On the convergence of solutions of globally modified magnetohydrodynamics equations with locally Lipschitz delay terms

Gabriel Deugoué, Jules Djoko Kamdem and Adèle Claire Fouapé

November 4, 2020

On the convergence of solutions of globally modified magnetohydrodynamics equations with locally Lipschitz delays terms

G. Deugoue[1], J. K. Djoko[2], and A.C. Fouape[3]

[1,3]Universite de Dschang, Departement de Mathematiques et Informatique, Cameroun
[2]African peer review mechanism, Johannesburg, South Africa

October 29, 2020

Abstract

Existence and uniqueness of strong solutions for three dimensional system of globally modified magnetohydrodynamics equations with locally Lipschitz delays terms are established in this article. Galerkin's method and Aubin Lions compactness theorem are the main mathematical tools we use to prove the existence result. Moreover, we prove that, from a sequence of weak solutions of globally modified magnetohydrodynamics equations with locally Lipschitz delays terms, we can extract a subsequence which converges in an adequate sense to a weak solution of three dimensional magnetohydrodynamics equations with locally Lipschitz delays terms.

Keywords: Magnetohydrodynamics equations; Globally modified Navier Stokes; Strong solutions; convergence; finite delays

1 Introduction and statement of the problem

Let $\Omega \subset \mathbb{R}^3$ be an open bounded set with a sufficient regular boundary $\Gamma = \partial\Omega$, and $N > 0$ be fixed. We define $F_N : (0, +\infty) \to (0, 1]$ by

$$F_N(r) = \min\left\{1, \frac{N}{r}\right\},\ r \in \mathbb{R}^+.$$

Recently the authors in [14] have discussed the existence of solutions and the asymptotic behavior of the globally modified magnetohydrodynamics equations (GMMHDE):

$$\begin{cases} \boldsymbol{u}_t + F_N(\|\boldsymbol{u}\|_{V_1})[(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}] - \frac{1}{R_e}\Delta\boldsymbol{u} \\ -SF_N\left(\|(\boldsymbol{u},\boldsymbol{B})\|_V\right)[(\boldsymbol{B}\cdot\nabla)\boldsymbol{B}] + \nabla\left(p + S\frac{|\boldsymbol{B}|^2}{2}\right) = \boldsymbol{f}_1(t) \text{ in } (0,T)\times\Omega, \\ \boldsymbol{B}_t + F_N(\|(\boldsymbol{u},\boldsymbol{B})\|_V)\left[(\boldsymbol{u}\cdot\nabla)\boldsymbol{B} - (\boldsymbol{B}\cdot\nabla)\boldsymbol{u}\right] + \frac{1}{R_m}\text{curl}(\text{curl}\boldsymbol{B}) = \boldsymbol{f}_2(t) \text{ in } (0,T)\times\Omega, \\ \text{div } \boldsymbol{u} = 0,\ \text{div } \boldsymbol{B} = 0 \text{ in}(0,T)\times\Omega, \\ \boldsymbol{u}(0,x) = \boldsymbol{u}_0(x),\ \boldsymbol{B}(0,x) = \boldsymbol{B}_0(x) \text{ for all } x\in\Omega, \\ \boldsymbol{u} = 0,\quad \boldsymbol{B}\cdot\boldsymbol{n} = 0 \text{ and curl}\boldsymbol{B}\times\boldsymbol{n} = 0 \text{ on } \Gamma, \end{cases} \tag{1.1}$$

where $\boldsymbol{u}, \boldsymbol{B}$ and p represent respectively the fluid velocity, the magnetic field and the pressure. $\boldsymbol{f}_1$ and $\boldsymbol{f}_2$ are given external forces field. R_e and R_m are the so-called Reynolds and magnetic

Reynolds numbers, respectively and $S = \frac{M^2}{R_e R_m}$ is a positive constant, where M is the Hartman number. $|\boldsymbol{B}|^2 = \boldsymbol{B} \cdot \boldsymbol{B}$ and represents the length of the magnetic field, $\boldsymbol{n}$ is the unit outward normal on Γ. This system is indeed a globally modified version of the following magnetohydrodynamics equations with locally Lipschitz delays terms

$$\begin{cases} \boldsymbol{u}_t + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} - \frac{1}{R_e}\Delta \boldsymbol{u} - S\left[(\boldsymbol{B} \cdot \nabla)\boldsymbol{B}\right] \\ +\nabla\left(p + S\frac{|\boldsymbol{B}|^2}{2}\right) = G_1(t, \boldsymbol{u}(t - \rho_1(t))) \text{ in } (0,T) \times \Omega, \\ \boldsymbol{B}_t + (\boldsymbol{u} \cdot \nabla)\boldsymbol{B} - (\boldsymbol{B} \cdot \nabla)\boldsymbol{u} + \frac{1}{R_m}\text{curl}(\text{curl}\boldsymbol{B}) = \\ G_2(t, \boldsymbol{B}(t - \rho_2(t))) \text{ in } (0,T) \times \Omega, \\ \text{div } \boldsymbol{u} = 0, \ \text{div } \boldsymbol{B} = 0 \text{ in}(0,T) \times \Omega, \\ \boldsymbol{u} = 0, \quad \boldsymbol{B} \cdot \boldsymbol{n} = 0 \text{ and } \text{curl}\boldsymbol{B} \times \boldsymbol{n} = 0 \text{ on } \Gamma, \\ \boldsymbol{u}(t,x) = \phi_1(t,x), \ \boldsymbol{B}(t,x) = \phi_2(t,x), \ t \in [-h, 0], \ x \in \Omega, \end{cases} \tag{1.2}$$

The magnetohydrodynamics (MHD) equations take into account the coupling between Maxwell's equations governing the magnetic field $\boldsymbol{B}$ and the Navier-Stokes equations (NSE) governing the fluid motion $\boldsymbol{u}$ (cf. [10]). They play a fundamental role in Astrophysics, Geophysics, Plasma Physics, and in many other areas in applied sciences.

The system (1.1) was introduced and studied in [14] where the authors established the existence of its unique strong solution and global attractor. However, there are some situations which are better described by mathematical equations containing delays terms. For instance, the delays terms may appear when we want to control the system by applying a force which takes into account not only the present state but also a part of the history of the system.

In [8], Caraballo and co-authors proved the existence and the uniqueness of strong solution of a three dimensional system of globally modified Navier-Stokes equations with a locally Lipschitz delay term. In [26] the convergence of solutions of globally modified Navier-Stokes equations with delays to solutions of Navier-Stokes equations with delays, is established. Motivated by these works, we introduce in the present paper the following three dimensional system of globally modified magnetohydrodynamics equations with locally Lipschitz delays terms (GMMHDEFD)

$$\begin{cases} \boldsymbol{u}_t + F_N(\|\boldsymbol{u}\|_{V_1})[(\boldsymbol{u} \cdot \nabla)\boldsymbol{u}] - \frac{1}{R_e}\Delta \boldsymbol{u} - SF_N\left(\|(\boldsymbol{u}, \boldsymbol{B})\|_V\right)\left[(\boldsymbol{B} \cdot \nabla)\boldsymbol{B}\right] \\ +\nabla\left(p + S\frac{|\boldsymbol{B}|^2}{2}\right) = G_1(t, \boldsymbol{u}(t - \rho_1(t))) \text{ in } (0,T) \times \Omega, \\ \boldsymbol{B}_t + F_N(\|(\boldsymbol{u}, \boldsymbol{B})\|_V)\left[(\boldsymbol{u} \cdot \nabla)\boldsymbol{B} - (\boldsymbol{B} \cdot \nabla)\boldsymbol{u}\right] + \frac{1}{R_m}\text{curl}(\text{curl}\boldsymbol{B}) = \\ G_2(t, \boldsymbol{B}(t - \rho_2(t))) \text{ in } (0,T) \times \Omega, \\ \text{div } \boldsymbol{u} = 0, \ \text{div } \boldsymbol{B} = 0 \text{ in}(0,T) \times \Omega, \\ \boldsymbol{u} = 0, \quad \boldsymbol{B} \cdot \boldsymbol{n} = 0 \text{ and } \text{curl}\boldsymbol{B} \times \boldsymbol{n} = 0 \text{ on } \Gamma, \\ \boldsymbol{u}(t,x) = \phi_1(t,x), \ \boldsymbol{B}(t,x) = \phi_2(t,x), \ t \in [-h, 0], \ x \in \Omega, \end{cases} \tag{1.3}$$

where $G_1(t, \boldsymbol{u}(t - \rho_1(t)))$ and $G_2(t, \boldsymbol{B}(t - \rho_2(t)))$ are external forces containing some hereditary characteristic (delays terms), where $0 \leq \rho_1(t), \rho_2(t) \leq h$. ϕ_1 and ϕ_2 are given functions defined in $[-h, 0] \times \Omega$. The GMMHDEFD (1.3) is inspired from the globally modified Navier-Stokes equations (GMNSE) with finite delays studied in [30, 31]. We refer the reader to [4, 5, 8, 27, 28, 37], just to cite some, for other models with delays.

The aim of this article is to establish the existence and uniqueness of solution of system (1.3). Moreover, we prove that, from a sequence of solutions of globally modified magnetohydrodynamics equations with delays, we can extract a subsequence which converges in an adequate sense to a weak solution of three dimensional magnetohydrodynamics equations with delays. This proves the existence of solutions for the three dimensional magnetohydrodynamics equations with locally

Lipschitz delays terms. This result is new in the literature.
The rest of the paper is structured as follows: in Section 2, we recall some spaces useful for the variational formulation of problem (1.3) and its resolution. We also present some mathematical properties and estimates related to the operators involved in the model. In Section 3 we establish the existence and the uniqueness of the solutions of the model. Section 4(the last one) is devoted to the study of the convergence of the weak solutions of (1.3) (depending on the parameter N) to a weak solution of (1.2).

2 Preliminaries

We recall from [14, 27, 35] the abstract spaces for model (1.3) and its abstract formulation. Bold notations will denote a vector or a tensor. We consider the well known Hilbert spaces $L^2(\Omega), H^m(\Omega), H_0^m(\Omega)$ and we set

$$\mathbb{L}^2(\Omega) := (L^2(\Omega))^3,\ \mathbb{H}^m(\Omega) := (H^m(\Omega))^3,\ \mathbb{H}_0^m(\Omega) := (H_0^m(\Omega))^3, \mathbb{L}_0^2(\Omega) := (L_0^2(\Omega))^3 \tag{2.1}$$

where $L_0^2(\Omega) = \left\{q \in L^2(\Omega);\ \int_\Omega q(x)dx = 0\right\}$. It is noted that for a vector $\boldsymbol{w}$ we set

$$\|\boldsymbol{w}\|_{\mathbb{L}^r(\Omega)}^r = \int_\Omega |\boldsymbol{w}(\boldsymbol{x})|^r dx\ ,$$

where $|\cdot|$ denotes the Euclidean norm $|\boldsymbol{w}|^2 = \boldsymbol{w}\cdot\boldsymbol{w}$. We shall frequently use Sobolev embedding: for a real number $p \in \mathbb{R}$, $1 \le p \le 6$, the space $\mathbb{H}^1(\Omega)$ is imbedded into $\mathbb{L}^p(\Omega)$. In particular, there exists a constant c_p which depends only on p, Ω and $d = 3$ such that

$$\text{for all}\ \ \boldsymbol{v} \in \mathbb{H}_0^1,\ \ \|\boldsymbol{v}\|_{\mathbb{L}^p(\Omega)} \le c_p|\nabla\boldsymbol{v}|\ . \tag{2.2}$$

When $p = 2$, this is Poincare's inequality and c_2 is Poincare's constant. In the case of the maximum norm, the following imbedding holds

$$\text{for all}\ r > d = 3,\ \ \mathbb{W}^{1,r}(\Omega) \subset \mathbb{L}^\infty(\Omega)$$

in particular, for each $r > d = 3$, there exists $c_{\infty,r}$ such that

$$\text{for all}\ \ \boldsymbol{v} \in \mathbb{H}_0^1(\Omega) \cap \mathbb{W}^{1,r},\ \ \|\boldsymbol{v}\|_{\mathbb{L}^\infty(\Omega)} \le c_{\infty,r}\|\nabla\boldsymbol{v}\|_{\mathbb{L}^r(\Omega)}\ . \tag{2.3}$$

Owing to Poincare's inequality, the semi-norm $|\cdot|$ is a norm on $\mathbb{H}_0^1(\Omega)$, equivalent to the full norm. As it is directly related gradient operator, we take this semi-norm as norm on $\mathbb{H}_0^1(\Omega)$, and we use it to define the dual norm on its dual space $\mathbb{H}^{-1}(\Omega)$:

$$\text{for all}\ \ \boldsymbol{f} \in \mathbb{H}^{-1}(\Omega),\qquad \|\boldsymbol{f}\|_{\mathbb{H}^{-1}(\Omega)} = \sup\frac{\langle\boldsymbol{f},\boldsymbol{v}\rangle}{|\nabla\boldsymbol{v}|}$$

where $\langle\cdot\rangle$ is the duality pairing between $\mathbb{H}^{-1}(\Omega)$ and $\mathbb{H}_0^1(\Omega)$. As usual for handling time dependent problems, it is convenient to consider functions defined on a time interval (a, b) with values in a functional space, say $\boldsymbol{Y}$ (see [2]). More precisely, we denote by $\|\cdot\|_{\boldsymbol{Y}}$ the norm on $\boldsymbol{Y}$ and for any number r with $1 \le r \le \infty$, we define

$$L^r(a, b; \boldsymbol{Y}) = \left\{w \text{ measurable in }\ (a, b)\ ;\ \ \int_a^b \|w(t)\|_{\boldsymbol{Y}}^r dt < \infty\right\}$$

equipped with the norm

$$\|w\|^r_{L^r(a,b;\boldsymbol{Y})} = \int_a^b \|w(t)\|^r_{\boldsymbol{Y}} dt$$

with the usual modification if $r = \infty$. It is a Banach space if $\boldsymbol{Y}$ is a Banach space, and when $r = 2$, it is a Hilbert space if $\boldsymbol{Y}$ is also a Hilbert space.
We also introduce the usual following spaces for MHD equations, (see [35])

$$\begin{aligned}
&\mathcal{V}_1 = \left\{\boldsymbol{u} \in (\mathcal{C}^\infty_c(\Omega))^3 : \mathrm{div}\boldsymbol{u} = 0\right\},\\
&V_1 = \text{the closure of } \mathcal{V}_1 \text{ in } \mathbb{H}^1_0(\Omega),\\
&H_1 = \left\{\boldsymbol{u} \in \mathbb{L}^2(\Omega) : \mathrm{div}\boldsymbol{u} = 0 \text{ and } \boldsymbol{u}\cdot\boldsymbol{n} = 0 \text{ on } \Gamma\right\}\\
&\mathcal{V}_2 = \left\{\boldsymbol{B} \in (\mathcal{C}^\infty(\bar{\Omega}))^3 : \mathrm{div}\boldsymbol{B} = 0,\ \boldsymbol{B}\cdot\boldsymbol{n} = 0 \text{ on } \Gamma\right\},\\
&V_2 = \left\{\boldsymbol{B} \in \mathbb{H}^1(\Omega) : \mathrm{div}\boldsymbol{B} = 0;\ \boldsymbol{B}\cdot\boldsymbol{n} = 0 \text{ on } \Gamma\right\},\\
&H_2 = \text{the closure of } \mathcal{V}_2 \text{ in } \mathbb{L}^2(\Omega).
\end{aligned} \tag{2.4}$$

