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August 17, 2019

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Wei Zhao [a], Fengli Ren [a*], Fenglan Wang[a], M.Abdel-Aty[b], Jinde Cao [c]
[a] Department of Mathematics, Nanjing University of Aeronautics and Astronautics, Nanjing 210016, China
[b] Mathematics Department, Faculty of Sciences, Sohag University, Egypt, amisaty@gmail.com
[c] Department of Mathematics, Southeast University, Nanjing 210096, China, jdcao@seu.edu.cn

Abstract: This paper investigates the problem of projective synchronization for fractional-order BAM neural networks with discrete and distributed delays. By fractional-order Razumikhin theorem and fractional differential inequality, sufficient conditions are derived to realize the Mittag-Leffler projective synchronization. To solve this problem, the state feedback and adaptive controllers are, respectively, designed. Numerical simulations are given to manifest the effectiveness of the proposed methods.

Key Words: Fractional-order BAM neural networks, Mixed time-varying delays, Synchronization

1 Introduction

It is well known that fractional-order differential systems has aroused the attention of numerous researchers because of its extensive applications in various areas such as applied in diffusion, electromagnetism, and quantum evolution of complex systems [1]. Fractional-order calculus can be thought as the extension of integer-order calculus. In addition, it has a great advantage than conventional integer-order calculus in depicting materials and process having infinite memory and hereditary [2]. Based on these features, some researchers has combined the fractional-order calculus with the neural networks to depicting the dynamic characteristics of neural networks.

Neural networks related to bidirectional associative memory(BAM)have been proposed by Kosko. BAM neural networks consist of neurons distributed in two layers, one of which is fully connected to the other, and there is no internal connection between the layers[3]. The dynamical analysis of BAM neural networks has attracted widely attention and Significant results have been invested [4][5]. It is mentioned that BAM neural networks focus on synchronization and finite time synchronization, stabilization for BAM models.

Over the past two decades, synchronization is a hot topic due to the importance of application in a variety of fields. Such as automatic control, electronic engineering, secure communication and biological systems[6]-[9]. At present several synchronization schemes such as projective synchronization [10][11], complete synchronization[12][13], anti-synchronization[14], lag-synchronization[15]-[17], however, projective synchronization is a interesting behavior not only the difference in phase between the master-slave systems is concerned, but also a proportional scale can be found in amplitudes of state vectors.

Time delays are inevitably in the neurons communication and finite switching speed of amplifiers, which cause the divergence, oscillation, instability and some complex dynamic behaviors in systems [9][19]. The synchronization of fractional order BAM neural networks with time delay are invested in[5]. However, the authors did not consider the distributed delays.

Motivated by the previous works and background, the objective of this paper is to investigate the fractional-order BAM neural networks of Mittag-Leffler projective synchronization with mixed time delays. To the best of our knowledge, there are no attempt have been taken on this topic. Two kinds of controllers namely feedback controller and adaptive controller, are employed for the Mittag-Leffler projective synchronization. The rest of the paper is outlined as follows. In section 2, some assumptions, preliminaries and useful lemmas are introduced. In section 3, two different control scheme are designed to accomplish projective synchronization. In section 4, several numerical examples and simulation are presented to show the effectiveness of the obtained results. Finally, conclusions are drawn in section 5.

1.1 Model description and preliminaries

In this paper, we consider a class of mixed time-varying delayed fractional-order BAM neural networks model

$$\begin{cases} D^{\alpha}x_i(t) = -c_i x_i(t) + \sum_{j=1}^{m} a_{ij} f_j(y_j(t)) \\ \qquad + \sum_{j=1}^{m} b_{ij} f_j(y_j(t-\tau^{(1)}(t))) \\ \qquad + \sum_{j=1}^{m} d_{ij} \int_{t-\tau^{(2)}(t)}^{t} f_j(y_j(s))ds + I_i, \\ D^{\alpha}y_j(t) = -\tilde{c}_j y_j(t) + \sum_{i=1}^{n} \tilde{a}_{ji} g_i(x_i(t)) \\ \qquad + \sum_{i=1}^{n} \tilde{b}_{ji} g_i(x_i(t-\sigma^{(1)}(t))) \\ \qquad + \sum_{i=1}^{n} \tilde{d}_{ji} \int_{t-\sigma^{(2)}(t)}^{t} g_i(x_i(\tilde{s}))d\tilde{s} + \tilde{I}_j. \end{cases} \tag{1}$$

This work is supported by National Natural Science Foundation (NNSF) of China under Grant no. 61104031.
* Correspoding author
E-mail address:flren@nuaa.edu.cn(F Ren)

with the initial conditions

$$x_i(t) = \varphi_i(t),\ y_i(t) = \Phi_j(t), \quad t \in [-\tau, 0].$$

in which $0 < \alpha < 1$, D^{α} represents Caputo type fractional-order derivatives of order α from 0 to t [1], $x_i(t)$ and $y_j(t)$ represent the membrane voltages of $i-th$ neuron and $j-th$ neuron in the X-and Y-layers, $g_i(x_i)$ and $f_j(y_j)$ is the neuron activation function of the $i-th$ and $j-th$ neuron at the time t respectively, b_{ij}, c_{ij}, d_{ij}, and $\tilde{b}_{ji}$, $\tilde{c}_{ji}$, $\tilde{d}_{ji}$ denotes the weight coefficients of the neurons. I_i and $\tilde{I}_j$ represent external input of X-layer and Y-layer, $\tau^{(1)}(t)$, $\tau^{(2)}(t)$ and $\sigma^{(1)}(t)$, $\sigma^{(2)}(t)$ are discrete and distributed time-varying delays.

Throughout this paper, the following assumption is considered.
Assumption 1: for any $z_i \in R^n (i = 1, 2)$ there exists constants $L_j^f > 0, L_i^g > 0 (i = 1, 2 \cdots, n. j = 1, 2, \cdots, m)$ such that

$$|f_j(z_1) - f_j(z_2)| \le L_j^f |z_1 - z_2|, j = 1, 2, \cdots$$
$$|g_i(z_1) - g_i(z_2)| \le L_i^g |z_1 - z_2|, i = 1, 2, \cdots$$

Assumption 2:

$$0 \le \tau^{(1)}(t), \tau^{(2)}(t) \le \tau, \qquad 0 \le \sigma^{(1)}(t), \sigma^{(2)}(t) \le \sigma.$$

where τ and σ are positive real constants.

