Distribution of Renyi Entropy from Nakagami Distribution Sample

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***Abstract*— This paper rigorously computes the precise quantiles associated with the Maximum Likelihood Estimation (MLE) method for Renyi entropy applied to the Nakagami distribution. Additionally, it approximates the quantiles of the asymptotic distribution of this statistic. Furthermore, the thesis introduces an enhanced version of this statistical measure. Remarkably, even in cases involving small sample sizes, the proposed method provides a formula for determining quantiles based on the chi-square distribution. This approach yields an exact distribution of statistics and reveals key characteristics of the MLE Renyi entropy applied to the Nakagami distribution.**

Keywords— Asymptotic Normality, Fisher Information, MLE, Nakagami Distribution, Renyi Entropy

Introduction

Our research delves into the domain of Renyi entropy, specifically focusing on conditional Renyi entropy (Baez, 2022).We demonstrate the validity of the chain rule for Renyi entropy and establish a relationship governing the rate of Renyi entropy (Mesfioui , Kayid, & Shrahili, 2023).This relationship proves instrumental in deriving the Renyi entropy rate for an irreducible-aperiodic Markov chain. Notably, our study reveals that the upper bound for Renyi entropy rate coincides with the Shannon entropy rate.

Advances in Renyi entropy have resulted in its utilization in various domains, including signal processing, machine learning, and statistical physics (Asafo-Adjei, Adam, Owusu Junior, Akorsu , & Arthur, 2022). The notion has been expanded by scholars to encompass continuous probability distributions, hence facilitating its utilization in situations that involve continuous data. Furthermore, Renyi entropy has been included in multivariate statistics, offering useful metrics for multidimensional dataset's dependence and uncertainty structures.

The study of conditional Renyi entropy in which the determination of a random variable's entropy is contingent upon the knowing of another variable—has garnered attention in recent years. Applications for this extension can be found in fields such as pattern recognition and image processing, which enable a more sophisticated comprehension of the information contained in large, complicated data sets.

In the analysis of age data and issues connected to the failure process, several parametric models are used (Nakagami, 1960). It is known that ultrasound modeling is used in medical imaging research, particularly when photographing various types of tumors, such as breast tumors. NAK may also be used to simulate the envelopes of high-frequency seismograms. Dr Mamoun told me to mention that in using result in diagnosis of some diseases associated with aging

Awad and Abu Hammad's paper (2007) made noteworthy contributions by investigating the distribution of the Shannon statistic derived from normal samples. Their research introduced modifications to existing statistics and provided exact distributions, calculated through the quantiles of the chi-square distribution.

In the contemporary research landscape, Afaf's work (Al Sous, 2023) stands out as a pioneering effort. This research meticulously calculated exact quantiles for the Maximum Likelihood Estimation (MLE) of Shannon entropy applied to Nakagami distributions. The study introduced a novel version of the statistic, demonstrating its precision even in scenarios involving limited sample sizes. Through rigorous analysis and application of chi-square distribution quantiles, the research established precise formulas for quantiles of the statistic's distribution and illuminated key characteristics of MLE Shannon entropy within the context of Nakagami distribution.

Renyi Entropy of NAK(a, b)

We will state the Renyi entropy of the NAK distribution.

Lemma 1

If X ~ NAK(a, b), then the Renyi entropy, is given by:

$$H_R(b) = \frac{1}{2(\xi-1)}\Big(log(4a\xi) + (\xi+1)\,log(b) - \xi\,log(4a) + (2a-1)\xi\,log(\xi) + 2\xi\,log\big(\Gamma(a)\big) - 2\xi\,log(\Gamma\left(\frac{1}{2} + \frac{2a-\xi}{2}\right)\Big)$$

Proof: In order to prove this result, we observe that the PDF of X is given by:

Hence, $\left[f[x]^{\xi}\right] = \frac{2^{\xi} a^{a\xi}}{\Gamma(a)^{\xi} b^{a\xi}}\ x^{(2a-1)^{\xi}}\ e^{-\left(\frac{ax^2}{b}\right)^{\xi}}$

Taking integration to both sides results in:

$$\int f(x)^{\xi}\,dx = (2)^{-1+\xi}(a)^{a\xi+\frac{1}{2}(-2a\xi+\xi-1)}(b)^{-a\xi+\frac{1}{2}(2a\xi-\xi+1)}$$
$$(\xi)^{\frac{1}{2}(-1+\xi-2a\xi)}\Gamma(a)^{-\xi}\Gamma\left(\left(a-\frac{1}{2}\right)\xi+\frac{1}{2}\right)$$

This immediately gives the desired result.

$$H_R(b) = \frac{1}{2(\xi-1)}\Bigg((log(4a\xi)) + (\xi+1)\,(log(b)) - (\xi\,log(4a)) + (2a-1)(\xi\,log(\xi)) + \left(2\xi\,log(\Gamma(a))\right) + \left(2\xi\log\left(\frac{2a-\xi+1}{2}\right)\right)\Bigg)$$