Thus $H_2 = H_1$. We endow $H_i, i = 1, 2$ with the inner product of $\mathbb{L}^2(\Omega)$ and the norm of $\mathbb{L}^2(\Omega)$ denote respectively by $(.,.)_{L^2}$ and $|.|_{L^2}$.
We equip V_1 with the following inner product

$$((\boldsymbol{u},\boldsymbol{v}))_1 = \sum_{i=1}^3 \left(\frac{\partial \boldsymbol{u}}{\partial x_i}, \frac{\partial \boldsymbol{v}}{\partial x_i}\right)_{L^2}. \tag{2.5}$$

We equip V_2 with the scalar product

$$((\boldsymbol{u},\boldsymbol{v}))_2 = (\mathrm{curl}\boldsymbol{u}, \mathrm{curl}\boldsymbol{v})_{L^2}. \tag{2.6}$$

Where $\mathrm{curl}\boldsymbol{u} = \nabla \wedge \boldsymbol{u}$. We note that by Poincaré's inequality, the scalar product $((.,.))_1$ defines in (2.5) coincides with the well known inner product in $\mathbb{H}^1_0(\Omega)$. The norm generated by $((.,.))_2$ is equivalent to the norm induced by $\mathbb{H}^1(\Omega)$ on V_2 (see [15, Chapter VII]).
Hereafter, we set

$$H = H_1 \times H_2,\ V = V_1 \times V_2. \tag{2.7}$$

The dual space of V is denoted by V'. We endow H with the inner products defined as: for all $\varphi = (\boldsymbol{u},\boldsymbol{B}),\ \psi = (\boldsymbol{v},\boldsymbol{C}) \in H$.

$$\begin{aligned}
(\varphi,\psi) &= (\boldsymbol{u},\boldsymbol{v})_{L^2} + (\boldsymbol{B},\boldsymbol{C})_{L^2},\\
[\varphi,\psi] &= (\boldsymbol{u},\boldsymbol{v})_{L^2} + S(\boldsymbol{B},\boldsymbol{C})_{L^2}.
\end{aligned} \tag{2.8}$$

They generate equivalent norms (for $0 < S < \infty$)

$$|\varphi|^2_H = (\varphi,\varphi) = |\boldsymbol{u}|^2_{L^2} + |\boldsymbol{B}|^2_{L^2},\ [\varphi]^2_H = [\varphi,\varphi] = |\boldsymbol{u}|^2_{L^2} + S|\boldsymbol{B}|^2_{L^2}. \tag{2.9}$$

We also endow V with the inner products

$$((\varphi,\psi)) = \frac{1}{R_e}((\boldsymbol{u},\boldsymbol{v}))_1 + \frac{1}{R_m}((\boldsymbol{B},\boldsymbol{C}))_2,\ [[\varphi,\psi]] = \frac{1}{R_e}((\boldsymbol{u},\boldsymbol{v}))_1 + \frac{S}{R_m}((\boldsymbol{B},\boldsymbol{C}))_2, \tag{2.10}$$

which in turn generate the equivalent norms on V

$$\|\varphi\|^2_V = ((\varphi,\varphi)),\ [[\varphi]]^2_V = [[\varphi,\varphi]]. \tag{2.11}$$

In order to give an abstract formulation of problem (1.3), we introduce the operators $\mathcal{A}_1 \in \mathcal{L}(V_1, V_1')$, $\mathcal{A}_2 \in \mathcal{L}(V_2, V_2')$, and $\mathcal{A} \in \mathcal{L}(V, V')$ defined by

$$\begin{aligned}
&\langle \mathcal{A}_1 \boldsymbol{u}, \boldsymbol{v}\rangle = ((\boldsymbol{u}, \boldsymbol{v}))_1, \text{ for all } \boldsymbol{u}, \boldsymbol{v} \in V_1,\\
&\langle \mathcal{A}_2 \boldsymbol{B}, \boldsymbol{C}\rangle = ((\boldsymbol{B}, \boldsymbol{C}))_2, \text{ for all } \boldsymbol{B}, \boldsymbol{C} \in V_2,\\
&\langle \mathcal{A}\varphi, \psi\rangle = ((\varphi, \psi)), \text{ for all } \varphi, \psi \in V\,.
\end{aligned} \tag{2.12}$$

with domains

$$\begin{aligned}
D(\mathcal{A}_1) &= \{\boldsymbol{u} \in V_1 : \mathcal{A}_1 \boldsymbol{u} \in H_1\}\ ,\\
D(\mathcal{A}_2) &= \{\boldsymbol{u} \in V_2 : \mathcal{A}_2 \boldsymbol{u} \in H_2\}\ ,\\
D(\mathcal{A}) &= D(\mathcal{A}_1) \times D(\mathcal{A}_2)\,.
\end{aligned}$$

By the regularity of Γ, $D(\mathcal{A}) = \mathbb{H}^2 \cap V$. From the continuity of the embedding of V_i into H_i, $i = 1, 2$, there exists constant $\kappa_i, i = 1, 2$ such that

$$|\boldsymbol{u}|_{L^2} \le \kappa_1 \|\boldsymbol{u}\|_{V_1} \text{ for all } \boldsymbol{u} \in V_1,\ |\boldsymbol{B}|_{L^2} \le \kappa_2 \|\boldsymbol{B}\|_{V_2} \text{ for all } \boldsymbol{B} \in V_2. \tag{2.13}$$

The best constant κ_i is equal to $\frac{1}{\sqrt{\lambda_1^i}}$, where λ_1^i is the first eigenvalue of the compact operator $\mathcal{A}_i^{-1}$ from H_i into itself.
As in [35], we introduce the trilinear form $\boldsymbol{B}_0$ on $V \times V \times V$ by

$$\mathcal{B}_0(\varphi_1, \varphi_2, \varphi_3) = b(\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3) - Sb(\boldsymbol{B}_1, \boldsymbol{B}_2, \boldsymbol{u}_3) + b(\boldsymbol{u}_1, \boldsymbol{B}_2, \boldsymbol{B}_3) - b(\boldsymbol{B}_1, \boldsymbol{u}_2, \boldsymbol{B}_3), \tag{2.14}$$

for all $\varphi_i = (\boldsymbol{u}_i, \boldsymbol{B}_i) \in V (i = 1, 2, 3)$, where b is a continuous trilinear form defined on $\mathbb{H}^1(\Omega) \times \mathbb{H}^1(\Omega) \times \mathbb{H}^1(\Omega)$ by

$$b(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) = \sum_{i,j=1}^{3} \int_\Omega u_i \frac{\partial v_j}{\partial x_i} w_j dx,$$

which satisfies the following relations, see for instance [14, 30]

$$\begin{aligned}
&b(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{v}) = 0,\ \forall \boldsymbol{u} \in V_1,\ \boldsymbol{v} \in \mathbb{H}^1(\Omega),\\
&b(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) = -b(\boldsymbol{u}, \boldsymbol{w}, \boldsymbol{v}),\ \forall \boldsymbol{u} \in V_1,\ \boldsymbol{v}, \boldsymbol{w} \in \mathbb{H}^1(\Omega),\\
&|b(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w})| \le c\|\boldsymbol{u}\|_{V_1}^{1/2} |\mathcal{A}_1 \boldsymbol{u}|_{L^2}^{1/2} \|\boldsymbol{v}\|_{V_1} |\boldsymbol{w}|_{L^2},\ \forall \boldsymbol{u} \in D(\mathcal{A}_1), \boldsymbol{v} \in V_1, \boldsymbol{w} \in H_1\\
&|b(\boldsymbol{b}_1, \boldsymbol{b}_2, \boldsymbol{u})| \le c|\boldsymbol{b}_1|_{L^2}^{1/4} \|\boldsymbol{b}_1\|_{V_2}^{3/4} \|\boldsymbol{u}\|_{V_1} \|\boldsymbol{b}_2\|_{V_2},\ \forall \boldsymbol{b}_1, \boldsymbol{b}_2 \in V_2, \boldsymbol{u} \in V_1,\\
&|b(\boldsymbol{b}_1, \boldsymbol{b}_2, \boldsymbol{u})| \le c\|\boldsymbol{b}_1\|_{V_2} |\mathcal{A}_2 \boldsymbol{b}_2|_{L^2} |\boldsymbol{u}|_{L^2},\ \forall \boldsymbol{b}_1 \in V_2, \boldsymbol{b}_2 \in D(\mathcal{A}_2), \boldsymbol{u} \in H_1,\\
&|b(\boldsymbol{b}_1, \boldsymbol{u}_1, \boldsymbol{b}_2)| \le c\|\boldsymbol{b}_1\|_{V_2} |\mathcal{A}_1 \boldsymbol{u}_1|_{L^2} |\boldsymbol{b}_2|_{L^2},\ \forall \boldsymbol{b}_1 \in V_2, \boldsymbol{u}_1 \in D(\mathcal{A}_1), \boldsymbol{b}_2 \in H_2.\\
&|b(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w})| \le |\boldsymbol{u}|_{L^6} |\nabla \boldsymbol{v}|_{L^2} |\boldsymbol{w}|_{L^2}^{1/2} |\boldsymbol{w}|_{L^6}^{1/2},\ \forall \boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in \mathbb{H}^1(\Omega)\,.
\end{aligned} \tag{2.15}$$

Remark 2.1 *Using the inclusion of $\mathbb{H}^1(\Omega)$ in $\mathbb{L}^p(\Omega)$ $1 \le p \le 6$, we infer that $b(\cdot, \cdot, \cdot)$ also satisfies*

$$|b(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w})| \le \|\boldsymbol{u}\|_{V_1} \|\boldsymbol{v}\|_{V_1} |\boldsymbol{w}|_{L^2}^{1/2} \|\boldsymbol{w}\|_{V_1}^{1/2},\ \forall \boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in V_1. \tag{2.16}$$

From (2.15), we infer that

$$\begin{aligned}
&\mathcal{B}_0(\varphi_1, \varphi_2, \varphi_2) = 0,\ \forall \varphi_1, \varphi_2 \in V,\\
&\mathcal{B}_0(\varphi_1, \varphi_2, \varphi_3) = -\mathcal{B}_0(\varphi_1, \varphi_3, \varphi_2),\ \forall \varphi_i \in V, i = 1, 2, 3.
\end{aligned} \tag{2.17}$$

Now we introduce the continuous bilinear form $\mathcal{B} : V \times V \to V'$ by

$$\langle \mathcal{B}(\varphi_1, \varphi_2), \varphi_3\rangle = \mathcal{B}_0(\varphi_1, \varphi_2, \varphi_3). \tag{2.18}$$

We also introduce a diagonal matrix $\mathbf{M}= (m_{ij})_{1\le i,j\le 6} \in M_6(\mathbb{R})$ defined by:

$$\begin{cases} m_{ii} = 1 \text{ if } 1 \le i \le 3, \\ m_{ii} = S \text{ if } 4 \le i \le 6, \\ m_{ij} = 0 \text{ if } i \ne j. \end{cases} \tag{2.19}$$

Note that

$$\begin{aligned} \mathcal{B}_0(\varphi_1, \varphi_2, \mathbf{M}\varphi_2) \quad &= b(\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_2) + Sb(\boldsymbol{u}_1, \boldsymbol{B}_2, \boldsymbol{B}_2) \\ &-S[b(\boldsymbol{B}_1, \boldsymbol{B}_2, \boldsymbol{u}_2) + b(\boldsymbol{B}_1, \boldsymbol{u}_2, \boldsymbol{B}_2)]. \end{aligned} \tag{2.20}$$

It follows from (2.15) and (2.20) that

$$\begin{aligned} &\mathcal{B}_0(\varphi_1, \varphi_2, \mathbf{M}\varphi_2) = 0 \ \forall \varphi_1, \varphi_2 \in V, \\ &\mathcal{B}_0(\varphi_1, \varphi_2, \mathbf{M}\varphi_3) = -\mathcal{B}_0(\varphi_1, \varphi_3, \mathbf{M}\varphi_2), \ \forall \varphi_i \in V, i = 1, 2, 3. \end{aligned} \tag{2.21}$$

We recall that (see [35]) $\mathcal{B}_0$ and $\mathcal{B}$ satisfy the following estimates

$$\begin{aligned} &|\mathcal{B}_0(\varphi_1, \varphi_2, \varphi_3)| \le c\|\varphi_1\|_V \|\varphi_2\|_V^{1/2} |\mathcal{A}\varphi_2|_H^{1/2} |\varphi_3|_H, \ \forall \varphi_1 \in V, \varphi_2 \in D(\mathcal{A}), \varphi_3 \in H, \\ &\|\mathcal{B}(\varphi, \varphi)\|_{V'} \le c|\varphi|_H^{1/2} \|\varphi\|_V^{3/2}. \end{aligned} \tag{2.22}$$

Hereafter we set

$$\begin{aligned} &\mathcal{B}_0^N(\varphi_1, \varphi_2, \varphi_3) \\ &= F_N(\|\boldsymbol{u}_2\|_{V_1}) b(\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3) - SF_N(\|(\boldsymbol{u}_2, \boldsymbol{B}_2)\|_V) b(\boldsymbol{B}_1, \boldsymbol{B}_2, \boldsymbol{u}_3) \\ &+F_N(\|(\boldsymbol{u}_2, \boldsymbol{B}_2)\|_V) b(\boldsymbol{u}_1, \boldsymbol{B}_2, \boldsymbol{B}_3) - F_N(\|(\boldsymbol{u}_2, \boldsymbol{B}_2)\|_V) b(\boldsymbol{B}_1, \boldsymbol{u}_2, \boldsymbol{B}_3) \\ &\langle \mathcal{B}^N(\varphi_1, \varphi_2), \varphi_3 \rangle = \mathcal{B}_0^N(\varphi_1, \varphi_2, \varphi_3), \ \forall \varphi_i = (\boldsymbol{u}_i, \boldsymbol{B}_i) \in V, \ i = 1, 2, 3. \end{aligned} \tag{2.23}$$

