
- [8] L. Kozma, Useful Inequalities, [http://www.lkozma.net/inequalities\\_cheat\\_sheet/ineq.pdf](http://www.lkozma.net/inequalities_cheat_sheet/ineq.pdf), accessed on 2021-10-11 (2021).
- [9] A. Ghosh, An Asymptotic Formula for the Chebyshev Theta Function, arXiv preprint arXiv:1902.09231.
- [10] G. Anderson, M. Vamanamurthy, M. Vuorinen, Monotonicity Rules in Calculus, *The American Mathematical Monthly* 113 (9) (2006) 805–816. doi:10.1080/00029890.2006.11920367.