In order to achieve the Mittag-Leffler projective synchronization between master and slave system, fractional-order BAM neural networks (1) is regarded as the master system, and the following form is denoted slave systems:

$$\begin{cases} D^{\alpha}\tilde{x}_i(t) = -c_i\tilde{x}_i(t) + \sum_{j=1}^{m} a_{ij} f_j(\tilde{y}_j(t)) \\ \qquad + \sum_{j=1}^{m} b_{ij} f_j(\tilde{y}_j(t - \tau^{(1)}(t))) \\ \qquad + \sum_{j=1}^{m} d_{ij} \int_{t-\tau^{(2)}(t)}^{t} f_j(\tilde{y}_j(s))ds \\ \qquad + I_i + u_i(t), \\ D^{\alpha}\tilde{y}_j(t) = -\tilde{c}_j\tilde{y}_j(t) + \sum_{i=1}^{n} \tilde{a}_{ji} g_i(\tilde{x}_i(t)) \\ \qquad + \sum_{i=1}^{n} \tilde{b}_{ji} g_i(\tilde{x}_i(t - \sigma^{(1)}(t))) \\ \qquad + \sum_{i=1}^{n} \tilde{d}_{ji} \int_{t-\sigma^{(2)}(t)}^{t} g_i(\tilde{x}_i(\tilde{s}))d\tilde{s} \\ \qquad + \tilde{I}_j + v_j(t). \end{cases} \tag{2}$$

The initial condition of slave system (2) are $\tilde{x}_i(t) = \tilde{\varphi}_i(t)$, $\tilde{y}_j(t) = \tilde{\Phi}_j(t)$, $t \in [-\tau, 0]$. We define the synchronization errors as $e_i^x(t) = \tilde{x}_i(t) - \beta x_i(t), i = 1, 2 \cdots, n$, $e_j^y(t) = \tilde{y}_j(t) - \beta y_j(t), j = 1, 2 \cdots, m$. From master system (1) and slave system (2), we have

$$\begin{cases} D^{\alpha}e_i^x(t) = -c_i e_i^x(t) + \sum_{j=1}^{m} a_{ij} \tilde{f}_j(e_j^y(t)) \\ \qquad + \sum_{j=1}^{m} b_{ij} \tilde{f}_j(e_j^y(t - \tau^{(1)}(t))) \\ \qquad + \sum_{j=1}^{m} d_{ij} \int_{t-\tau^{(2)}(t)}^{t} \tilde{f}_j(e_j^y(s))ds + u_i(t) + H(t), \\ D^{\alpha}e_j^y(t) = -\tilde{c}_j e_j^y(t) + \sum_{i=1}^{n} \tilde{a}_{ji} \tilde{g}_i(e_i^x(t)) \\ \qquad + \sum_{i=1}^{n} \tilde{b}_{ji} \tilde{g}_i(e_i^x(t - \sigma^{(1)}(t))) \\ \qquad + \sum_{i=1}^{n} \tilde{d}_{ji} \int_{t-\sigma^{(2)}(t)}^{t} \tilde{g}_i(e_i^x(\tilde{s}))d\tilde{s} + v_j(t) + F(t). \end{cases} \tag{3}$$

where

$$\tilde{f}(e_j^y(t)) = f(\tilde{y}_j(t)) - f(\beta y_j(t))$$
$$\tilde{f}(e_j^y(t - \tau^{(1)}(t)))$$
$$= f(\tilde{y}_j(t - \tau^{(1)}(t))) - f(\beta y_j(t - \tau^{(1)}(t)))$$
$$\tilde{f}(e_j^y(s)) = f(\tilde{y}_j(s)) - f(\beta y_j(s))$$
$$\tilde{g}(e_i^x(t)) = g(\tilde{x}_i(t)) - g(\beta x_i(t))$$
$$\tilde{g}(e_i^x(t - \sigma^{(1)}(t)))$$
$$= g(\tilde{x}_i(t - \sigma^{(1)}(t))) - g(\beta x_i(t - \sigma^{(1)}(t)))$$
$$\tilde{g}(e_i^x(\tilde{s})) = g(x_i(\tilde{s})) - g(\beta x_i(\tilde{s}_i))$$
$$H(y(t), \tau^{(1)}(t), \tau^{(2)}(t), \beta)$$
$$= \sum_{j=1}^{m} a_{ij}[\beta f(y_j(t)) - f(\beta y_j(t))]$$
$$+ \sum_{j=1}^{m} b_{ij}[\beta f(y_j(t - \tau^{(1)}(t))) - f(\beta y_j(t - \tau^{(1)}(t)))]$$
$$+ \sum_{j=1}^{m} d_{ij} \int_{t-\tau^{(2)}(t)}^{t} \beta f(y_j(s)) - f(\beta y_j(s))ds + \beta(I_i - 1)$$
$$F(x(t), \sigma^{(1)}(t), \sigma^{(2)}(t), \beta)$$
$$= \sum_{i=1}^{n} \tilde{a}_{ji}[\beta g(x_i(t)) - g(\beta x_i(t))]$$
$$+ \sum_{i=1}^{n} \tilde{b}_{ji}[\beta g(x_i(t - \sigma^{(1)}(t))) - g(\beta x_i(t - \sigma^{(1)}(t)))]$$
$$+ \sum_{i=1}^{n} \tilde{d}_{ji} \int_{t-\sigma^{(2)}(t)}^{t} \beta g(x_i(\tilde{s})) - g(\beta\tilde{s})d\tilde{s} + \beta(\tilde{I}_j - 1)$$

Remark 1: a wide range of nonlinear functions are covered in assumption 1,which involves a class of fractional chaotic systems. In addition, time-varying delays is only required to be bounded without any derivative constriant[31].

2 Preliminaries

Definition 1.([1]) The fractional integral of order for a function f is defined as $I^{\alpha} f(t)$

$$I^\alpha f(t)=\frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-\tau)^{\alpha-1}f(\tau)d\tau$$

where $t\geq t_0$ and $\alpha>0$.

Definition 2. ([2]) The Caputo's fractional derivative of order α for a function $f\in C^{n+1}([t_0,+\infty],R)$ is defined by

$$D^\alpha f(t)=\frac{1}{\Gamma(n-\alpha)}\int_{t_0}^{t}\frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha-n+1}}d\tau$$

where $t\geq t_0$ and n is a positive interger such that $n-1<\alpha<n$. Particulary, when $0<\alpha<1$

$$D^\alpha f(t)=\frac{1}{\Gamma(1-\alpha)}\int_{t_0}^{t}\frac{f'(\tau)}{(t-\tau)^{\alpha}}d\tau$$

Definition 3. ([2]) The Mittag-Leffler function with two parameters is defined as

$$E_{\alpha,\beta}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(k\alpha+\beta)}$$

where $\alpha,\beta>0$, and $z\in C$(complexes set). when $\beta=1$, $E_\alpha(z)=E_{\alpha,1}(z)$.