Arguing similarly as in the proof of (2.22), we can check that the following inequalities hold

$$\begin{aligned} |\mathcal{B}_0^N(\varphi_1, \varphi_2, \varphi_3)| \quad &\le cN\|\varphi_1\|_V^{1/2} |\mathcal{A}\varphi_1|_H^{1/2} |\varphi_3|_H \\ &+cSN\|\varphi_1\|_V^{1/2} |\mathcal{A}\varphi_1|_H^{1/2} |\varphi_3|_H, \ \forall \varphi_1 \in V, \varphi_2 \in D(\mathcal{A}), \varphi_3 \in H, \end{aligned} \tag{2.24}$$

$$\begin{aligned} |\mathcal{B}_0^N(\varphi_1, \varphi_1, \varphi_2)| \quad &\le cN|\varphi_1|_H^{1/4} |\mathcal{A}\varphi_1|_H^{3/4} |\varphi_2|_H \\ &+cSN|\varphi_1|_H^{1/4} |\mathcal{A}\varphi_1|_H^{3/4} |\varphi_2|_H, \ \forall \varphi_1 \in D(\mathcal{A}), \ \varphi_2 \in H, \end{aligned} \tag{2.25}$$

$$\begin{aligned} \|\mathcal{B}^N(\varphi_1, \varphi_2)\|_{V'} \quad &\le c|\varphi_1|_H^{1/4} \|\varphi_1\|_V^{3/4} |\varphi_2|_H^{1/4} \|\varphi_2\|_V^{3/4} \\ &+cS|\varphi_1|_H^{1/4} \|\varphi_1\|_V^{3/4} |\varphi_2|_H^{1/4} \|\varphi_2\|_V^{3/4}, \ \forall \varphi_i = (\boldsymbol{u}_i, \boldsymbol{B}_i) \in V, \end{aligned} \tag{2.26}$$

$$\|\mathcal{B}^N(\varphi_1, \varphi_2)\|_{V'} \quad \le cN\|\varphi_1\|_V + cNS\|\varphi_1\|_V, \tag{2.27}$$

$$\begin{aligned} |\mathcal{B}_0^N(\varphi_1, \varphi_1, \varphi_2)| \quad &\le cN\|\varphi_1\|_V^{1/2} |\mathcal{A}\varphi_1|_H^{1/2} |\varphi_2|_H \\ &+c\|\varphi_1\|_V^{3/2} |\mathcal{A}\varphi_1|_H^{1/2} |\varphi_2|_H, \ \forall \varphi_1 \in D(\mathcal{A}), \ \varphi_2 \in H. \end{aligned} \tag{2.28}$$

The analysis of (1.3) will also required the following version of Gronwall's lemma, where its proof can be found in [34].

Lemma 2.1 *Let $T > 0$ and let κ be a non-negative function in $L^1(0, T)$. Let $c > 0$ be a constant and $\psi \in \mathcal{C}^0(0, T)$ a function that satisfies*

$$\textit{for all } t \in [0, T], \quad 0 \le \psi(t) \le c + \int_0^t \kappa(s)\psi(s)ds,$$

then ψ satisfies the bound

$$\psi(t) \le ce^{\int_0^t \kappa(s)ds}.$$

Here, $\mathcal{C}^0(0, T)$ denotes the set of continuous functions on $[0, T]$.

More assumptions on G_1 and G_2 are still required (see [26]); We assume for that
(h_1) : The mapping

$$\begin{array}{llll} G_1(.,\boldsymbol{u}): & (0,T) & \to & \mathbb{L}^2(\Omega) \\ & t & \mapsto & G_1(t,\boldsymbol{u}(t-\rho_1(t))) \end{array}$$

is measurable.
(h_2) : there exists a non-negative function $g_1 \in L^p_{\text{loc}}(\mathbb{R})$ for some $1 \le p \le +\infty$ and a non-decreasing function $l_1 : (0,+\infty) \to (0,+\infty)$ such that, for all $R > 0$, if $|\boldsymbol{u}|_{L^2}, |\boldsymbol{v}|_{L^2} \le R$, then

$$|G_1(.,\boldsymbol{u}) - G_1(.,\boldsymbol{v})|_{L^2} \le l_1(R) g_1^{1/2}(t) |\boldsymbol{u}-\boldsymbol{v}|_{L^2}. \tag{2.29}$$

(h_3) : there exists a non-negative function $\zeta_1 \in L^1_{\text{loc}}(\mathbb{R})$ such that for any $\boldsymbol{u} \in H_1$,

$$|G_1(.,\boldsymbol{u})|^2_{L^2} \le g_1(t) |\boldsymbol{u}|^2_{L^2} + \zeta_1(t). \tag{2.30}$$

(h_4) : The mapping

$$\begin{array}{llll} G_2(.,\boldsymbol{B}): & (0,T) & \to & \mathbb{L}^2(\Omega) \\ & t & \to & G_2(t,\boldsymbol{B}(t-\rho_2(t))) \quad \text{is measurable}. \end{array}$$

(h_5) : there exists a non-negative function $g_2 \in L^p_{\text{loc}}(\mathbb{R})$ for some $1 \le p \le +\infty$ and a non-decreasing function $l_2 : (0,+\infty) \to (0,+\infty)$ such that, for all $R > 0$, if $|\boldsymbol{B}_1|_{L^2}, |\boldsymbol{B}_2|_{L^2} \le R$, then

$$|G_2(.,\boldsymbol{B}_1) - G_2(.,\boldsymbol{B}_2)|_{L^2} \le l_2(R) g_2^{1/2}(t) |\boldsymbol{B}_1-\boldsymbol{B}_2|_{L^2}. \tag{2.31}$$

(h_6) : there exists a non-negative function $\zeta_2 \in L^1_{\text{loc}}(\mathbb{R})$ such that for any $\boldsymbol{B} \in H_1$,

$$|G_2(.,\boldsymbol{B})|^2_{L^2} \le g_2(t) |\boldsymbol{B}|^2_{L^2} + \zeta_2(t). \tag{2.32}$$

Finally, we suppose that $\phi_i \in L^{2p'}(-h,0;H_i)$ where p' satisfies $\frac{1}{p}+\frac{1}{p'}=1$. Since we will use the same techniques as in [26, 30], we suppose moreover that the delays functions $\rho_i \in \mathcal{C}^1([0,T])$ are such that $0 \le \rho_i \le h$ for all $t \in [0,T]$ and there exists two constants $\rho_1^*,\ \rho_2^*$ satisfying

$$\rho_i'(t) \le \rho_i^* < 1,\ \forall t \in [0,T],\ i = 1,\ 2. \tag{2.33}$$

Now, we are able to give a definition of a weak solution of (1.3).

Definition 2.1 *Let $(\boldsymbol{u}(0),\boldsymbol{B}(0)) = (\boldsymbol{u}_0,\boldsymbol{B}_0) \in H$, $\phi_i \in L^{2p'}(-h,0;H_i)$ be given; $\frac{1}{p}+\frac{1}{p'}=1$; G_1 and G_2 satisfying $(h_1)-(h_3)$ and $(h_4)-(h_6)$ respectively.*
A weak solution of (1.3) is any function $y = (\boldsymbol{u},\boldsymbol{B}) \in L^{2p'}(-h,T;H) \cap L^2(0,T;V) \cap L^\infty(0,T;H)$ such that, for a.e. $t \in (0,T)$

$$\left\{ \begin{array}{ll} \dfrac{d}{dt}y(t) + \mathcal{A}y(t) + \mathcal{B}^N(y(t),y(t)) & = G_1(t,\boldsymbol{u}(t-\rho_1(t))) + G_2(t,\boldsymbol{B}(t-\rho_2(t))) \quad on \quad V' \\ y(s,x) & = \phi(s,x) = (\phi_1(s,x),\phi_2(s,x)),\ s \in [-h,0],\ x \in \Omega \end{array} \right. \tag{2.34}$$

or equivalently for all $\varphi = (\boldsymbol{v},\boldsymbol{C}) \in V$

$$\left\{ \begin{array}{l} \left(\dfrac{d}{dt}y(t),\varphi\right) + ((y(t),\varphi)) + \mathcal{B}_0^N(y(t),y(t),\varphi) \\ = \langle G_1(t,\boldsymbol{u}(t-\rho_1(t))),\boldsymbol{v}\rangle + \langle G_2(t,\boldsymbol{B}(t-\rho_2(t))),\boldsymbol{C}\rangle, \\ y(s,x) = (\phi_1(s,x),\phi_2(s,x)),\ s \in [-h,0],\ x \in \Omega. \end{array} \right. \tag{2.35}$$

Remark 2.2 • *Definition* (2.1) *also provide the weak formulation of (1.3) which is, due to G. De Rham theorem, equivalent to it.*

- *If* $(\boldsymbol{u}, \boldsymbol{B})$ *is a weak solution of (1.3) and we define* $\widetilde{g}_i = g_i(\theta_i^{-1}(t))$, *where* $\theta_i : [0,T] \to [-\rho_i(0), T-\rho_i(T)]$ *is the differentiable and strictly increasing function given by* $\theta_i(s) = s - \rho_i(s)$, *then taking into account that* $\widetilde{g}_i \in L^p(-\rho_i(0), T)$ *for all* $T > 0$ *and* $(\boldsymbol{u}, \boldsymbol{B}) \in L^{2p'}(-h, T; H)$, *we infer that* $G_1(t, \boldsymbol{u}(t-\rho_1(t)))$ *and* $G_2(t, \boldsymbol{B}(t-\rho_2(t)))$ *belong to* $L^2(0,T;H_1)$ *and* $L^2(0,T;H_2)$ *respectively.*
  Indeed,

$$\begin{aligned}\int_0^T |G_1(t, \boldsymbol{u}(t-\rho_1(t)))|_{L^2}^2\,dt &\leq \int_0^T g_1(t)\,|\boldsymbol{u}(t-\rho_1(t))|_{L^2}^2\,dt + \int_0^T \zeta_1(t)dt\\ &\leq \tfrac{1}{1-\rho_1^*}\int_{-\rho_1(0)}^0 \widetilde{g}_1(t)\,|\phi_1(t)|_{L^2}^2\,dt +\\ &\tfrac{1}{1-\rho_1^*}\int_0^T \widetilde{g}_1(t)\,|\boldsymbol{u}(t)|_{L^2}^2\,dt + \int_0^T \zeta_1(t)dt\\ &\leq +\infty\end{aligned}$$

  We can also prove that $\displaystyle\int_0^T |G_2(t, \boldsymbol{B}(t-\rho_2(t)))|_{L^2}^2\,dt \leq +\infty$, *then* $\dfrac{d}{dt}y \in L^2(0,T;V')$ *and consequently* $y \in \mathcal{C}([0,T];H)$.

- *In addition, by taking* $\varphi = \boldsymbol{M}y$ *in* $(2.35)_1$ *and using* $(2.21)_1$ *we infer that* y *satisfies the following energy equality*

$$\begin{aligned}&|\boldsymbol{u}(t)|_{L^2}^2 + S|\boldsymbol{B}(t)|_{L^2}^2 + \tfrac{2}{R_e}\int_0^t \|\boldsymbol{u}(\xi)\|_{V_1}^2 d\xi + \frac{2S}{R_m}\int_0^t \|\boldsymbol{B}(\xi)\|_{V_2}^2 d\xi\\ &= |\boldsymbol{u}_0|_{L^2}^2 + S|\boldsymbol{B}_0|_{L^2}^2 + 2\int_0^t (G_1(t, \boldsymbol{u}(\xi-\rho_1(\xi))), \boldsymbol{u}(\xi))d\xi\\ &+2S\int_0^t (G_2(\xi, \boldsymbol{B}(\xi-\rho_2(\xi))), \boldsymbol{B}(\xi))d\xi.\end{aligned} \tag{2.36}$$

3 Existence and uniqueness result

In this section, we prove that problem (2.34) has a unique weak solution which is, indeed a strong solution. Before doing this, we recall from [6, 33, 36] the following properties of F_N, where the proof can be found in [6, 33]. These properties are the main tools in the proof of the uniqueness result.

$$\begin{aligned}&|F_N(p) - F_N(r)| \leq \tfrac{|p-r|}{r},\ \forall p, r \in \mathbb{R}^+,\ r \neq 0,\\ &|F_N(\|\boldsymbol{u}\|_{V_1}) - F_N(\|\boldsymbol{v}\|_{V_1})| \leq \tfrac{\|\boldsymbol{u}-\boldsymbol{v}\|_{V_1}}{\|\boldsymbol{v}\|_{V_1}},\ \boldsymbol{u}, \boldsymbol{v} \in V_1,\ \boldsymbol{v} \neq 0,\\ &|F_M(p) - F_N(r)| \leq \tfrac{|M-N|}{r} + \tfrac{|p-r|}{r},\ \forall p, r, M, N \in \mathbb{R}^+, r \neq 0\\ &|F_N(\|\boldsymbol{u}\|_{V_1}) - F_N(\|\boldsymbol{v}\|_{V_1})| \leq \tfrac{1}{N}F_N(\|\boldsymbol{u}\|_{V_1})F_N(\|\boldsymbol{v}\|_{V_1})\|\boldsymbol{u}-\boldsymbol{v}\|_{V_1},\ \boldsymbol{u}, \boldsymbol{v} \in V_1.\end{aligned} \tag{3.1}$$

In the rest of this work, we will denote by c, a generic positive constant (possibly depending on $S, R_e, R_m, \kappa_1, \kappa_2, \Omega$,) which can vary even within the same line. However, this constant is always independent of time and initial data. We start by proving the uniqueness result; for this purpose, we have:

Theorem 3.1 *There exists at most one weak solution* $(\boldsymbol{u}, \boldsymbol{B})$ *of (2.34) in the sense of Definition 2.1.*

proof. Let $y_i = (\boldsymbol{u}_i, \boldsymbol{B}_i), i = 1, 2$ be weak solutions to (2.34) that belong to $L^2(0, T; V)$ and $R > 0$ such that $|\boldsymbol{u}_i|_{L^2}$, $|\boldsymbol{B}_i|_{L^2} \leq R$. We set $\delta y = (\delta \boldsymbol{u}, \delta \boldsymbol{B}) = y_1 - y_2$. Then $(\delta \boldsymbol{u}, \delta \boldsymbol{B})$ satisfies for $a.e.t \in (0, T)$,

$$\begin{cases} \dfrac{d}{dt}\delta y(t) + \mathcal{A}\delta y(t) = -\left(\mathcal{B}^N(y_1(t), y_1(t)) - \mathcal{B}^N(y_2(t), y_2(t))\right) + G_1(t, \boldsymbol{u}_1(t-\rho_1(t))) \\ -G_1(t, \boldsymbol{u}_2(t-\rho_1(t))) + G_2(t, \boldsymbol{B}_1(t-\rho_2(t))) - G_2(t, \boldsymbol{B}_2(t-\rho_2(t))) \ \text{ in } \ V', \\ \delta y(0) = 0. \end{cases} \tag{3.2}$$