MLE for Renyi Entropy of NAK Distribution

It is well known that the MLE for b based on random sample $X_1, X_2, \ldots, X_n$ from NAK (a, b) is

$$\hat{b} = \frac{\sum_{i=1}^{n} x_i^2}{n}$$

we get:

$$H_R(\hat{b}) = \frac{1}{2(\xi-1)}\Bigg((log(4a\xi)) + (\xi+1)\left(log(\hat{b})\right) - (\xi\,log(4a)) + (2a-1)(\xi\,log(\xi)) + \left(2\xi\,log(\Gamma(a))\right) + \left(2\xi\log\left(\frac{2a-\xi+1}{2}\right)\right)\Bigg)$$

$$H_R(\hat{b}) - H_R(b) = \frac{(\xi+1)\,log(\hat{b}) - (\xi+1)\,log(b)}{2(\xi-1)}$$

Where, $\frac{2\alpha}{\beta}\sum_{i=1}^{n} x_i^2 : G\left(2a, 2, \sqrt{\frac{b}{a}}\right) \equiv G(an, 2) \equiv \mathcal{X}^2{}_{[2\alpha n]}$

Lemma 2 Properties of this estimator

1) $E\left(H_R(\hat{b})\right) = \psi(an) + \log(2) + H_R(b)$
2) $Var(\log H_R(\hat{b})) = \psi'(an)$

Proof:

1) $E\left(H_R(\hat{b})\right) = H_R(b) + \frac{(\xi+1)}{2(\xi-1)} E(\log(\chi^2_{(2na)})) + \frac{(\xi+1)(\log(2na))}{2(\xi-1)}$

We want to find $E\log(\chi^2_{(2an)})$

Now, $X{:}\,\chi^2_{(2na)} \equiv G(na, 2)$

So,

$$E(\log(X)) = \psi(an) + \log(2)$$

$$E\left(H_R(\hat{b})\right) = H_R(b) + \frac{(\xi+1)}{2(\xi-1)} E(\log(2na) + \frac{(\xi+1)}{2(\xi-1)}\left[\left(\psi(an)\right) + (\log(2))\right]$$

$$= H_R(b) + \psi(an) + (\log(2))$$

2) $Var(log\,H_R(\hat{b}) = \frac{(\xi+1)}{2(\xi-1)}\,Var\left(\log\chi^2_{(2na)}\right)$

Now, $X{:}\,\chi^2_{(2na)} \equiv G(na, 2)$

So,

$$Var(\log(X)) = \frac{(\xi+1)}{2(\xi-1)}\left(\psi'(an)\right)$$

Exact Distribution of $H(\hat{b})$ when α known

Lemma 3 The random variable

$W = \frac{\sqrt{n}\left(H(\hat{b})-H(b)\right)}{\sqrt{\sigma^2(b)}}$ has a Gumbel distribution of type one, where $\sigma^2(b) = \frac{t^2}{I(b)}$; $t = \frac{\partial H(b)}{\partial b}$; (Pardo L. , 2006)

Proof:

$$H(\hat{b}) - H(b) = \frac{(\xi+1)}{2(\xi-1)}\left(\left(log\left(\frac{\sum_{i=1}^{n} x_i^2}{n}\right)\right) - (log(b))\right).$$

$$= \frac{(\xi+1)}{2(\xi-1)}\left(\left(log\left(\frac{2a\sum_{i=1}^{n} x_i^2}{2an}\right)\right) - (log(b))\right).$$

$$= \frac{(\xi+1)}{2(\xi-1)}\left((log(2a\sum_{i=1}^{n} x_i^2)) - (\log(2a)) - (\log(n)) - (log(b))\right).$$

$$= \frac{(\xi+1)}{2(\xi-1)}\left(\left(log\left(\frac{2a\sum_{i=1}^{n} x_i^2}{b}\right)\right) - (log(2an))\right).$$

we have:

$$t = \frac{\partial H(b)}{\partial b} = \frac{(\zeta+1)}{2(\zeta-1)b}.$$

and from,

$$I(b) = \frac{a}{b^2}.$$

It is clear from tables that:

1) The Nakagami approximation is acceptable for values in the middle of the distribution.
2) In the tails of the distribution, the approximated quantiles are larger than those of the exact distribution. It also seems that this is valid even for large values of n.

3) Since testing and construction of confidence interval depend on quantiles near the tails, we conclude that the asymptotic distribution is not applicable for small sample.

Table1: Percentiles of Renyi entropy statistic based on sample form $NK(b,b)$, $b = 5$, number of simulated samples $n_i = 1000$

TABLE I.