Definition 4. The master system (1) and slave system (2) are said to be Mittag-Leffler projective synchronization with projective coefficient β, if there exist positive constants $\lambda>0$ and $b>0$, such that

$$\|y(t)-\beta x(t)\|\leq M(\delta)E_\alpha(-\lambda(t-t_0)^\alpha)$$

for any $t\geq t_0+T$, where $\|y(t_0)-\beta x(t_0)\|\leq\delta$.

Remark 2. Based on the definition of the Mittag-Leffler function, which implies the asymptotic projective synchronization. The projective synchronization is complete synchronization, if $\beta=1$; the projective synchronization is anti-synchronization, if $\beta=-1$.

Lemma 1. ([20]) Let $v(t)$ and $w(t)$ be two continuous nonnegative functions and satisfy
$D^\alpha_{t_0}(v(t)+w(t))\leq-\lambda v(t)$
where $0<\alpha<1$ and $\lambda>0$. there exists a $T>0$ such that
$v(t)\leq(v(t_0)+w(t_0)+h)E_\alpha(-\lambda(t-t_0)^\alpha)),t\geq t_0+T$.
where h is any positive constant.

3 Mittag-Leffler projective synchronization

In this section, we will derive some conditions to accomplish the Mittag-Leffler projective synchronization under two kinds of controllers respectively.

3.1 Feedback control scheme

Choose the control input u_i,v_j in the slave system as the following form

$$\begin{cases}u_i(t)=-H(y(t),\tau^{(1)}(t),\tau^{(2)}(t),\beta)+w_i(t),\\ w_i(t)=-k_i(t)[\tilde{x}_i(t)-\beta x_i(t)],\\ v_j(t)=-F(x(t),\sigma^{(1)}(t),\sigma^{(2)}(t),\beta)+\tilde{w}_j(t),\\ \tilde{w}_j(t)=-\tilde{k}_j(t)[\tilde{y}_j(t)-\beta y_j(t)].\end{cases}\tag{4}$$

Theorem 1: Under the assumption $1-2$, the system (1) and the system (2) are Mittag-leffler projective synchronization based on the control scheme (4) and in which the parameters satisfy

$$\begin{cases}\xi_1=min\Big\{\underset{i}{min}\Big(c_i-\sum_{i=1}^{n}\tilde{a}_{ji}L_i^g+k_i(t)\Big),\\ \underset{j}{min}\Big(\tilde{c}_j-\sum_{j=1}^{m}a_{ij}L_j^f+\tilde{k}_j(t)\Big)\Big\}\geq\xi_1,\\ \xi_2=max\Big\{\underset{i}{max}\Big(\sum_{j=1}^{m}d_{ij}L_j^f\tau^{(2)}(t)+\sum_{j=1}^{m}b_{ij}L_j^f\Big),\\ \underset{j}{max}\Big(\sum_{i=1}^{n}\tilde{d}_{ji}L_i^g\sigma^{(2)}(t)+\sum_{i=1}^{n}\tilde{b}_{ji}L_i^g\Big)\Big\}\geq 0.\end{cases}\tag{5}$$

Proof. Consider the following Lyapunov-Krasovskii functional candidate

$$V(t,e(t))=\sum_{i=1}^{n}|e_i^x(t)|+\sum_{j=1}^{m}|e_j^y(t)|$$

Calculating the derivative of $V(t,e(t))$ with respect to t along solution of error system (3)

$$\begin{aligned}D^\alpha V(t,e(t))\leq&\sum_{i=1}^{n}sign(e_i^x(t))D^\alpha e_i^x(t)+\sum_{j=1}^{m}sign(e_j^y(t))D^\alpha e_j^y(t)\\ \leq&\sum_{i=1}^{n}sign(e_i^x(t))\big\{-c_ie_i^x(t)+\sum_{j=1}^{m}a_{ij}\tilde{f}_j(e_j^y(t))\\ &+\sum_{j=1}^{m}b_{ij}\tilde{f}_j(e_j^y(t-\tau^{(1)}(t))\\ &+\sum_{j=1}^{m}d_{ij}\int_{t-\tau^{(2)}(t)}^{t}\tilde{f}_j(e_j^y(s))ds-k_i(t)e_i^x(t)\big\}\\ &+\sum_{i=1}^{n}sign(e_j^y(t))\big\{-\tilde{c}_je_j^y(t)+\sum_{i=1}^{n}\tilde{a}_{ji}\tilde{g}_i(e_i^x(t))\\ &+\sum_{i=1}^{n}\tilde{b}_{ji}\tilde{g}_i(e_i^x(t-\sigma^{(1)}(t))\\ &+\sum_{i=1}^{n}\tilde{d}_{ji}\int_{t-\sigma^{(2)}(t)}^{t}\tilde{g}_i(e_i^x(\tilde{s}))d\tilde{s}-\tilde{k}_j(t)e_j^y(t)\big\}\end{aligned}$$

Applying assumption $(1)-(2)$, we can obtain that

$$\begin{aligned}D^\alpha V(t,e(t))\leq&-\sum_{i=1}^{n}(c_i+k_i(t))|e_i^x(t)|\\ &+\sum_{i=1}^{n}\sum_{j=1}^{m}a_{ij}L_j^f|e_j^y(t)|\\ &+\sum_{i=1}^{n}\sum_{j=1}^{m}b_{ij}L_j^f|e_j^y(t-\tau^{(1)}(t))|\\ &+\sum_{i=1}^{n}\sum_{j=1}^{m}d_{ij}L_j^f\int_{t-\tau^{(2)}(t)}^{t}|e_j^y(s)|ds\end{aligned}\tag{6}$$