Taking the scalar product in H of (3.2) with $M\delta y$, we obtain

$$\begin{array}{l} \dfrac{d}{dt}\mathcal{Y}(t) + \dfrac{2}{R_e}\|\delta \boldsymbol{u}(t)\|_{V_1}^2 + \dfrac{2S}{R_m}\|\delta \boldsymbol{B}(t)\|_{V_2}^2 = -2(\mathcal{B}^N(y_1(t), y_1(t)) - \mathcal{B}^N(y_2(t), y_2(t)), M\delta y(t)) \\ +2\left(G_1(t, \boldsymbol{u}_1(t-\rho_1(t))), \delta \boldsymbol{u}\right) - 2\left(G_1(t, \boldsymbol{u}_2(t-\rho_1(t))), \delta \boldsymbol{u}\right) \\ +2\left(G_2(t, \boldsymbol{B}_1(t-\rho_2(t))) - G_2(t, \boldsymbol{B}_2(t-\rho_2(t))), S\delta \boldsymbol{B}\right) \end{array} \tag{3.3}$$

with $\mathcal{Y} = |\delta \boldsymbol{u}(t)|_{L^2}^2 + S|\delta \boldsymbol{B}(t)|_{L^2}^2$ and $2(-\mathcal{B}^N(y_1(t), y_1(t)) + \mathcal{B}^N(y_2(t), y_2(t)), \boldsymbol{M}\delta y(t))$ satisfies the following (see [14] for the details)

$$2\left(-\mathcal{B}^N(y_1(t), y_1(t)) + \mathcal{B}^N(y_2(t), y_2(t)), \boldsymbol{M}y(t)\right) \leq \left(cN^4 + cN^8\right)\mathcal{Y}(t). \tag{3.4}$$

Using hypotheses (h_2) and (h_5), we obtain

$$\begin{array}{l} 2\left(G_1(t, \boldsymbol{u}_1(t-\rho_1(t))) - G_1(t, \boldsymbol{u}_2(t-\rho_1(t))), \delta \boldsymbol{u}\right) + \\ 2\left(G_2(t, \boldsymbol{B}_1(t-\rho_2(t))) - G_2(t, \boldsymbol{B}_2(t-\rho_2(t))), S\delta \boldsymbol{B}\right) \\ \leq 2l_1(R)g_1^{1/2}(t)\left|\delta \boldsymbol{u}(t-\rho_1(t))\right|_{L^2}\left|\delta \boldsymbol{u}\right|_{L^2} + 2Sl_2(R)g_2^{1/2}(t))\left|\delta \boldsymbol{B}(t-\rho_2(t))\right|_{L^2}\left|\delta \boldsymbol{B}\right|_{L^2} \end{array} \tag{3.5}$$

Integrating (3.3) after dropping momentarily the term $\frac{2}{R_e}\|\delta \boldsymbol{u}(t)\|_{V_1}^2 + \frac{2S}{R_m}\|\delta \boldsymbol{B}(t)\|_{V_2}^2$ and using (3.4)-(3.5) we have

$$\begin{aligned} \mathcal{Y}(t) \quad & \leq \left(cN^4 + cN^8\right)\int_0^T \mathcal{Y}(\xi)d\xi + 2l_1(R)\int_0^T g_1^{1/2}(\xi)\left|\delta \boldsymbol{u}(\xi - \phi_1(\xi))\right|_{L^2}\left|\delta \boldsymbol{u}(\xi)\right|_{L^2}d\xi \\ & +2Sl_2(R)\int_0^T g_2^{1/2}(\xi))\left|\delta \boldsymbol{B}(\xi - \phi_2(\xi))\right|_{L^2}\left|\delta \boldsymbol{B}(\xi)\right|_{L^2}d\xi \\ & \leq \tfrac{2l_1(R)}{1-\rho_1^*}\int_{-\rho_1(0)}^{T-\rho_1(T)}\widetilde{g}_1^{1/2}(\xi)\left|\delta \boldsymbol{u}(\xi)\right|_{L^2}^2 d\xi + \frac{2Sl_1(R)}{1-\rho_2^*}\int_{-\rho_2(0)}^{T-\rho_2(T)}\widetilde{g}_2^{1/2}(\xi)\left|\delta \boldsymbol{B}(\xi)\right|_{L^2}^2 d\xi + \\ & \left(cN^4 + cN^8\right)\textstyle\int_0^T \mathcal{Y}(\xi)d\xi \\ & \leq l\int_0^T\left(\widetilde{g}_1^{1/2}(\xi) + \widetilde{g}_2^{1/2}(\xi)\right)\mathcal{Y}(\xi)d\xi + \left(cN^4 + cN^8\right)\int_0^T \mathcal{Y}(\xi)d\xi \\ & \leq \int_0^T\left(l\left(\widetilde{g}_1^{1/2}(\xi) + \widetilde{g}_2^{1/2}(\xi)\right) + cN^4 + cN^8\right)\mathcal{Y}(\xi)d\xi \end{aligned} \tag{3.6}$$

where we have used the fact that $\delta \boldsymbol{u}(t) = 0$ and $\delta \boldsymbol{B}(t) = 0$ in $[-\rho_1(0), 0]$ and $[-\rho_2(0), 0]$ respectively and set $l = \max\left\{\frac{2l_1(R)}{1-\rho_1^*}, \frac{2l_2(R)}{1-\rho_2^*}\right\}$
Then, from lemma 2.1, we infer that $\mathcal{Y} \leq 0$, consequently, $\boldsymbol{u}_1 = \boldsymbol{u}_2$ and $\boldsymbol{B}_1 = \boldsymbol{B}_2$. □

Now, we state the existence result.

Theorem 3.2 *Let* $(\boldsymbol{u}(0), \boldsymbol{B}(0)) \in H$, $\phi_i \in L^{2p'}(-h, 0; H_i)$ *be given;* $\frac{1}{p} + \frac{1}{p'} = 1$; G_1 *and* G_2 *satisfying* $(h_1) - (h_3)$ *and* $(h_4) - (h_6)$ *respectively.*

Then there exists a weak solution $(\boldsymbol{u}, \boldsymbol{B})$ *of* (2.34), *which is in fact a strong solution in the sense that it belongs to*

$$\mathcal{C}(0, T; V) \cap L^2(\epsilon, T; D(\mathcal{A}_1) \times D(\mathcal{A}_2)) \quad \textit{for all } 0 < \epsilon < T. \tag{3.7}$$

Moreover, if $(\boldsymbol{u}(0), \boldsymbol{B}(0)) \in V$, *then* $(\boldsymbol{u}, \boldsymbol{B})$ *satisfies*

$$(\boldsymbol{u}, \boldsymbol{B}) \in \mathcal{C}(0, T; V) \cap L^2(0, T; D(\mathcal{A}_1) \times D(\mathcal{A}_2)). \tag{3.8}$$

Proof We split it in several steps.
Step1: A Galerkin scheme. Since the injection $V \subset H$ is compact,
let $\{(w_i, \psi_i), i = 1, 2, ...\} \subset V$ be an orthonormal basis of H, where $\{w_i, i = 1, 2, ...\}$, $\{\psi_i, i = 1, 2, ...\}$ are eigenfunctions of $\mathcal{A}_1$ and $\mathcal{A}_2$, respectively. We set
$V_n = H_n = \text{span}\,\{(w_1, \psi_1), ..., (w_n, \psi_n)\}$ and denote by $P_n = (P_n^1, P_n^2)$, the orthogonal projector from H onto V_n for the scalar product $(.,.)$ defined by $(2.8)_1$. Note that P_n is also the orthogonal projector from $D(\mathcal{A}), V, V'$ onto V_n. We look for $y_n = P_n(\boldsymbol{u}, \boldsymbol{B}) = (\boldsymbol{u}_n, \boldsymbol{B}_n) \in H_n$ solution to the ordinary differential equations with delay

$$\begin{cases} \dfrac{d}{dt} y_n(t) + P_n \mathcal{A} y_n(t) + P_n \mathcal{B}^N(y_n(t), y_n(t)) \\ = P_n^1 G_1(t, \boldsymbol{u}(t - \rho_1(t))) + P_n^2 G_2(t, \boldsymbol{B}(t - \rho_2(t))) \quad \text{on} \quad \mathcal{D}'(0, T; V_n') \\ \\ y_n(s) = P_n(\phi_1(s), \phi_2(s)) = (P_n^1 \phi_1(s), P_n^2 \phi_2(s)), \ s \in [-h, 0]. \end{cases} \tag{3.9}$$

According to $(h_1)-(h_6)$, the above system of the ordinary differential equations with delay satisfies the conditions for existence and uniqueness of solution y_n on an interval $[0, T_n]$, $T_n \leq T$ (see Theorem 3.10 of [16], page 20). It will follow from a priori estimates that y_n exists on the interval $[0, T]$.
Step2: A priori estimates.
As in remark 2.2, y_n satisfies the following energy inequality:

$$\begin{aligned} &\frac{d}{dt} |\boldsymbol{u}_n(t)|^2_{L^2} + S \frac{d}{dt} |\boldsymbol{B}_n(t)|^2_{L^2} + \frac{2}{R_e} \|\boldsymbol{u}_n(t)\|^2_{V_1} + \frac{2S}{R_m} \|\boldsymbol{B}_n(t)\|^2_{V_2} = \\ &2(P_n^1 G_1(t, \boldsymbol{u}_n(t - \rho_1(t))), \boldsymbol{u}_n(t)) + 2S(P_n^2 G_2(t, \boldsymbol{B}_n(t - \rho_2(t))), \boldsymbol{B}_n(t)). \end{aligned} \tag{3.10}$$

Note that by $(h_3), (h_6)$, Young's and Cauchy-Schwartz's inequalities, we have

$$\begin{aligned} |2(P_n^1 G_1(t, \boldsymbol{u}_n(t - \rho_1(t))), \boldsymbol{u}_n(t))| &\leq 2\|G_1(t, \boldsymbol{u}_n(t - \rho_1(t)))\|_{V_1'} \|\boldsymbol{u}_n(t)\|_{V_1} \\ &\leq 2c\,|G_1(t, \boldsymbol{u}_n(t - \rho_1(t)))|_{H_1} \|\boldsymbol{u}_n(t)\|_{V_1} \\ &\leq c\,|G_1(t, \boldsymbol{u}_n(t - \rho_1(t)))|^2_{H_1} + \tfrac{1}{R_e} \|\boldsymbol{u}_n(t)\|^2_{V_1} \\ &\leq \tfrac{1}{R_e} \|\boldsymbol{u}_n(t)\|^2_{V_1} + c\left(g_1(t)\,|\boldsymbol{u}_n(t - \rho_1(t))|^2_{L^2} + \zeta_1(t)\right), \end{aligned} \tag{3.11}$$

$$\begin{aligned} |2(P_n^2 G_2(t, \boldsymbol{B}_n(t - \rho_2(t))), \boldsymbol{B}_n(t))| &\leq 2\|G_2(t, \boldsymbol{B}_n(t - \rho_2(t)))\|_{V_2'} \|\boldsymbol{B}_n(t)\|_{V_2} \\ &\leq 2c\,|G_2(t, \boldsymbol{B}_n(t - \rho_2(t)))|_{H_2} \|\boldsymbol{B}_n(t)\|_{V_2} \\ &\leq c\,|G_2(t, \boldsymbol{B}_n(t - \rho_2(t)))|^2_{H_2} + \tfrac{S}{R_m} \|\boldsymbol{B}_n(t)\|^2_{V_2} \\ &\leq \tfrac{S}{R_m} \|\boldsymbol{B}_n(t)\|^2_{V_2} + c\left(g_2(t)\,|\boldsymbol{B}_n(t - \rho_2(t))|^2_{L^2} + \zeta_2(t)\right). \end{aligned} \tag{3.12}$$

Inserting the estimates (3.11) and (3.12) in (3.10) and integrating over the interval $(0,t)$ with $t \le T$, , we obtain by using also remark 2.2:

$$
\begin{aligned}
&|\boldsymbol{u}_n(t)|_{L^2}^2 + S|\boldsymbol{B}_n(t)|_{L^2}^2 + \tfrac{1}{R_e}\int_0^t \|\boldsymbol{u}_n(\xi)\|_{V_1}^2 d\xi + \frac{S}{R_m}\int_0^t \|\boldsymbol{B}_n(\xi)\|_{V_2}^2 d\xi \\
&\le |\boldsymbol{u}_0|_{L^2}^2 + S|\boldsymbol{B}_0|_{L^2}^2 + \int_0^T c\left(g_1(\xi)\,|\boldsymbol{u}_n(\xi-\rho_1(\xi))|_{L^2}^2 + \zeta_1(\xi)\right)d\xi + \\
&\int_0^T c\left(g_2(\xi)\,|\boldsymbol{B}_n(\xi-\rho_2(\xi))|_{L^2}^2 + \zeta_2(\xi)\right)d\xi \\
&\le |\boldsymbol{u}_0|_{L^2}^2 + S|\boldsymbol{B}_0|_{L^2}^2 + \int_0^T c\left(\zeta_1(\xi)+\zeta_2(\xi)\right)d\xi + \frac{c}{1-\rho_1^*}\int_{-\rho_1(0)}^{T-\rho_1(T)} \tilde{g}_1(\xi)\,|\boldsymbol{u}_n(\xi)|_{L^2}^2\,d\xi \\
&+\tfrac{c}{1-\rho_2^*}\int_{-\rho_2(0)}^{T-\rho_2(T)} \widetilde{g}_2(\xi)\,|\boldsymbol{B}_n(\xi)|_{L^2}^2\,d\xi \\
&\le |\boldsymbol{u}_0|_{L^2}^2 + S|\boldsymbol{B}_0|_{L^2}^2 + \int_0^T c\left(\zeta_1(\xi)+\zeta_2(\xi)\right)d\xi + \frac{c}{1-\rho_1^*}\int_{-\rho_1(0)}^{0} \widetilde{g}_1(\xi)\,|\phi_1(\xi)|_{L^2}^2\,d\xi + \\
&\tfrac{c}{1-\rho_1^*}\int_0^T \widetilde{g}_1(\xi)\,|\boldsymbol{u}_n(\xi)|_{L^2}^2\,d\xi + \frac{c}{1-\rho_2^*}\int_{-\rho_2(0)}^{0} \widetilde{g}_2(\xi)\,|\phi_2(\xi)|_{L^2}^2\,d\xi + \frac{c}{1-\rho_2^*}\int_0^T \widetilde{g}_2(\xi)\,|\boldsymbol{B}_n(\xi)|_{L^2}^2\,d\xi \\
&\le |\boldsymbol{u}_0|_{L^2}^2 + S|\boldsymbol{B}_0|_{L^2}^2 + l\int_0^T \left[\widetilde{g}_1(\xi) + \frac{\widetilde{g}_2(\xi)}{S}\right]\left(|\boldsymbol{u}_n(\xi)|_{L^2}^2 + S|\boldsymbol{B}_n(\xi)|_{L^2}^2\right)d\xi \\
&+\int_0^T c\left(\zeta_1(\xi)+\zeta_2(\xi)\right)d\xi + l\left(\int_{-\rho_1(0)}^0 |\widetilde{g}_1(\xi)|^p d\xi\right)^{\frac{1}{p}}\left(\int_{-\rho_1(0)}^0 |\phi_1(\xi)|_{L^2}^{2p'}\,d\xi\right)^{\frac{2}{2p'}} d\xi + \\
&l\left(\int_{-\rho_2(0)}^0 |\widetilde{g}_2(\xi)|^p d\xi\right)^{\frac{1}{p}}\left(\int_{-\rho_2(0)}^0 |\phi_2(\xi)|_{L^2}^{2p'}\,d\xi\right)^{\frac{2}{2p'}} d\xi
\end{aligned}
\tag{3.13}
$$