	0.005	0.01	0.025	0.05	0.1	0.25	0.75	0.9	0.95	0.97
1	-8.15	-8.16	-5.55	-5.51	-4.53	-2.31	0.98	1.97	2.32	2.41
2	-9.57	-9.57	-6.52	-5.85	-5.24	-2.42	0.82	1.96	2.83	3.10
3	-6.97	-6.97	-4.38	-4.04	-2.84	-1.33	1.80	2.52	3.27	3.7
4	-8.65	-8.65	-4.61	-3.44	-1.71	-0.52	1.99	3.11	3.81	4.60
5	-6.33	-6.33	-4.45	-3.21	-2.13	-0.94	1.74	2.71	3.22	3.81
6	-4.10	-4.1	-2.78	-2.51	-1.70	-0.42	2.02	2.95	3.65	4.36
7	-5.56	-5.56	-3.6	-3.17	-2.52	-0.54	2.81	3.80	4.48	5.14
8	-4.94	-4.94	-3.79	-3.09	-1.65	-0.13	2.40	3.91	4.29	4.62
9	-5.42	-5.42	-2.85	-2.31	-1.17	0.42	3.28	4.35	4.68	5.51
10	-6.12	-6.12	-2.76	-2.67	-1.86	0.23	2.57	4.06	4.75	5.18
11	-5.18	-5.18	-2.33	-1.54	-1.02	0.52	3.30	4.60	5.27	5.39
12	-2.57	-2.57	-2.15	-2.05	-1.38	-0.02	3.20	4.26	5.66	6.47
13	-2.66	-2.66	-1.46	-1.16	-0.43	0.88	3.39	4.72	5.43	5.77
14	-4.35	-4.35	-2.46	-1.34	-0.51	0.23	3.20	4.72	5.59	5.81
15	-5.14	-5.14	-2.57	-1.91	-0.76	0.58	3.60	4.55	5.68	6.36
16	-3.12	-3.12	-1.93	-1.45	-0.41	0.73	3.80	5.35	5.88	6.50
17	-2.93	-2.93	-1.91	-1.63	-0.49	0.96	4.37	5.78	6.57	7.13
18	-4.13	-4.13	-1.18	-1.04	-0.57	1.70	3.96	5.06	5.61	5.88
19	-2.55	-2.55	-2.21	-1.69	-0.16	0.70	3.73	5.19	5.94	6.60
20	-1.85	-1.85	-0.62	0	0.54	1.51	4.42	5.55	5.78	6.70
21	-4.86	-4.86	-2.34	-1.60	-0.49	1.12	4.11	4.94	5.72	7.30
22	-2.66	-2.66	-1.61	-0.74	-0.16	1.72	4.74	5.72	6.13	6.85
23	-3.76	-3.76	-1.99	-1.88	-0.63	0.88	4.30	5.48	5.89	6.37
24	-2.61	-2.61	-1.61	-0.82	0.48	1.74	4.37	5.52	6.53	6.90
25	-2.80	-2.80	-0.95	-0.78	-0.34	1.82	4.54	5.32	5.98	6.70
26	-1.76	-1.76	-1.40	-0.91	0.05	1.54	4.66	5.52	6.40	7.78
27	-3.70	-3.7	-2.16	-1.36	-0.17	1.74	5.04	6.57	7.08	8.78
28	-1.47	-1.47	-0.57	0.35	1.13	2.11	5.10	6.50	6.85	7.72
29	-2.23	-2.23	-0.77	-0.12	0.19	1.39	4.79	6.32	6.75	7.40
30	-3.12	-3.12	-0.87	-0.42	0.88	2.09	5.12	7.01	7.49	8.52

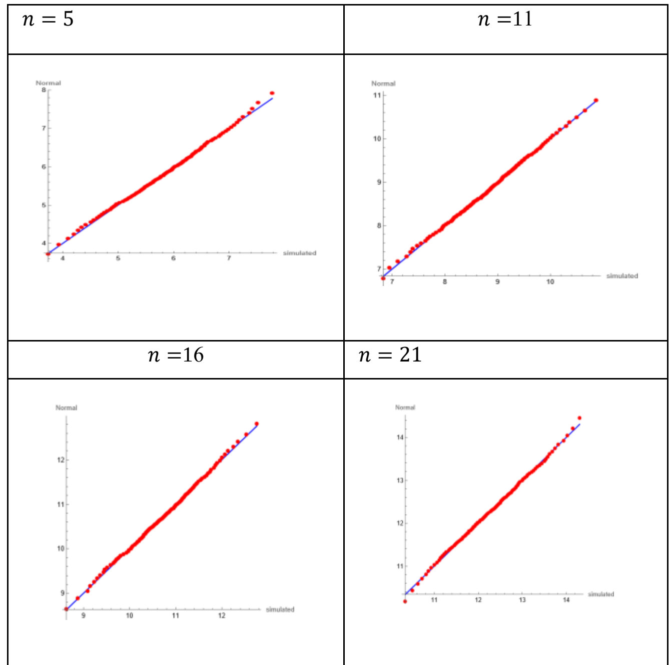

Figure 1: Q-Q-Plot of Nakagami Approximation vs. Exact Distribution of Standard Renyi, when $b = 3$, $n_i = 10000$.

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