$$
\begin{aligned}
&-\sum_{j=1}^{m}(\tilde{c}_j+\tilde{k}_j(t))|e_i^x(t)| \\
&+\sum_{i=1}^{n}\sum_{j=1}^{m}\tilde{a}_{ji}L_i^g|e_i^x(t)| \\
&+\sum_{i=1}^{n}\sum_{j=1}^{m}\tilde{b}_{ji}L_i^g|e_i^x(t-\sigma^{(1)}(t))| \\
&+\sum_{i=1}^{n}\sum_{j=1}^{m}\tilde{d}_{ji}L_i^g\int_{t-\sigma^{(2)}(t)}^{t}|e_i^x(\tilde{s})|d\tilde{s} \\
\le&-\sum_{i=1}^{n}(c_i+k_i(t)-\sum_{i=1}^{n}\tilde{a}_{ji}L_i^g)|e_i^x(t)| \\
&+\sup_{t-\tau\le s\le t}(\sum_{j=1}^{m}d_{ij}L_j^f\tau^{(2)}(t) \\
&+\sum_{j=1}^{m}b_{ij}L_j^f)|e_j^y(s)| \\
&-\sum_{j=1}^{m}(c_j+\tilde{k}_j(t)-\sum_{j=1}^{m}a_{ij}L_j^f)|e_j^y(t)| \\
&+\sup_{t-\sigma\le\tilde{s}\le t}(\sum_{i=1}^{n}\tilde{d}_{ji}L_i^g\sigma^{(2)}(t) \\
&+\sum_{i=1}^{n}\tilde{b}_{ji}L_i^g)|e_i^x(\tilde{s})|.
\end{aligned}
$$

Where

$$
\xi_1\triangleq min\Big\{\min_i\Big(c_i+k_i(t)-\sum_{i=1}^{n}\tilde{a}_{ji}L_i^g\Big),\ \min_j\Big(\tilde{c}_j+\tilde{k}_j(t)-\sum_{j=1}^{m}a_{ij}L_j^f\Big)\Big\},
$$

$$
\xi_2\triangleq max\Big\{\max_i\Big(\sum_{j=1}^{m}d_{ij}L_j^f\tau^{(2)}(t)+\sum_{j=1}^{m}b_{ij}L_j^f\Big),\ \max_j\Big(\sum_{i=1}^{n}\tilde{d}_{ji}L_i^g\sigma^{(2)}(t)+\sum_{i=1}^{n}\tilde{b}_{ji}L_i^g\Big)\Big\}
$$

From the condition (5), we can get a parameter ξ , which satisfies $0<\xi\le\xi_1-\xi_2$, and the equation (6) can be written as

$$
\begin{aligned}
D^\alpha V(t,e(t))&\le-\xi\Big(\sum_{i=1}^{n}|e_i^x(t)|+\sum_{j=1}^{m}|e_j^y(t)|\Big)\\
&=-\xi V(t,e(t)).
\end{aligned}
$$

whenever

$$
V(t+s,e(t+s))\le V(t,e(t)),-\tau\le s\le 0 \tag{7}
$$

Then $V(t,e(t))\le V(0,e_i^x(0),e_j^y(0))E_\alpha(-\xi t^\alpha)$.

That is to say

$$
\|e_i^x(t)\|+\|e_j^y(t)\|\le E_\alpha(-\xi t^\alpha)\Big\{\|\varphi-\tilde{\varphi}\|+\|\Phi-\tilde{\Phi}\|\Big\}.
$$

Thus, from definition 2, the master-slave system (1) and (2) are Mittag-Leffler projective synchronized under the feedback control scheme (4). the proof is completed.

Remark 3. The inequality (7) is called Razumikhin condition[5], which is usually used in integer order for functional differential equation, and also can be utilized in fractional-order systems with time delays in[30].

Remark 4. The methods of using the state feedback controller, can be used for both tracking control problems, stabilization and synchronization. In the following theorem we address this problem by designing a adaptive control scheme.

3.2 Adaptive control scheme

Theorem 2. Under the assumption (1)-(2), if the adaptive controller be chosen the following form

$$
\left\{
\begin{aligned}
u_i(t)=&-H_i(t)-l_i(t)sign(e_i^x(t))\sum_{j=1}^{m}|e_j^y(t-\tau^{(1)}(t))|\\
&-p_i(t)e_i(t)-\sum_{j=1}^{m}h_i(t)\int_{t-\tau^{(2)}(t)}^{t}|e_j^y(s)|ds,\\
D^\alpha l_i(t)=&\eta_i|e_i^x(t)|\sum_{j=1}^{m}|e_j^y(t-\tau^{(1)}(t))|,\\
D^\alpha p_i(t)=&\rho_i e_i^x(t)^2,\\
D^\alpha h_i(t)=&\varepsilon_i|e_i^x(t)|\sum_{j=1}^{m}\int_{t-\tau^{(2)}(t)}^{t}|e_j^y(s)|ds,
\end{aligned}
\right. \tag{8}
$$

and

$$
\left\{
\begin{aligned}
v_j(t)=&-F_j(t)-\bar{l}_j(t)sign(e_j^y(t))\sum_{i=1}^{n}|e_i^x(t-\sigma^{(1)}(t))|\\
&-\bar{p}_j(t)e_j^y(t)-\sum_{i=1}^{n}\bar{h}_j(t)\int_{t-\sigma^{(2)}(t)}^{t}|e_i^x(\tilde{s})|d\tilde{s},\\
D^\alpha\bar{l}_j(t)=&\varrho_j|e_j^y(t)|\sum_{i=1}^{n}|e_i^x(t-\sigma^{(1)}(t))|,\\
D^\alpha\bar{p}_j(t)=&\omega_j e_j^y(t)^2,\\
D^\alpha\bar{h}_j(t)=&\delta_j|e_j^y(t)|\sum_{i=1}^{n}\int_{t-\sigma(2)(t)}^{t}|e_i^x(\tilde{s})|d\tilde{s},
\end{aligned}
\right. \tag{9}
$$

where $i=1,2,\cdots,n$, $j=1,2,\cdots,m$. $\eta_i,\rho_i,\varepsilon_i,\varrho_j,\omega_j,\delta_j$ are arbitrary positive constants, then the master -slave system (1) and (2) are Mittag-Leffler projective synchronization.