where $l = \max\left\{\frac{c}{1-\rho_1^*}, \frac{c}{1-\rho_2^*}\right\}$. Then, from the assumptions, there exits a constant κ_1 such that

$$
\begin{aligned}
&|\boldsymbol{u}_n(t)|_{L^2}^2 + S|\boldsymbol{B}_n(t)|_{L^2}^2 + \tfrac{1}{R_e}\int_0^t \|\boldsymbol{u}_n(\xi)\|_{V_1}^2 d\xi + \frac{S}{R_m}\int_0^t \|\boldsymbol{B}_n(\xi)\|_{V_2}^2 d\xi \\
&\le |\boldsymbol{u}_0|_{L^2}^2 + S|\boldsymbol{B}_0|_{L^2}^2 + \kappa_1 + l\int_0^T \left[\widetilde{g}_1(\xi) + \frac{\widetilde{g}_2(\xi)}{S}\right]\left(|\boldsymbol{u}_n(\xi)|_{L^2}^2 + S|\boldsymbol{B}_n(\xi)|_{L^2}^2\right)d\xi
\end{aligned}
\tag{3.14}
$$

where

$$
\begin{aligned}
\kappa_1 = &\int_0^T c\left(\zeta_1(\xi)+\zeta_2(\xi)\right)d\xi + l\left(\int_{-\rho_1(0)}^0 |\widetilde{g}_1(\xi)|^p d\xi\right)^{\frac{1}{p}}\left(\int_{-\rho_1(0)}^0 |\phi_1(\xi)|_{L^2}^{2p'}\,d\xi\right)^{\frac{2}{2p'}} d\xi + \\
&l\left(\int_{-\rho_2(0)}^0 |\widetilde{g}_2(\xi)|^p d\xi\right)^{\frac{1}{p}}\left(\int_{-\rho_2(0)}^0 |\phi_2(\xi)|_{L^2}^{2p'}\,d\xi\right)^{\frac{2}{2p'}} d\xi.
\end{aligned}
\tag{3.15}
$$

Dropping momentarily the term $\frac{1}{R_e}\int_0^t \|\boldsymbol{u}_n(\xi)\|_{V_1}^2 d\xi + \frac{S}{R_m}\int_0^t \|\boldsymbol{B}_n(\xi)\|_{V_2}^2 d\xi$ and using lemma 2.1 in (3.13), we obtain

$$
\begin{aligned}
&|\boldsymbol{u}_n(t)|_{L^2}^2 + S|\boldsymbol{B}_n(t)|_{L^2}^2 \le \\
&\left[|\boldsymbol{u}_0|_{L^2}^2 + S|\boldsymbol{B}_0|_{L^2}^2 + \kappa_1\right]\exp\left\{\int_0^T \left(l g_1(\xi) + \frac{l}{S} g_2(\xi)\right)d\xi\right\}.
\end{aligned}
\tag{3.16}
$$

Using (h_2), (h_5), we infer that the right hand side of (3.16) is bounded independently of n; taking into account this bound in (3.13), we conclude that
$\frac{1}{R_e}\int_0^t \|\boldsymbol{u}_n(\xi)\|_{V_1}^2 d\xi + \dfrac{S}{R_m}\int_0^t \|\boldsymbol{B}_n(\xi)\|_{V_2}^2 d\xi$ is also uniformly bounded in n.
Moreover, using the equality $\frac{1}{p}+\frac{1}{p'}=1$, we infer that

$$\begin{aligned}\int_{-h}^{T} |y_n(t)|_H^{2p'}\, dt &= \textstyle\int_{-h}^{0} |P_n\phi(t)|_H^{2p'}\, dt + \displaystyle\int_0^T |y_n(t)|_H^{2p'}\, dt \\ &\le \|\phi\|_{L^{2p'}(-h,0;H)}^{2p'} + T\,\|y_n\|_{L^\infty(0,T;H)}^{2p'} \\ &= \|\phi\|_{L^{2p'}(-h,0;H)}^{\frac{2p-1}{2p}} + T\,\|y_n\|_{L^\infty(0,T;H)}^{\frac{2p-1}{2p}} \\ &\le +\infty.\end{aligned} \tag{3.17}$$

Consequently, the sequence

$$y_n = (\boldsymbol{u}_n, \boldsymbol{B}_n) \in L^\infty(0,T;H)\cap L^2(0,T;V)\cap L^{2p'}(-h,T;H). \tag{3.18}$$

Hence, we can use a compactness argument (see [41]) to extract a subsequence from $y_n = (\boldsymbol{u}_n, \boldsymbol{B}_n)$ still denoted by $y_n = (\boldsymbol{u}_n, \boldsymbol{B}_n)$ satisfying

$$y_n \to y \begin{cases} \text{weak-star in } L^\infty(0,T;H), \\ \text{weakly in } L^2(0,T;V), \\ \text{strongly in } L^2(0,T;H), \\ \text{a.e., in } (0,T)\times\Omega, \\ \text{weak-star in } L^{2p'}(-h,T;H), \\ \text{strongly in } L^{\frac{2p}{2p-1}}(-h,T;H), \end{cases} \tag{3.19}$$

with $y=(\boldsymbol{u},\boldsymbol{B}) \in L^\infty(0,T;H)\cap L^2(0,T;V)\cap L^{2p'}(-h,T;H)$.
According to (3.19), we can assume ([45], proposition 21.23, page 258) without loss of generality, that there exists two functions $z_1 \in L^{\frac{2p}{2p-1}}(-h,T)$ and $z_2 \in L^{\frac{2p}{2p-1}}(-h,T)$ such that

$$|\boldsymbol{u}_n(t)|_{H_1} \le z_1(t),\ |\boldsymbol{B}_n(t)|_{H_2} \le z_2(t) \tag{3.20}$$

But the estimates (3.18) are not enough to pass to the limit in (2.34) and deduce the solution of (1.3). Indeed, we need stronger estimates to prove the following:

$$\begin{aligned} &F_N(\|\boldsymbol{u}_n\|_{V_1}) \to F_N(\|\boldsymbol{u}\|_{V_1}) \text{ as } n\to\infty, \\ &F_N(\|(\boldsymbol{u}_n,\boldsymbol{B}_n)\|_V) \to F_N(\|(\boldsymbol{u},\boldsymbol{B})\|_V) \text{ as } n\to\infty\,. \end{aligned} \tag{3.21}$$

First, we take the inner product in H of $(3.9)_1$ with $\mathcal{A}y_n$, we obtain

$$\begin{aligned} &\frac{d}{dt}\|y_n(t)\|_V^2 + 2|\mathcal{A}y_n(t)|_H^2 = 2 - 2\mathcal{B}_0^N(y_n(t), y_n(t), \mathcal{A}y_n(t)) \\ &+2(G_1(t,\boldsymbol{u}_n(t-\rho_1(t))), \mathcal{A}_1\boldsymbol{u}_n(t))) + (G_2(t,\boldsymbol{B}_n(t-\rho_2(t)), \mathcal{A}_2\boldsymbol{B}_n(t))). \end{aligned} \tag{3.22}$$

Now using (2.24) and Young's inequality with the exponents $(4, 4/3)$, we have

$$\begin{aligned} 2|\mathcal{B}_0^N(y_n(t), y_n(t), \mathcal{A}y_n(t))| &\le cN\|y_n(t)\|_V^{1/2}|\mathcal{A}y_n(t)|_H^{3/2} \\ &\le \tfrac{1}{2}|\mathcal{A}y_n(t)|_H^2 + cN^4\|y_n(t)\|_V^2. \end{aligned} \tag{3.23}$$

In addition, using again (h_3), (h_6), Young's and Cauchy-Schwartz's inequalities, we obtain

$$\begin{aligned}
&|2(P_n^1 G_1(t, \boldsymbol{u}_n(t-\rho_1(t))), \mathcal{A}_1 \boldsymbol{u}_n(t))| \\
&\leq 2|G_1(t, \boldsymbol{u}_n(t-\rho_1(t)))|_{L^2} |\mathcal{A}_1 \boldsymbol{u}_n(t)|_{L^2} \\
&\leq c\,|G_1(t, \boldsymbol{u}_n(t-\rho_1(t)))|_{L^2}^2 + \tfrac{1}{2R_e} |\mathcal{A}_1 \boldsymbol{u}_n(t)|_{L^2}^2 \\
&\leq \tfrac{1}{2R_e} |\mathcal{A}_1 \boldsymbol{u}_n(t)|_{L^2}^2 + c\left(g_1(t)\,|\boldsymbol{u}_n(t-\rho_1(t))|_{L^2}^2 + \zeta_1(t)\right),
\end{aligned} \tag{3.24}$$

$$\begin{aligned}
&|2(P_n^2 G_2(t, \boldsymbol{B}_n(t-\rho_2(t))), \mathcal{A}_2 \boldsymbol{B}_n(t))| \\
&\leq 2|G_2(t, \boldsymbol{B}_n(t-\rho_2(t)))|_{L^2} |\mathcal{A}_2 \boldsymbol{B}_n(t)|_{L^2} \\
&\leq c\,|G_2(t, \boldsymbol{B}_n(t-\rho_2(t)))|_{L^2}^2 + \tfrac{1}{2R_m} |\mathcal{A}_2 \boldsymbol{B}_n(t)|_{L^2}^2 \\
&\leq \tfrac{1}{2R_m} |\mathcal{A}_2 \boldsymbol{B}_n(t)|_{L^2}^2 + c\left(g_2(t)\,|\boldsymbol{B}_n(t-\rho_2(t))|_{L^2}^2 + \zeta_2(t)\right),
\end{aligned} \tag{3.25}$$

Inserting the estimates (3.23) – (3.25) in (3.22), we obtain

$$\begin{aligned}
&\frac{d}{dt}\|y_n(t)\|_V^2 + |\mathcal{A}y_n(t)|_H^2 \leq cN^4 \|y_n(t)\|_V^2 + c g_1(t)\,|\boldsymbol{u}_n(t-\rho_1(t))|_{L^2}^2 + c\zeta_1(t) + \\
&c g_2(t)\,|\boldsymbol{B}_n(t-\rho_2(t))|_{L^2}^2 + c\zeta_2(t).
\end{aligned} \tag{3.26}$$

Now we distinguish two cases:
Case 1: $y(0) = (\boldsymbol{u}(0), \boldsymbol{B}(0)) \in H$.
Integrating (3.26) between s and t for $0 < s \leq t \leq T$ and use the continuous injection $V \hookrightarrow H$ and (3.13), we obtain

$$\begin{aligned}
&\|y_n(t)\|_V^2 + \tfrac{1}{R_e}\int_s^t |\mathcal{A}y_n(\xi)|_H^2 d\xi \\
&\leq \|y_n(s)\|_V^2 + cN^4 \int_0^T \|y_n(\xi)\|_V^2 d\xi + \kappa_1 + l\int_0^T c\,[\widetilde{g}_1(\xi) + \widetilde{g}_2(\xi)]\left(\|\boldsymbol{u}_n(\xi)\|_{V_1}^2 + \|\boldsymbol{B}_n(\xi)\|_{V_2}^2\right) d\xi \\
&\leq \|y_n(s)\|_V^2 + cN^4 \int_0^T \|y_n(\xi)\|_V^2 d\xi + \kappa_1 + lc\int_0^T [\widetilde{g}_1(\xi) + \widetilde{g}_2(\xi)]\,\|y_n(\xi)\|_V^2 d\xi.
\end{aligned} \tag{3.27}$$

Dropping momentarily the term $\frac{1}{R_e}\int_s^t |\mathcal{A}y_n(\xi)|_H^2 d\xi$ and using lemma 2.1 in (3.27), we obtain

$$\begin{aligned}
&\|\boldsymbol{u}_n(t)\|_{V_1}^2 + \|\boldsymbol{B}_n(t)\|_{V_2}^2 \leq \\
&\left[\|y_n(s)\|_V^2 + \kappa_1\right] \exp\left\{\int_0^T \left[cN^4 + lc\,(g_1(\xi) + g_2(\xi))\right] d\xi\right\}
\end{aligned} \tag{3.28}$$

Now, integrating (3.28) between 0 and ϵ for some $\epsilon \in (0, T)$, we have

$$\left[\epsilon\|y_n(t)\|_V^2 \leq \int_0^T \|y_n(s)\|_V^2 ds + T\kappa_1\right] \exp\left\{\int_0^T \left[cN^4 + lc\,(g_1(\xi) + g_2(\xi))\right] d\xi\right\} \tag{3.29}$$

Using the assumptions (h_2), (h_5) and (3.18), we infer from (3.29) that $\|y_n\|_{L^\infty(\epsilon,T;V)}$ is bounded independently of n.
Coming back to (3.27) and dropping the term $\|y_n(t)\|_V^2$, we get for some $\epsilon \in [0, T]$

$$\int_\epsilon^T |\mathcal{A}y_n(\xi)|_H^2 d\xi \leq \|y_n\|_{L^\infty(\epsilon,T;V)}^2 + \kappa_1 + \|y_n\|_{L^\infty(\epsilon,T;V)}^2 \int_0^T \left[cN^4 + lc\,(g_1(\xi) + g_2(\xi))\right] d\xi. \tag{3.30}$$

Therefore,

$$y_n \in L^\infty(\epsilon, T; V) \cap L^2(\epsilon, T; D(\mathcal{A}_1) \times D(\mathcal{A}_2)) \text{ for all } 0 < \epsilon < T. \tag{3.31}$$

Note that from (3.9),

$$\frac{d}{dt}y_n(t) = -\mathcal{A}y_n(t) - P_n\mathcal{B}^N(y_n, y_n)(t) + P_n^1 G_1(t, \boldsymbol{u}(t-\rho_1(t))) + P_n^2 G_2(t, \boldsymbol{B}(t-\rho_2(t))).$$

Then using (2.24) we deduce that the sequence $\left\{P_n\mathcal{B}^N(y_n, y_n)\right\}$ is bounded in $L^2(\epsilon, T; H)$. Therefore, from (3.30) and remark 2.2, we infer that the sequence

$$\frac{d}{dt}(\boldsymbol{u}_n, \boldsymbol{B}_n) \text{ is also bounded in } L^2(\epsilon, T; H). \tag{3.32}$$