Proof. with the adaptive controllers (8) and (9) the error dynamics system can be described by the following equations

$$
\begin{aligned}
D^\alpha e_i^x(t)=&-c_ie_i^x(t)+\sum_{j=1}^{m}a_{ij}\tilde{f}_j(e_j^y(t))\\
&+\sum_{j=1}^{m}b_{ij}\tilde{f}_j(e_j^y(t-\tau^{(1)}(t)))\\
&+\sum_{j=1}^{m}d_{ij}\int_{t-\tau^{(2)}(t)}^{t}\tilde{f}_j(e_j^y(s))ds
\end{aligned} \tag{10}
$$

$$- l_i(t)sign(e_i^x(t))\sum_{j=1}^{m}|e_j^y(t-\tau^{(1)}(t))|$$
$$- p_i(t)e_i^x(t) - \sum_{j=1}^{m}h_i(t)\int_{t-\tau^{(2)}(t)}^{t}|e_j^y(s)|ds$$
$$\begin{aligned} D^{\alpha}e_j^y(t) =& - \tilde{c}_j e_j^y(t) + \sum_{i=1}^{n}\tilde{a}_{ji}\tilde{g}_i(e_i^x(t)) \\ &+ \sum_{i=1}^{n}\tilde{b}_{ji}\tilde{g}_i(e_i^x(t-\sigma^{(1)}(t))) \\ &+ \sum_{i=1}^{n}\tilde{d}_{ji}\int_{t-\sigma^{(2)}(t)}^{t}\tilde{g}_i(e_i^x(\tilde{s}))d\tilde{s} \\ &- \bar{l}_j(t)sign(e_j^y(t))\sum_{i=1}^{n}|e_i^x(t-\sigma^{(1)}(t))| \\ &- \bar{p}_j(t)e_j^y(t) - \sum_{i=1}^{n}\bar{h}_j(t)\int_{t-\sigma^{(2)}(t)}^{t}e_i^x(\tilde{s})d\tilde{s} \end{aligned} \tag{11}$$

Define the Lyapunov function candidate as the following form

$$\tilde{V}(t,e(t)) = \frac{1}{2}\sum_{i=1}^{n}e_i^x(t)^2 + \frac{1}{2}\sum_{j=1}^{m}e_j^y(t)^2$$

The Caputo's fractional derivative of $\tilde{V}(t)$ with respect to time t along the solution of the error equation (10) and (11), we get

$$\begin{aligned} D^{\alpha}\tilde{V}(t) \leq& \sum_{i=1}^{n}e_i^x(t)D^{\alpha}e_i^x(t) + \sum_{j=1}^{m}e_j^y(t)D^{\alpha}e_j^y(t) \\ \leq& \sum_{i=1}^{n}e_i^x(t)\{ - c_i e_i^x(t) + \sum_{j=1}^{m}a_{ij}\tilde{f}_j(e_j^y(t)) \\ &+ \sum_{j=1}^{m}b_{ij}\tilde{f}_j(e_j^y(t-\tau^{(1)}(t))) \\ &+ \sum_{j=1}^{m}d_{ij}\int_{t-\tau^{(2)}(t)}^{t}\tilde{f}_j(e_j^y(s))ds \\ &-l_i(t)sign(e_i^x(t))\sum_{j=1}^{m}|e_j^y(t-\tau^{(1)}(t))| \\ &-p_i(t)e_i^x(t) - \sum_{j=1}^{m}h_i(t)\int_{t-\tau^{(2)}(t)}^{t}|e_j^y(s)|ds\} \\ &+ \sum_{j=1}^{m}e_j^y(t)\{ - \tilde{c}_j e_j^y(t) + \sum_{i=1}^{n}\tilde{a}_{ji}\tilde{g}_i(e_i^x(t)) \\ &+ \sum_{i=1}^{n}\tilde{b}_{ji}\tilde{g}_i(e_i^x(t-\sigma^{(1)}(t)) \\ &+ \sum_{i=1}^{n}\tilde{d}_{ji}\int_{t-\sigma^{(2)}(t)}^{t}\tilde{g}_i(e_i^x(\tilde{s}))d\tilde{s} \\ &-\bar{l}_j(t)sign(e_j^y(t))\sum_{i=1}^{n}|e_i^x(t-\sigma^{(1)}(t))| \\ &-\bar{p}_j(t)e_j^y(t) - \sum_{i=1}^{n}\bar{h}_j(t)\int_{t-\sigma^{(2)}(t)}^{t}|e_i^x(\tilde{s})|d\tilde{s}\} \\ \leq& -\sum_{i=1}^{n}c_i e_i^x(t)^2 + \sum_{i=1}^{n}\sum_{j=1}^{m}a_{ij}L_j^f|e_i^x(t)||e_j^y(t)| \\ &+\sum_{i=1}^{n}\sum_{j=1}^{m}b_{ij}L_j^f|e_i^x(t)||e_j^y(t-\tau^{(1)}(t))| \\ &+\sum_{i=1}^{n}\sum_{j=1}^{m}d_{ij}L_j^f\int_{t-\tau^{(2}(t)}^{t}|e_i^x(t)||e_j^y(s)|ds \\ &-\sum_{j=1}^{m}\tilde{c}_j e_j^y(t)^2 + \sum_{i=1}^{n}\sum_{j=1}^{m}\tilde{a}_{ji}L_i^g|e_i^x(t)||e_j^y(t)| \\ &+\sum_{i=1}^{n}\sum_{j=1}^{m}\tilde{b}_{ji}L_i^g|e_i^x(t-\sigma^{(1)}(t))||e_j^y(t)| \\ &+\sum_{i=1}^{n}\sum_{j=1}^{m}\tilde{d}_{ji}L_i^g\int_{t-\sigma^{(2)}(t)}^{t}|e_j^y(t)||e_i^x(\tilde{s})|d\tilde{s} \end{aligned}$$

Note that

$$\sum_{i=1}^{n}\sum_{j=1}^{m}|e_i^x(t)||e_j^y(t)| \leq \sum_{i=1}^{n}\sum_{j=1}^{m}\frac{1}{2}e_i^x(t)^2 + \sum_{i=1}^{n}\sum_{j=1}^{m}\frac{1}{2}e_j^y(t)^2$$

Constructing the following auxiliary function:

$$\begin{aligned} W(t) =& \sum_{i=1}^{n}\{\frac{1}{2\eta_i}(l_i(t)-l_i^*)^2 + \frac{1}{2\rho_i}(p_i(t)-p_i^*)^2 \\ &+ \frac{1}{2\varepsilon_i}(h_i(t)-h_i^*)^2\} \\ &+ \sum_{j=1}^{m}\{\frac{1}{2\varrho_j}(\bar{l}_j(t)-\bar{l}_j^*)^2 + \frac{1}{2\omega_j}(\bar{p}_j(t)-\bar{p}_j^*)^2 \\ &+ \frac{1}{2\delta_j}(\bar{h}_j(t)-\bar{h}_j^*)^2\} \end{aligned}$$