Since $D(\mathcal{A}) = D(\mathcal{A}_1) \times D(\mathcal{A}_2) \subset V \subset H$ with compact injection, we derive from [24, Theorem 5.1, Chapter 1] that there exists an element $(\boldsymbol{u}, \boldsymbol{B}) \in L^\infty(\epsilon, T; V) \cap L^2(\epsilon, T; D(\mathcal{A}))$, and a subsequence of $(\boldsymbol{u}_n, \boldsymbol{B}_n)$ (still) denoted $(\boldsymbol{u}_n, \boldsymbol{B}_n)$ such that for all $T > \epsilon$, we have

$$(\boldsymbol{u}_n, \boldsymbol{B}_n) \to (\boldsymbol{u}, \boldsymbol{B}) \begin{cases} \text{weak-star in } L^\infty(\epsilon, T; V), \\ \text{weakly in } L^2(\epsilon, T; D(\mathcal{A})), \\ \text{strongly in } L^2(\epsilon, T; V), \\ \text{a.e., in } (\epsilon, T) \times \Omega, \end{cases} \tag{3.33}$$

and

$$\frac{d}{dt}(\boldsymbol{u}_n, \boldsymbol{B}_n) \to \frac{d}{dt}(\boldsymbol{u}, \boldsymbol{B}) \text{ weakly in } L^2(\epsilon, T; H). \tag{3.34}$$

From (3.33), we can assume, eventually extracting a subsequence of $\{y_n\}$ still denoted $\{y_n\}$ such that

$$\begin{aligned} &\|\boldsymbol{u}_n\|_{V_1} \to \|\boldsymbol{u}\|_{V_1} \text{ a.e. in } (\epsilon, T), \\ &\|(\boldsymbol{u}_n, \boldsymbol{B}_n)\|_V \to \|(\boldsymbol{u}, \boldsymbol{B})\|_V \text{ a.e. in } (\epsilon, T), \end{aligned} \tag{3.35}$$

and therefore

$$\begin{aligned} &F_N(\|\boldsymbol{u}_n\|_{V_1}) \to F_N(\|\boldsymbol{u}\|_{V_1}) \text{ a.e. in } (\epsilon, T), \\ &F_N(\|(\boldsymbol{u}_n, \boldsymbol{B}_n)\|_V) \to F_N(\|(\boldsymbol{u}, \boldsymbol{B})\|_V) \text{ a.e. in } (\epsilon, T). \end{aligned} \tag{3.36}$$

Case 2: $(\boldsymbol{u}(0), \boldsymbol{B}(0)) \in V$.

Integrating (3.26) between 0 and t for $0 < t \leq T$ and use the continuous injection $V \hookrightarrow H$ and (3.13), we obtain

$$\begin{aligned} &\|y_n(t)\|_V^2 + \tfrac{1}{R_e}\int_0^t |\mathcal{A}y_n(\xi)|_H^2 d\xi \\ &\leq \|y_n(0)\|_V^2 + cN^4\int_0^T \|y_n(\xi)\|_V^2 d\xi + \kappa_1 + l\int_0^T c\left[\widetilde{g}_1(\xi) + \widetilde{g}_2(\xi)\right]\left(\|\boldsymbol{u}_n(\xi)\|_{V_1}^2 + \|\boldsymbol{B}_n(\xi)\|_{V_2}^2\right) d\xi \\ &\leq \|y_n(0)\|_V^2 + cN^4\int_0^T \|y_n(\xi)\|_V^2 d\xi + \kappa_1 + lc\int_0^T \left[\widetilde{g}_1(\xi) + \widetilde{g}_2(\xi)\right]\|y_n(\xi)\|_V^2 d\xi \end{aligned} \tag{3.37}$$

Dropping momentarily the term $\frac{1}{R_e}\int_0^t |\mathcal{A}y_n(\xi)|_H^2 d\xi$ and using lemma 2.1 in (3.37), we obtain

$$\|\boldsymbol{u}_n(t)\|_{V_1}^2 + \|\boldsymbol{B}_n(t)\|_{V_2}^2 \leq$$

$$\left[\|y(0)\|_V^2 + \kappa_1\right] \exp\left\{\int_0^T \left[cN^4 + lc\left(g_1(\xi) + g_2(\xi)\right)\right] d\xi\right\}. \tag{3.38}$$

Hence, $\|y_n\|^2_{L^\infty(0,T;V)}$ is bounded independently of n.
Using (3.38) in (3.37), we infer that

$$\begin{aligned}&\tfrac{1}{R_e}\int_0^T |\mathcal{A}y_n(\xi)|^2_H d\xi\\ &\le \|y_n(0)\|^2_V + cN^4\int_0^T \|y_n(\xi)\|^2_V d\xi + lc\|y_n\|^2_{L^\infty(0,T;V)}\int_0^T [\widetilde{g}_1(\xi)+\widetilde{g}_2(\xi)]\,d\xi.\end{aligned} \tag{3.39}$$

Consequently, we derive from (3.38) and (3.39) that $(y_n) = (\boldsymbol{u}_n, \boldsymbol{B}_n)$ satisfies

$$\|(\boldsymbol{u}_n,\boldsymbol{B}_n)(t)\|^2_V \le \mathcal{K}_2, \quad \int_0^T \left(|\mathcal{A}_1\boldsymbol{u}_n(\xi)|^2_{L^2} + |\mathcal{A}_2\boldsymbol{B}_n(\xi)|^2_{L^2}\right) d\xi \le \mathcal{K}_3, \tag{3.40}$$

which proves that $(\boldsymbol{u}_n, \boldsymbol{B}_n)$ is bounded in $L^\infty(0,T;V)\cap L^2(0,T;D(\mathcal{A}_1)\times D(\mathcal{A}_2))$.
Note that in (3.40), $\mathcal{K}_2$ and $\mathcal{K}_3$ are positive constants independent of n and depending only on data $\Omega, R_e, R_m, S, T, L_1(R), L_2(R), \boldsymbol{u}_0, \boldsymbol{B}_0, \zeta_1$ and ζ_2.
Moreover from (3.9),

$$\frac{d}{dt}y_n(t) = -\mathcal{A}y_n(t) - P_n\mathcal{B}^N(y_n(t),y_n(t)) + P_n^1 G_1(t,\boldsymbol{u}(t-\rho_1(t))) + P_n^2 G_2(t,\boldsymbol{B}(t-\rho_2(t))).$$

Then using (2.24) we deduce that the sequence $\left\{P_n\mathcal{B}^N(y_n,y_n)\right\}$ is bounded in $L^2(0,T;H)$.
Therefore, from (3.39) and remark 2.2, we infer that the sequence

$$\frac{d}{dt}(\boldsymbol{u}_n,\boldsymbol{B}_n) \text{ is also bounded in } L^2(0,T;H). \tag{3.41}$$

Since $D(\mathcal{A}) = D(\mathcal{A}_1)\times D(\mathcal{A}_2) \subset V \subset H$ with compact injection, we derive from [24, Theorem 5.1, Chapter 1] that there exists an element $(\boldsymbol{u},\boldsymbol{B}) \in L^\infty(0,T;V)\cap L^2(0,T;D(\mathcal{A}))$, and a subsequence of $(\boldsymbol{u}_n,\boldsymbol{B}_n)$ (still) denoted $(\boldsymbol{u}_n,\boldsymbol{B}_n)$ such that for all $T>0$, we have

$$(\boldsymbol{u}_n,\boldsymbol{B}_n) \to (\boldsymbol{u},\boldsymbol{B}) \begin{cases}\text{weak-star in } L^\infty(0,T;V),\\ \text{weakly in } L^2(0,T;D(\mathcal{A})),\\ \text{strongly in } L^2(0,T;V),\\ \text{a.e., in } (0,T)\times\Omega,\end{cases} \tag{3.42}$$

and

$$\frac{d}{dt}(\boldsymbol{u}_n,\boldsymbol{B}_n) \to \frac{d}{dt}(\boldsymbol{u},\boldsymbol{B}) \text{ weakly in } L^2(0,T;H). \tag{3.43}$$

From (3.42), we infer that

$$\begin{aligned}&\|\boldsymbol{u}_n\|_{V_1} \to \|\boldsymbol{u}\|_{V_1} \text{ a.e. in } (0,T),\\ &\|(\boldsymbol{u}_n,\boldsymbol{B}_n)\|_V \to \|(\boldsymbol{u},\boldsymbol{B})\|_V \text{ a.e. in } (0,T),\end{aligned} \tag{3.44}$$

and therefore

$$\begin{aligned}&F_N(\|\boldsymbol{u}_n\|_{V_1}) \to F_N(\|\boldsymbol{u}\|_{V_1}) \text{ a.e. in } (0,T),\\ &F_N(\|(\boldsymbol{u}_n,\boldsymbol{B}_n)\|_V) \to F_N(\|(\boldsymbol{u},\boldsymbol{B})\|_V) \text{ a.e. in } (0,T).\end{aligned} \tag{3.45}$$

Step3: Passage to the limit.
We want to take the limit in (3.9) when n goes to $+\infty$. More precisely, we want to prove that

$$G_1(t,\boldsymbol{u}_n(t-\rho_1(t))) \to G_1(t,\boldsymbol{u}(t-\rho_1(t))) \ \text{ as } n\to+\infty \tag{3.46}$$

and

$$G_2(t,\boldsymbol{B}_n(t-\rho_2(t))) \to G_2(t,\boldsymbol{B}(t-\rho_2(t))) \ \text{ when } n\to+\infty. \tag{3.47}$$

We refer the reader to [6, 14] for the other terms involved in (3.9). We will use the dominated convergence theorem [1] to prove (3.46) and (3.47).
Note that by (h_2) and (h_5), for *a.e.* $t\in(0,T)$, $G_1(t,.) : H_1 \to H_1$ and $G_2(t,.) : H_2 \to H_2$ are continuous; then since $\boldsymbol{u}_n \to u$ and $\boldsymbol{B}_n \to \boldsymbol{B}$ strongly in $L^{\frac{2p}{2p-1}}(-h,T;H_1)$ and $L^{\frac{2p}{2p-1}}(-h,T;H_2)$ respectively, we have that

$$G_1(t,\boldsymbol{u}_n(t-\rho_1(t))) \to G_1(t,\boldsymbol{u}(t-\rho_1(t))) \ \text{ in } H_1 \ a.e. \text{ in } (0,T) \tag{3.48}$$

and

$$G_2(t,\boldsymbol{B}_n(t-\rho_2(t))) \to G_2(t,\boldsymbol{B}(t-\rho_2(t))) \ \text{ in } H_2 \ a.e. \text{ in } (0,T). \tag{3.49}$$

On the other hand, using (3.20), (h_3) and (h_6), we have for *a.e.* $t\in(0,T)$

$$\begin{aligned} |G_1(t,\boldsymbol{u}_n(t-\rho_1(t)))|_{L^2} &\le g_1^{1/2}(t)\,|\boldsymbol{u}_n(t-\rho_1(t))|_{L^2} + \zeta_1^{1/2}(t) \\ &\le g_1^{1/2}(t) z_1(t-\rho_1(t)) + \zeta_1^{1/2}(t) \end{aligned} \tag{3.50}$$

and

$$\begin{aligned} |G_2(t,\boldsymbol{B}_n(t-\rho_2(t)))|_{L^2} &\le g_2^{1/2}(t)\,|\boldsymbol{B}_n(t-\rho_2(t))|_{L^2} + \zeta_2^{1/2}(t) \\ &\le g_2^{1/2}(t) z_2(t-\rho_2(t)) + \zeta_2^{1/2}(t). \end{aligned} \tag{3.51}$$

Then (3.46) and (3.47) follow from the dominated convergence theorem.

4 Convergence to weak solutions of MHD equations with locally Lipschitz delays terms

The solution of problem (1.3) given by theorem (3.2) depends on the parameter N; so for each N, we denote by y^N a such corresponding weak solution; we then obtain a sequence $\{y^N\}_{N>0}$ of such solutions. In this section, we prove that from the sequence $\{y^N\}_{N>0}$, we can extract a subsequence $\{y_j^N\}_{j>0}$ which converges when N_j goes to $+\infty$ to a weak solution of (1.2). For this purpose, we have the following result.

Theorem 4.1 *The assumptions* $(h_1)-(h_6)$ *are satisfied, let* $y^{N_k} = (\boldsymbol{u}^{N_k}, \boldsymbol{B}^{N_k})$ *where* $\boldsymbol{u}^{N_k} = \boldsymbol{u}^{N_k}(\boldsymbol{u}_0^{N_k}, \phi_1^{N_k}))$ *and* $\boldsymbol{B}^{N_k} = \boldsymbol{B}^{N_k}(\boldsymbol{B}_0^{N_k}, \phi_2^{N_k})$, $k=1,2,...$ *with* $N_k\to+\infty$ *as* $k\to+\infty$, *be a sequence of weak solutions of* (1.3) *with* $N=N_k$, *and with the initial data such that* $\boldsymbol{u}_0^{N_k}\to\boldsymbol{u}_0$ *weakly in* H_1, $\boldsymbol{B}_0^{N_k}\to\boldsymbol{B}_0$ *weakly in* H_2, $\phi_i^{N_k}\to\phi_i$ *strongly in* $L^{\frac{2p}{2p-1}}(-h,T;H_i)$, $i=1,2$ *as* $k\to+\infty$, *and the sequence* $\{\phi_i^{N_k} : k=1,2,...\}$ *is bounded in* $L^{2p'}(-h,0;H_i)$, *where* $i=1,2$ *and* $\frac{1}{p}+\frac{1}{p'}=1$.
Then, there exists a subsequence $\{y_j^N\}_{j>0}$ *of* $\{y_k^N\}_{k>0}$ *which converges weakly-star in* $L^\infty(0,T;H)$, *weakly in* $L^2(0,T;V)$, *weakly-star in* $L^{2p'}(-h,T;H)$, *strongly in* $L^{\frac{2p}{2p-1}}(-h,T;H)$ *and strongly in* $L^2(0,T;H)$, *to a solution* y *of* (1.2).

proof. We know by remark 2.2 that each of these solutions $y^{N_k} = (\boldsymbol{u}^{N_k}, \boldsymbol{B}^{N_k})$ satisfies the following energy equality

$$
\begin{aligned}
&|\boldsymbol{u}^{N_k}(t)|_{L^2}^2 + S|\boldsymbol{B}^{N_k}(t)|_{L^2}^2 + \tfrac{2}{R_e}\int_0^t \|\boldsymbol{u}^{N_k}(\xi)\|_{V_1}^2 d\xi + \frac{2S}{R_m}\int_0^t \|\boldsymbol{B}^{N_k}(\xi)\|_{V_2}^2 d\xi \\
&= |\boldsymbol{u}_0^{N_k}|_{L^2}^2 + S|\boldsymbol{B}_0^{N_k}|_{L^2}^2 + 2\int_0^t (G_1(t, \boldsymbol{u}^{N_k}(\xi - \rho_1(\xi))), \boldsymbol{u}^{N_k}(\xi))d\xi + \\
&2S\int_0^t (G_2(\xi, \boldsymbol{B}^{N_k}(\xi - \rho_2(\xi))), \boldsymbol{B}^{N_k}(\xi))d\xi.
\end{aligned}
\tag{4.1}
$$

Now, following the proof of (3.14), we obtain

$$
\begin{aligned}
&|\boldsymbol{u}^{N_k}(t)|_{L^2}^2 + S|\boldsymbol{B}^{N_k}(t)|_{L^2}^2 + \tfrac{2}{R_e}\int_0^t \|\boldsymbol{u}^{N_k}(\xi)\|_{V_1}^2 d\xi + \frac{2S}{R_m}\int_0^t \|\boldsymbol{B}^{N_k}(\xi)\|_{V_2}^2 d\xi \\
&\leq |\boldsymbol{u}_0^{N_k}|_{L^2}^2 + S|\boldsymbol{B}_0^{N_k}|_{L^2}^2 + l\int_0^T \left[\widetilde{g}_1(\xi) + \frac{\widetilde{g}_2(\xi)}{S}\right]\left(|\boldsymbol{u}_n^{N_k}(\xi)|_{L^2}^2 + S|\boldsymbol{B}_n^{N_k}(\xi)|_{L^2}^2\right) d\xi \\
&+ \int_0^T c\left(\zeta_1(\xi) + \zeta_2(\xi)\right) d\xi + l\left(\int_{-\rho_1(0)}^0 |\widetilde{g}_1(\xi)|^p d\xi\right)^{\frac{1}{p}} \left(\int_{-\rho_1(0)}^0 \left|\phi_1^{N_k}(\xi)\right|_{L^2}^{2p'} d\xi\right)^{\frac{2}{2p'}} d\xi \\
&+ l\left(\int_{-\rho_2(0)}^0 |\widetilde{g}_2(\xi)|^p d\xi\right)^{\frac{1}{p}} \left(\int_{-\rho_2(0)}^0 |\phi_2(\xi)|_{L^2}^{2p'} d\xi\right)^{\frac{2}{2p'}} d\xi
\end{aligned}
\tag{4.2}
$$

where $l = \max\left\{\frac{c}{1-\rho_1^*}, \frac{c}{1-\rho_2^*}\right\}$. From the assumptions of theorem 4.1, there exists a constant β such that

$$
|\boldsymbol{u}_0^{N_k}|_{L^2}^2 + S|\boldsymbol{B}_0^{N_k}|_{L^2}^2 \leq \beta; \ \left\|\phi_i^{N_k}\right\|_{L^{2p'}(-h,0;H_i)} \leq \beta, \ i = 1, 2. \tag{4.3}
$$

Consequently, we infer from (4.2) that

$$
\begin{aligned}
&|\boldsymbol{u}^{N_k}(t)|_{L^2}^2 + S|\boldsymbol{B}^{N_k}(t)|_{L^2}^2 + \tfrac{2}{R_e}\int_0^t \|\boldsymbol{u}^{N_k}(\xi)\|_{V_1}^2 d\xi + \frac{2S}{R_m}\int_0^t \|\boldsymbol{B}^{N_k}(\xi)\|_{V_2}^2 d\xi \\
&\leq K_0 + l\int_0^T \left[\widetilde{g}_1(\xi) + \frac{\widetilde{g}_2(\xi)}{S}\right]\left(|\boldsymbol{u}_n^{N_k}(\xi)|_{L^2}^2 + S|\boldsymbol{B}_n^{N_k}(\xi)|_{L^2}^2\right) d\xi
\end{aligned}
\tag{4.4}
$$

where

$$
K_0 = \beta^2 + l\beta^2\left(\|\widetilde{g}_1\|_{L^p(-\rho_1(0),0)} + \|\widetilde{g}_2\|_{L^p(-\rho_2(0),0)}\right) + \int_0^T c\left(\zeta_1(\xi) + \zeta_2(\xi)\right) d\xi.
$$

Dropping momentarily the term $\frac{2}{R_e}\int_0^t \|\boldsymbol{u}^{N_k}(\xi)\|_{V_1}^2 d\xi + \frac{2S}{R_m}\int_0^t \|\boldsymbol{B}^{N_k}(\xi)\|_{V_2}^2 d\xi$ in (4.4) and using lemma 2.1, we have for all $t \in [0, T]$,

$$
|\boldsymbol{u}^{N_k}(t)|_{L^2}^2 + S|\boldsymbol{B}^{N_k}(t)|_{L^2}^2 \leq K_0 \exp\left[\int_0^T l\left(\widetilde{g}_1(\xi) + \frac{\widetilde{g}_2(\xi)}{S}\right) d\xi.\right] \tag{4.5}
$$

In addition, as in (3.17), we have

$$
\begin{aligned}
\int_{-h}^T \left|y^{N_k}(t)\right|_H^{2p'} dt &= \int_{-h}^0 \left|\phi^{N_k}(t)\right|_H^{2p'} dt + \int_0^T \left|y^{N_k}(t)\right|_H^{2p'} dt \\
&\leq \left\|\phi^{N_k}\right\|_{L^{2p'}(-h,0;H)}^{2p'} + T\left\|y^{N_k}\right\|_{L^\infty(0,T;H)}^{2p'} \\
&\leq +\infty.
\end{aligned}
\tag{4.6}
$$

Using (4.5) in (4.4) and taking into account (4.7), we infer that the sequence

$$\left\{y^{N_k}\right\}_{N_k\geq 0} \subset L^\infty(0,T;H)\cap L^2(0,T;V)\cap L^{2p'}(-h,T;H). \tag{4.7}$$

Moreover, from (2.34), we have

$$\frac{d}{dt}y^{N_k}(t) = -\mathcal{A}y^{N_k}(t) - \mathcal{B}^N(y^{N_k}(t), y^{N_k}(t)) + G_1(t, \boldsymbol{u}^{N_k}(t-\rho_1(t))) + G_2(t, \boldsymbol{B}^{N_k}(t-\rho_2(t))).$$

Then using (2.24) we deduce that the sequence $\left\{\mathcal{B}^N(y^{N_k}, y^{N_k})\right\}$ is bounded in

$L^2(0,T;H)$. Therefore, from (4.7) and remark 2.2,we infer that the sequence

$$\frac{d}{dt}(\boldsymbol{u}^{N_k}, \boldsymbol{B}^{N_k}) \text{ is also bounded in } L^2(0,T;H). \tag{4.8}$$

Hence, we use again a compactness argument (see [41]) to extract a subsequence from $\left\{y^{N_k} = (\boldsymbol{u}^{N_k}, \boldsymbol{B}^{N_k})\right\}$ denoted by $\left\{y^{N_j} = (\boldsymbol{u}^{N_j}, \boldsymbol{B}^{N_j})\right\}$ satisfying

$$y^{N_j} \to y \begin{cases} \text{weak-star in } L^\infty(0,T;H), \\ \text{weakly in } L^2(0,T;V), \\ \text{strongly in } L^2(0,T;H), \\ \text{weak-star in } L^{2p'}(-h,T;H), \\ \text{strongly in } L^{\frac{2p}{2p-1}}(-h,T;H), \\ \text{a.e., in } (-h,T)\times\Omega, \end{cases} \tag{4.9}$$

and

$$\frac{d}{dt}y^{N_j} \to \frac{d}{dt}y \;\text{ weakly in } L^2(0,T;H). \tag{4.10}$$

with $y = (\boldsymbol{u}, \boldsymbol{B})$ being in the spaces involved in (4.9).
According to (4.9), we can assume, as before, without loss of generality, that there exists two functions $z_3 \in L^{\frac{2p}{2p-1}}(-h,T)$ and $z_4 \in L^{\frac{2p}{2p-1}}(-h,T)$ such that

$$\left|\boldsymbol{u}^{N_j}(t)\right|_{L^2} \leq z_3(t), \; \left|\boldsymbol{B}^{N_j}(t)\right|_{L^2} \leq z_4(t). \tag{4.11}$$

In the following lines, we are going to prove that $(\boldsymbol{u}, \boldsymbol{B})$ is the solution of (1.2). More precisely, we want to prove that $(\boldsymbol{u}, \boldsymbol{B})$ Satisfies the following: for all $(\boldsymbol{v}, \boldsymbol{C}) \in \mathcal{D}(\mathcal{A}_1)\times\mathcal{D}(\mathcal{A}_2)$,

$$\begin{aligned} &(\boldsymbol{u}(t), \boldsymbol{v})_{L^2} + (\boldsymbol{B}(t), \boldsymbol{C})_{L^2} + \tfrac{1}{R_e}\int_0^T ((\boldsymbol{u}(t), \boldsymbol{v}))_1 dt + \frac{1}{R_m}\int_0^T ((\boldsymbol{B}(t), \boldsymbol{C}))_2 dt + \\ &\int_0^T \mathcal{B}_0^{N_j}(y(t), y(t), (v, \boldsymbol{C}))dt = \int_0^T (G_1(t, \boldsymbol{u}(t-\rho_1(t))), \boldsymbol{v})_{L^2} dt + \\ &\int_0^T (G_2(t, \boldsymbol{B}(t-\rho_2(t))), \boldsymbol{C})_{L^2} dt + (\boldsymbol{u}_0, \boldsymbol{v})_{L^2} + (\boldsymbol{B}_0, \boldsymbol{C})_{L^2}. \end{aligned} \tag{4.12}$$

This will be done by taking the limit when $N_j \to +\infty$ and, for all $(\boldsymbol{v}, \boldsymbol{C}) \in \mathcal{D}(\mathcal{A}_1)\times\mathcal{D}(\mathcal{A}_2)$, in

$$
\begin{aligned}
&(\boldsymbol{u}^{N_j}(t),\boldsymbol{v})_{L^2}+(\boldsymbol{B}^{N_j}(t),\boldsymbol{C})_{L^2}+\tfrac{1}{R_e}\int_0^T((\boldsymbol{u}^{N_j}(t),\boldsymbol{v}))_1dt+\frac{1}{R_m}\int_0^T((\boldsymbol{B}^{N_j}(t),\boldsymbol{C}))_2dt+\\
&=\int_0^T\mathcal{B}_0^{N_j}(y^{N_k}(t),y^{N_k}(t),(\boldsymbol{v},\boldsymbol{C}))dt+\int_0^T(G_1(t,\boldsymbol{u}^{N_k}(t-\rho_1(t))),\boldsymbol{v})_{L^2}dt+\\
&\int_0^T(G_2(t,\boldsymbol{B}^{N_k}(t-\rho_2(t))),\boldsymbol{C})_{L^2}dt+(\boldsymbol{u}_0^{N_j},\boldsymbol{v})_{L^2}+(\boldsymbol{B}_0^{N_j},\boldsymbol{C})_{L^2}\,.
\end{aligned}
\tag{4.13}
$$

By the assumptions,

$$
(\boldsymbol{u}_0^{N_j},\boldsymbol{v})_{L^2}\to(\boldsymbol{u}_0,\boldsymbol{v})_{L^2}\ \text{ and }\ (\boldsymbol{B}_0^{N_j},\boldsymbol{C})_{L^2}\to(\boldsymbol{B}_0,\boldsymbol{C})_{L^2}. \tag{4.14}
$$

In addition, since $y^{N_j}\to y$ weakly in $L^2(0,T;V)$, we claim that

$$
\frac{1}{R_e}\int_0^T((\boldsymbol{u}^{N_j}(t),\boldsymbol{v}))_1dt\to\frac{1}{R_e}\int_0^T((\boldsymbol{u}(t),\boldsymbol{v}))_1dt \tag{4.15}
$$

and

$$
\frac{1}{R_m}\int_0^T((\boldsymbol{B}^{N_j}(t),\boldsymbol{C}))_2dt\to\frac{1}{R_m}\int_0^T((\boldsymbol{B}(t),\boldsymbol{C}))_2dt\,. \tag{4.16}
$$

Now, for the delay terms, we will use the dominated convergence theorem, as in the proof of the existence result.
Note that by (h_2) and (h_5), for *a.e.* $t\in(0,T)$, $G_1(t,.):H_1\to H_1$ and $G_2(t,.):H_2\to H_2$ are continuous functions; then by (3.19), $\boldsymbol{u}^{N_j}\to u$ and $\boldsymbol{B}^{N_j}\to\boldsymbol{B}$ strongly in $L^{\frac{2p}{2p-1}}(-h,T;H_1)$ and $L^{\frac{2p}{2p-1}}(-h,T;H_2)$ respectively; consequently

$$
G_1(t,\boldsymbol{u}^{N_j}(t-\rho_1(t)))\to G_1(t,\boldsymbol{u}(t-\rho_1(t)))\ \text{ in } H_1\ a.e. \text{ in } (0,T) \tag{4.17}
$$

and

$$
G_2(t,\boldsymbol{B}^{N_j}(t-\rho_2(t)))\to G_2(t,\boldsymbol{B}(t-\rho_2(t)))\ \text{ in } H_2\ a.e. \text{ in } (0,T). \tag{4.18}
$$

On the other hand, using (4.11), (h_3) and (h_6), we have for *a.e.* $t\in(0,T)$

$$
\begin{aligned}
\left|G_1(t,\boldsymbol{u}^{N_j}(t-\rho_1(t)))\right|_{L^2}\quad&\le g_1^{1/2}(t)\left|\boldsymbol{u}^{N_j}(t-\rho_1(t))\right|_{L^2}+\zeta_1^{1/2}(t)\\
&\le g_1^{1/2}(t)z_1(t-\rho_1(t))+\zeta_1^{1/2}(t)\,,
\end{aligned}
\tag{4.19}
$$

and

$$
\begin{aligned}
\left|G_2(t,\boldsymbol{B}^{N_j}(t-\rho_2(t)))\right|_{L^2}\quad&\le g_2^{1/2}(t)\left|\boldsymbol{B}^{N_j}(t-\rho_2(t))\right|_{L^2}+\zeta_2^{1/2}(t)\\
&\le g_2^{1/2}(t)z_2(t-\rho_2(t))+\zeta_2^{1/2}(t)\,.
\end{aligned}
\tag{4.20}
$$

Then, by using a dominated convergence theorem, we have

$$
\int_0^T(G_1(t,\boldsymbol{u}^{N_k}(t-\rho_1(t))),v)_{L^2}dt\to\int_0^T(G_1(t,\boldsymbol{u}(t-\rho_1(t))),\boldsymbol{v})_{L^2}dt. \tag{4.21}
$$

$$
\int_0^T(G_2(t,\boldsymbol{B}^{N_k}(t-\rho_2(t))),v)_{L^2}dt\to\int_0^T(G_2(t,\boldsymbol{B}(t-\rho_2(t))),\boldsymbol{v})_{L^2}dt. \tag{4.22}
$$