Then, we have

$$\begin{aligned} & D^{\alpha}(\tilde{V}(t)+W(t)) \\ \leq& \sum_{i=1}^{n}\{ - c_i + \frac{1}{2}\sum_{j=1}^{m}\tilde{a}_{ji}L_i^g + \frac{1}{2}\sum_{j=1}^{m}a_{ij}L_j^f\}e_i^x(t)^2 \\ &+\sum_{i=1}^{n}\sum_{j=1}^{m}b_{ij}L_j^f|e_i^x(t)||e_j^y(t-\tau^{(1)}(t))| \\ &+\sum_{i=1}^{n}\sum_{j=1}^{m}d_{ij}L_j^f\int_{t-\tau^{(2)}(t)}^{t}|e_i^x(t)||e_j^y(s)|ds \\ &+\sum_{j=1}^{m}\{ - \tilde{c}_j + \frac{1}{2}\sum_{i=1}^{n}a_{ij}L_j^f + \frac{1}{2}\sum_{i=1}^{n}\tilde{a}_{ji}L_i^g\}e_j^y(t)^2 \\ &+\sum_{i=1}^{n}\sum_{j=1}^{m}\tilde{b}_{ji}L_i^g|e_i^x(t-\sigma^{(1)}(t))||e_j^y(t)| \\ &+\sum_{i=1}^{n}\sum_{j=1}^{m}\tilde{d}_{ji}L_i^g\int_{t-\sigma^{(2)}(t)}^{t}|e_i^x(\tilde{s})||e_j^y(t)|d\tilde{s} \\ &-l_i^*\sum_{i=1}^{n}\sum_{j=1}^{m}|e_j^y(t-\tau^{(1)}(t))||e_i^x(t)| \\ &-\bar{l}_j^*\sum_{i=1}^{n}\sum_{j=1}^{m}|e_j^y(t)||e_i^x(t-\sigma^{(1)}(t))| \end{aligned}$$

$$
\begin{aligned}
&-p_i^* \sum_{i=1}^{n} e_i^x(t)^2 - \bar{p}_j^* \sum_{j=1}^{m} e_j^y(t)^2 \\
&-h_i^* \sum_{i=1}^{n}\sum_{j=1}^{m} \int_{t-\tau^{(2)}(t)}^{t} |e_j^y(t)||e_i^x(s)|ds \\
&-\bar{h}_j^* \sum_{i=1}^{n}\sum_{j=1}^{m} \int_{t-\sigma^{(2)}(t)}^{t} |e_i^x(t)||e_j^y(\tilde{s})|d\tilde{s}
\end{aligned}
$$

for all $t \in [0, \infty)$ choosing p_i^*, l_i^*, h_i^*,and $\bar{p}_j^*$, $\bar{l}_j^*$, $\bar{h}_j^*$ large enough such that

$$
\begin{cases}
p_i^* \geq -c_i + \dfrac{1}{2}\displaystyle\sum_{j=1}^{m} a_{ij} L_j^f + \dfrac{1}{2}\sum_{j=1}^{m} \tilde{a}_{ji} L_i^g, \\
l_i^* \geq \max\limits_{1\leq j\leq m} b_{ij} L_j^f, \\
h_i^* \geq \max\limits_{1\leq j\leq m} d_{ij} L_j^f,
\end{cases}
$$

and

$$
\begin{cases}
\bar{p}_j^* \geq -\tilde{c}_j + \dfrac{1}{2}\displaystyle\sum_{i=1}^{n} a_{ij} L_j^f + \dfrac{1}{2}\sum_{i=1}^{n} \tilde{a}_{ji} L_i^g, \\
\bar{l}_j^* \geq \max\limits_{1\leq i\leq n} \tilde{b}_{ji} L_i^g, \\
\bar{h}_j^* \geq \max\limits_{1\leq i\leq n} \tilde{d}_{ji} L_i^g,
\end{cases}
$$

where λ is a positive constant, then

$$
D^\alpha(\tilde{V}(t) + W(t)) \leq -\lambda \tilde{V}(t).
$$

From lemma (1), it follows that for any $h \geq 0$,
there exists a $T \leq 0$ such that

$$
\begin{aligned}
\|e(t)\| &\leq (V(t_0 + w(t_0) + h))E_\alpha(-\lambda(t-t_0)^\alpha) \\
&\leq M(\delta)E_\alpha(-\lambda(t-t_0)^\alpha),
\end{aligned}
$$

for any $t \geq t_0 + T$.
Therefore, we can prove that the drive-response (1) and (2) are Mittag-Leffler projective synchronization under the adaptive control.

Corollary 1. Under assumptions (1)-(2), the projective synchronization can be achieved without adaptive distribute controller. That is, the adaptive controllers can be chosen as the following form

$$
\begin{cases}
\tilde{u}_i(t) = -H_i(t) - l_i(t) sign(e_i^x(t)) \displaystyle\sum_{j=1}^{m} |e_j^y(t - \tau^{(1)}(t))| \\
\qquad -p_i(t) e_i^x(t), \\
D^\alpha l_i(t) = \eta_i e_i^x(t) \displaystyle\sum_{j=1}^{m} |e_j^y(t-\tau^{(1)}(t))|, \\
D^\alpha p_i(t) = \rho_i e_i^x(t)^2,
\end{cases}
$$

and

$$
\begin{cases}
v_j(t) = -E_j(t) - \bar{l}_j(t) sign(e_j^y(t)) \displaystyle\sum_{i=1}^{n} |e_i^x(t-\sigma^{(1)}(t))| \\
\qquad -\bar{p}_j(t) e_j^y(t), \\
D^\alpha \bar{l}_i(t) = \varrho_j e_j^y(t) \displaystyle\sum_{i=1}^{n} |e_i^x(t-\sigma^{(1)}(t))|, \\
D^\alpha \bar{p}_j(t) = \omega_j e_j^y(t)^2,
\end{cases}
$$

where $i = 1, 2, \cdots, n$, $j = 1, 2, \cdots, m$. $\eta_i, \rho_i, \varrho_j, \omega_j$ are arbitrary positive constants, then the drive-response system (1) and (2) are Mittag-Leffler projective synchronization.

Similar to the proof of theorem 1 and theorem 2, here we omit it.

4 Illustrative example

In this section, we consider two examples of fractional-order BAM neural networks to show the effectiveness of the theoretical results given in the previous sections The equations of Caputo fractiona-order BAM neural networks with master system can be written as

$$
\begin{aligned}
D^\alpha x(t) =& -Cx(t) + Af(y(t)) + Bf(y(t-\tau^{(1)}(t)) \\
&+ D\int_{t-\tau^{(2)}(t)}^{t} f(y(s))ds \\
D^\alpha y(t) =& -\tilde{C}y(t) + \tilde{A}g(x(t)) + \tilde{B}g(x(t-\sigma^{(1)}(t))) \\
&+ \tilde{D}\int_{t-\sigma^{(2)}(t)}^{t} g(y(\tilde{s}))d\tilde{s}
\end{aligned}
$$

and slave system

$$
\begin{aligned}
D^\alpha \tilde{x}(t) =& -C\tilde{x}(t) + Af(\tilde{y}(t)) + Bf(\tilde{y}(t-\tau^{(1)}(t))) \\
&+ D\int_{t-\tau^{(2)}(t)}^{t} f(\tilde{y}(s))ds + I \\
D^\alpha \tilde{y}(t) =& -\tilde{C}\tilde{y}(t) + \tilde{A}g(\tilde{x}(t)) + \tilde{B}g(\tilde{x}(t-\sigma^{(1)}(t))) \\
&+ \tilde{D}\int_{t-\sigma^{(2)}(t)}^{t} g(\tilde{x}(\tilde{s}))d\tilde{s} + \tilde{I}
\end{aligned}
$$

where

$$
C = \begin{bmatrix} 0.7 & 0 \\ 0 & 0.6 \end{bmatrix} \quad A = \begin{bmatrix} 0.35 & 0.1 \\ 0.2 & 0.1 \end{bmatrix}
$$

$$
B = \begin{bmatrix} 0.35 & -0.5 \\ 0.4 & 0.45 \end{bmatrix} \quad D = \begin{bmatrix} 0.5 & 0.1 \\ -0.2 & 0.6 \end{bmatrix}
$$

$$
\tilde{C} = \begin{bmatrix} 0.65 & 0 \\ 0 & 0.4 \end{bmatrix} \quad \tilde{A} = \begin{bmatrix} 0.35 & 0.12 \\ 0.9 & 0.15 \end{bmatrix}
$$

$$
\tilde{B} = \begin{bmatrix} 0.62 & 0.35 \\ 0.4 & 1 \end{bmatrix} \quad \tilde{D} = \begin{bmatrix} -0.5 & 1 \\ 2 & -1.8 \end{bmatrix}
$$

$\tau^{(1)}(t) = \sigma^{(1)}(t) = \frac{e^t}{1+e^t}$, $\sigma^{(2)}(t) = \sigma^{(2)}(t) = 1$, and the activation functions $f_1(z) = f_2(z) = tanh(z)$, $g_1(z) = g_2(z) = tanh(z)$, for any $z \in R$, $I = \tilde{I} = (0,0)^T$, $\alpha = 0.95$, and projective factor $\beta = 1$.
It is obviously that the assumption 1 and 2 hold with $L_j^f = L_i^g = 1$, for $i, j = 1, 2$. with initial values $x(t) = (0.8, 0.7)^T$, $y(t) = (-0.25, -0.20)^T$ and $\tilde{x}(t) = (-1.2, -0.6)^T$, $\tilde{y}(t) = (0.7, 0.8)^T$, for $t \in [-1, 0]$. Let $K(t) = 0$, the control term disappears, which is numerically simulated in Fig1.