For the last term, we recall the definition of $\mathcal{B}_0^{N_j}$

$$\mathcal{B}_0^{N_j}(y^{N_j}(t), y^{N_j}(t), (\boldsymbol{v}, \boldsymbol{C}))$$
$$= F_{N_j}(\|\boldsymbol{u}^{N_j}(t)\|_{V_1}) b(\boldsymbol{u}^{N_j}(t), \boldsymbol{u}^{N_j}(t), \boldsymbol{v}) - SF_{N_j}(\|(\boldsymbol{u}^{N_j}(t), \boldsymbol{B}^{N_j}(t))\|_V) b(\boldsymbol{B}^{N_j}(t), \boldsymbol{B}^{N_j}(t), \boldsymbol{v})$$
$$+F_{N_j}(\|(\boldsymbol{u}^{N_j}(t), \boldsymbol{B}^{N_j}(t))\|_V) b(\boldsymbol{u}^{N_j}(t), \boldsymbol{B}^{N_j}(t), \boldsymbol{C}) -$$
$$F_{N_j}(\|(\boldsymbol{u}^{N_j}(t), \boldsymbol{B}^{N_j}(t))\|_V) b(\boldsymbol{B}^{N_j}(t), \boldsymbol{u}^{N_j}(t), \boldsymbol{C}) \,. \tag{4.23}$$

It is manifest that in order to finish the proof of theorem 4.1, we need to prove that

$$F_{N_j}(\|(\boldsymbol{u}^{N_j}(t), \boldsymbol{B}^{N_j}(t))\|_V) b(\boldsymbol{u}^{N_j}(t), \boldsymbol{B}^{N_j}(t), \boldsymbol{C}) \to b(\boldsymbol{u}(t), \boldsymbol{B}(t), \boldsymbol{C}) \,.$$

Similar reasoning will be made for other terms of (4.23). As in [26], we need to show that

$$F_{N_j}(\|(\boldsymbol{u}^{N_j}(t), \boldsymbol{B}^{N_j}(t))\|_V) \to 1 \;\text{ in }\; L^2(0,T). \tag{4.24}$$

and

$$b(\boldsymbol{u}^{N_j}(t), \boldsymbol{B}^{N_j}(t), \boldsymbol{C}) \to b(\boldsymbol{u}(t), \boldsymbol{B}(t), \boldsymbol{C}). \tag{4.25}$$

Thanks to ([6], lemma 12), (4.24) is true. On the other hand, for $a.e.t \in (0,T)$,

$$\begin{aligned} b(\boldsymbol{u}^{N_j}(t), \boldsymbol{B}^{N_j}(t), \boldsymbol{C}) &= \textstyle\sum_{i,\tau=1}^{3} \displaystyle\int_\Omega \boldsymbol{u}^{N_{j_i}}(t) \partial_i \boldsymbol{B}^{N_{j_\tau}}(t) \boldsymbol{C}^\tau dx \\ &= \textstyle\sum_{i,\tau=1}^{3} \displaystyle\int_\Omega \boldsymbol{u} N_{j_i}(t) \partial_i \boldsymbol{B}^{N_{j_\tau}}(t) \boldsymbol{C}^\tau dx \\ &= -\textstyle\sum_{i,\tau=1}^{3} \displaystyle\int_\Omega \boldsymbol{u} N_{j_i}(t) \partial_i \boldsymbol{C}^\tau \boldsymbol{B}^{N_{j_\tau}}(t) dx \\ &\to -\textstyle\sum_{i,\tau=1}^{3} \displaystyle\int_\Omega \boldsymbol{u}^i(t) \partial_i \boldsymbol{C}^\tau s \boldsymbol{B}^\tau(t) dx \\ &= b(\boldsymbol{u}(t), \boldsymbol{B}(t), \boldsymbol{C}) \,, \end{aligned}$$

where $\partial_i = \frac{\partial}{\partial x_i}$. Then

$$\int_0^T b(\boldsymbol{u}^{N_k}(t), \boldsymbol{B}^{N_k}(t), \boldsymbol{C}) dt \to \int_0^T b(\boldsymbol{u}(t), \boldsymbol{B}(t), \boldsymbol{C}) dt \,. \tag{4.26}$$

Now , we introduce the abbreviations , just to simplify our expressions

$$F^{N_j}(t) = F_{N_j}(\|(\boldsymbol{u}^{N_j}(t), \boldsymbol{B}^{N_j}(t))\|_V); \quad b^{N_j}(t) = b(\boldsymbol{u}^{N_j}(t), \boldsymbol{B}^{N_j}(t), \boldsymbol{C}) \,,$$
$$b(t) = b(\boldsymbol{u}(t), \boldsymbol{B}(t), \boldsymbol{C}) \,.$$

We want to prove that

$$\int_0^T F^{N_j}(t) b^{N_j}(t) dt \to \int_0^T b(t) dt \,. \tag{4.27}$$

We have

$$\begin{aligned} \int_0^T \left(F_{N_j}(t) b^{N_j}(t) - b(t)\right) dt &= \int_0^T \left(F_{N_j}(t) - 1\right) b^{N_j}(t) dt + \\ &\int_0^T \left(b^{N_j}(t) - b(t)\right) dt \,. \end{aligned} \tag{4.28}$$

From (4.26), we infer that

$$\left|\int_0^T \left(F_{N_j}(t)b^{N_j}(t) - b(t)\right) dt\right|^2 \leq \int_0^T \left|F_{N_j}(t) - 1\right|^2 dt \int_0^T \left|b^{N_j}(t)\right|^2 dt\,. \tag{4.29}$$

But

$$\begin{aligned}\int_0^T \left|b^{N_k}(t)\right|^2 dt &\leq c\|\boldsymbol{C}\|_{V_2}|\mathcal{A}_2\boldsymbol{C}|_{L^2}\int_0^T |\boldsymbol{u}^{N_j}(t)|_{L^2}\|\boldsymbol{B}\|_{V_2} \\ &\leq c\|\boldsymbol{C}\|_{V_2}|\mathcal{A}_2\boldsymbol{C}|_{L^2}\|\boldsymbol{u}^{N_j}(t)\|_{L^\infty(0,T;H_1)}\|\boldsymbol{B}^{N_j}\|_{L^2(0,T;V_2)} \\ &< +\infty\,.\end{aligned} \tag{4.30}$$

Then we deduce (4.27) and consequently the proof of theorem 4.1 since $\mathcal{D}(\mathcal{A})$ is dense in V. □

References

[1] R. A. Adams, Sobolev spaces; Academic Press, New York San Francisco London, 1975.

[2] H. Brezis. *Functional Analysis, Sobolev Spaces and partial differential equations.*

[3] T. Caraballo and J. Real. Navier-Stokes equations with delays. R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci., **457** (2001), 2441-2453.

[4] T. Caraballo and J. Real. Asymptotic behavior of two-dimensional Navier-Stokes equations with delays. R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci., **459** (2003), 3181-3194.

[5] T. Caraballo and J. Real. Attractors for $2D$-Navier-Stokes models with delays. J. Differential Equations, **205** (2004), 271-297.

[6] T. Caraballo, J. Real, P. E. Kloeden. Unique strong solutions and V-attractors of a three dimensional system of globally modified Navier-Stokes equations. Adv. Nonlinear Stud., **6** (3) (2006), 411-436.

[7] T. Caraballo, P. E. Kloeden and J. Real, Invariant measures and statistical solutions of the globally modified Navier-Stokes equations. Discrete Continuous Dyn. Syst. Ser. B, **10** (4) (2008), 761-781.

[8] T. Caraballo, J. Real and A. M. Márquez, Three-dimensional system of globally modified Navier-Stokes equations with delay. Int. J. Bifurcat. Chaos Appl. Sci. Eng., **20** (9) (2010), 2869-2883.

[9] T. Caraballo and P. E. Kloeden, The three-dimensional globally modified Navier-Stokes equations: recent developments. In: A. Johann, H. P. Kruse, F. Rupp (eds.) Recent Trends in Dynamical Systems: Proceedings of a Conference in Honor of Jürgen Scheurle, pp. 473-492. Springer Proceedings in Mathematics and Statistics, vol. **35** (2013).

[10] S. Chandrasekhar, Hydrodynamic and Hydromagnetic Stability, in: The International Series of Monographs on Physics, Clarendon Press, Oxford, 1961.

[11] P. Constantin, C. Foias. Navier-Stokes Equations, in: Chicago Lectures in Mathematics, University of Chicago Press, Chicago, IL, 1988.

[12] T. G. Cowling, Magnetohydrodynamics, in: Interscience Tracts on Physics and Astronomy, vol. **4**, Interscience Publishers, Inc., New York, 1957, Interscience Publishers, Ltd., London.

[13] G. Deugoué and J. K. Djoko, On the time discretization for the globally modified three-dimensional Navier-Stokes equations. J. Comput. Appl. Math., **235** (8) (2011), 2015-2029.

[14] G. Deugoué, J. K. Djoko, A. C. Fouapé, A. N. Ngana, Unique strong solutions and V-attractor of a three dimensional globally modified magnetohydrodynamics equations, Communications on Pure and Applied Analysis 2020, 19(3):1505-1535. doi: 103934/cpaa.2020076.

[15] G. Duvaut, J. L. Lions. Inequalities in Mechanics and Physics (Transl. from the French by C. W. John), in: Grundlehren der Mathematischen Wissenschaften, vol. **219**, Springer-Verlag, Berlin, New York, 1976.

[16] Y. Hino, S. Murakami, and T. Naito. Functional Differential Equations with Infinite Delay, Lecture Notes in Mathematics, 1473, Berlin: Springer-Verlag, 1991.

[17] D. D. Holm, J. E. Marsden and T. S. Ratiu, Euler-Poincaré models of ideal fluids with nonlinera dispersion. Phys. Rev. Lett., **349** (1998), 4173-4177.

[18] D. D. Holm, J. E. Marsden and T. S. Ratiu, The Euler-Poincaé equations and semi-direct products with applications to continuum theories. Adv. Math., vol. **137** (1998), 1-81.

[19] P. E. Kloeden, J. A. Langa and J. Real. Pullback V-attractors of the three-dimensional globally modified Navier-Stokes equations. Commun. Pure Appl. Anal., (6) (2007), 937-955.

[20] P. E. Kloeden and J. Valero, The weak connectnedness of the attainability set of weak solutions of the three-dimensional Navier-Stokes equations. Proc. R. Soc. Lond. Ser. A Math. Phys Engs. Sci., **463** (2082) (2007), 1491-1508.

[21] P. E. Kloeden and J. A. Langa. Flattening, squeezing and the existence of random attractors. Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci., **463** (2007), 163-181.

[22] P. E. Kloeden, P. Marín-Rubio and J. Real, Equivalence of invariant measures and stationary statistical solutions for the autonomous globally modified Navier-Stokes equations. Commun. Pure Appl. Anal., **8** (3) (2009), 785-802.

[23] O. A. Ladyzenskaya. The Mathematical Theory of Viscous Incompressible Flow, in: Mathematics and its Applucations, vol. **2**, Gordon and Breach, Science Publishers, New York, London, Paris, 1969.

[24] J. L. Lions. Quelques Méthodes de Résolution des Problémes aux Limites Non Linéaires, Dunod, Paris, 1969.

[25] J. E. Marsden and S. Shkoller, Global well-posedness for the Lagrangian averaged Navier-Stokes (LANS-α) equations on bounded domains. Phil. Trans. R. Soc. Lond. A., **359** (2001), 1449-1468.

[26] P. Marín-Rubio, J. Real and A. M. Márquez-Durán, On the convergence of solutions of globally modified Navier-Stokes equations with delays to solutions of Navier-Stokes equations with delays. Adv. Nonlinear Stud., **11** (4) (2011), 917-927.

[27] P. Marín-Rubio, A. M. Márquez and J. Real. Three dimensional system of globally modified Navier - Stokes equations with infinite delays. Disc and Cont. Dynam. Syst. Series B, **14** (2) (2010), 655-673.

[28] P. Marín-Rubio, J. Real and J. Valero. $2D$ Navier-Stokes equations with infinite delays: existence of solutions and pullback attractors. preprint.

[29] P. Marín-Rubio, A. M. Márquez and J. Real. Pullback attractor for globally modified Navier - Stokes equations with infinite delays, Disc. and Cont. Dynam. Syst.**31** (3) (2011), 779-796.

[30] P. Marín-Rubio, A. M. Márquez and J. Real. Asymptotic behavior of solutions for a three dimensional system of globally modified Navier-Stokes equations with a locally Lipschitz delay term. Nonlinear Analysis 79 (2013), 68-79.

[31] A. M. Márquez, Existence and uniqueness of solutions, and pullback attractor for a system of globally modified 3D-Navier-Stokes equations with finite delay. SeMA J., vol. **51** (2010), 117-124.

[32] Shih.-I. Pai, Magnetogasdynamics and Plasma Dynamics, Springer-Verlag, Vienna, 1962, Prentice-Hall, Inc., Englewood Cliffs, NJ.

[33] M. Romito. The uniqueness of weak solutions of the Globally modified Navier-Stokes equations. Adv. Nonlinear Stud., **9** (2) (2009), 425-427.

[34] T. Silvertru Sever, Some Gronwall Type Inequalities and Applications. School of Communications and Informatics , November7, 2002

[35] M. Sermange, R. Temam. Some mathematical questions related to the MHD equations, Comm. Pure Appl. Math., **36** (5) (1983), 635-664.

[36] T. Tachim Medjo. Unique strong and $\mathbb{V}$-attractor of a three dimensional globally modified Cahn-Hilliard-Navier-Stokes model, Applicable Analysis, **96** (16) (2017).

[37] T. Tachim Medjo. A two-phase flow model with delays, Disc. and Cont. Dynam. Syst. Series B, **22** (2017), 3273-3294.

[38] T. Tachim Medjo. Unique strong and $\mathbb{V}$-attractor of a three-dimensional globally modified two-phase flow model, Annali di Mathematica **197** (2018), 843-868.

[39] R. Temam. Navier-Stokes Equations. Theory and Numerical Analysis, in: Studies in Mathematics and its Applications, vol. **2**, North-Holland Publishing Co., Amsterdam, New York, Oxford, 1977.

[40] R. Temam. Navier-Stokes Equations and Nonlinear Functional Analysis, Second Edition, SIAM, Philadelphia, 1995.

[41] R. Temam. Infinite dimensional dynamical systems in mechanics and physics. 2nd ed. vol. **68**, Applied mathematics at science: Springer-Verlag, New York; 1997.

[42] R. Temam. Navier-Stokes equations, Theory and Numerical Analysis. AMS-Chelsea Series. AMS, Providence 2001.

[43] F. Tone, On the Long-Time H^2-Stability of the Implicit Euler Scheme for the 2D Magnetohydrodynamics Equations, J. Sci. Comput., **38** (2009), 331-348.

[44] M. I. Vishik, A. I. Komech and A. V. Fursikov, Some mathematical problems of statistical hydromechanics. Uspekhi Mat. Nauk. 209 (5) (1979), 135-210.

[45] E. Zeidler. Nonlinear Analysis and its Applications. II/A. Mathematics subject classification (1980) 46xx.

[46] Z. Zhang, Renormalized notes for Navier-Stokes equations by Roger Temam, Charl. Univ. Math. J., vol. 2 (2011).