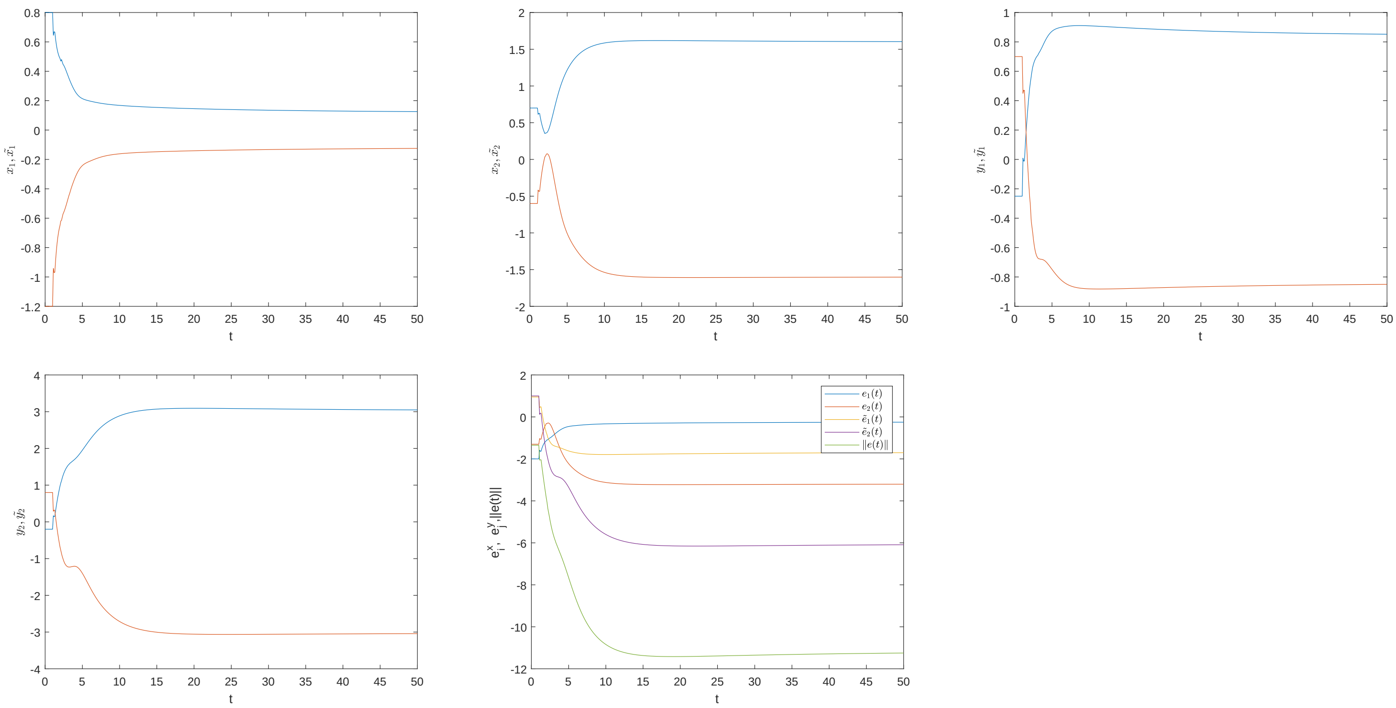

Fig. 1: The state response of the uncontrolled neural network

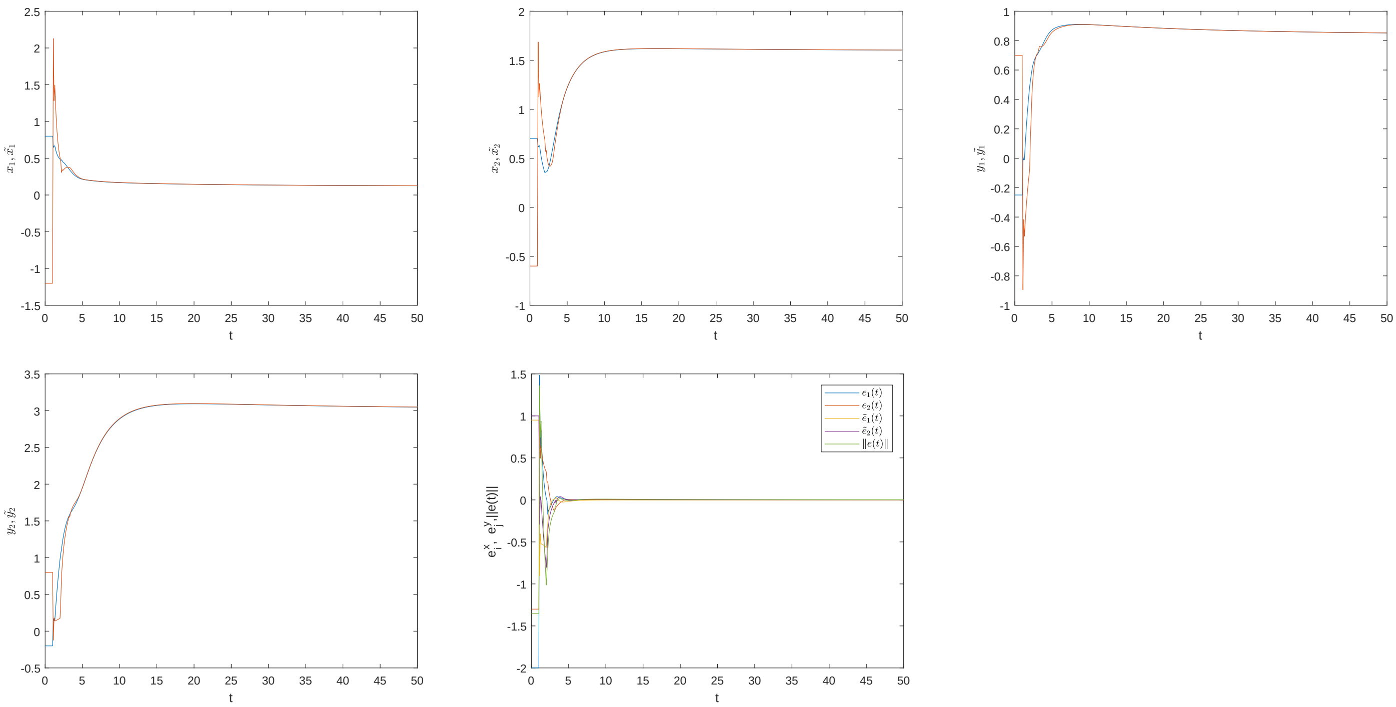

Fig. 2: The state response of the feedback controlled neural network

when we take control term in the response term and from the condition (5), we can take the control term $k(t) = [-1.5, -1.6, -1.5, -0.6]^T$, The projective Mittag-Leffler synchronization motion of the neural network under the feedback control was shown Fig. 2.

Consider the adaptive synchronization for systems (10) and (11), For convenience we take the $\tau^{(1)}(t) = \tau^{(2)}(t) = 1, \sigma^{(1)}(t) = \sigma^{(2)}(t) = 1$ other parameters$A$, B, C, $\tilde{A}$, $\tilde{B}$, $\tilde{C}$, f, g, $\tilde{f}$, $\tilde{g}$, I and $\tilde{I}$ are all the same as above parameters, It is not difficult to check that all condition in corolloary 1 are satisfied, Taking $\eta_i = \varepsilon_i = 0.1$, $\varrho_j = \omega_j = 0.1$. $l_1(0) = 0.1$, $p_1(0) = 0$, $\bar{l}_1(0) = 0.1$, $\bar{p}_1(0) = 0$ the $l_i(t)$, $p_i(t)$ and $\bar{l}_j(t)$, $\bar{p}_j(t)$ are presented in Fig. 3.

5 Conclusions

In this paper, the problem of projective synchronization for BAM neural networks with discrete and distributed de-

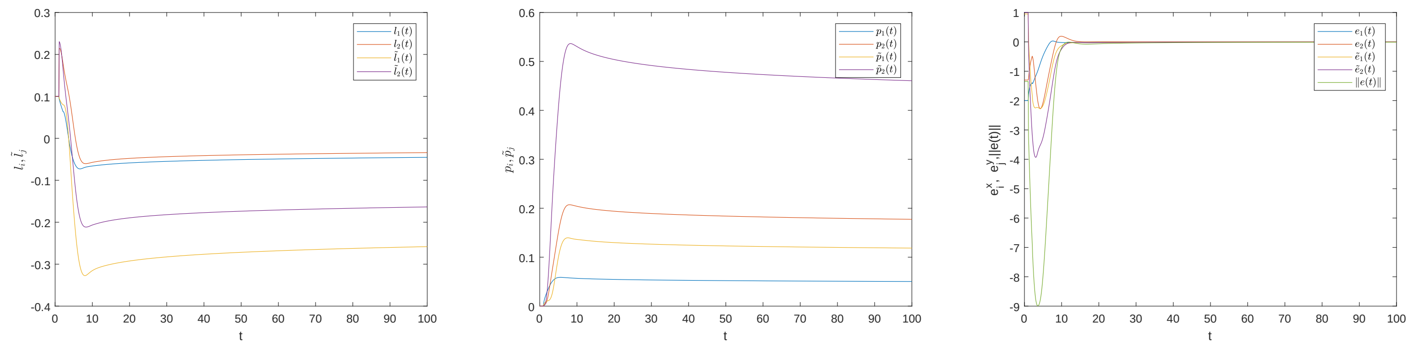

Fig. 3: The state response of the adaptive controlled neural network

lays is first investigated in detail.we have established two kinds of controllers' feedback controller and adaptive controller to achieve the Mittag-Leffler synchronization. if $\beta = 1$ the projective synchronization is complete synchronization. complete synchronization has received widely attention in[20]-[22]. if $\beta = -1$ the projective synchronization is anti-synchronization. It is convenient that the methods of this paper can put in to practice. At the end of the paper, the numerical examples are presented to illustrate the feasibility of the proposed methods.

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