Miscellaneous Extensions of Four-Valued Expansions of Belnap's Logic

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Abstract. As a generic tool, we prove that the poset of (axiomatic) disjunctive [non-pseudo-axiomatic] extensions of the logic of a finite set M of [(truth-non-empty)] finite disjunctive matrices is dual to the distributive lattice of relative universal (positive) Horn model subclasses of the set S of [truth-non-empty] consistent submatrices of members of M [(the duality preserving axiomatic relative axiomatizations)]. If M consists of a single matrix with equality determinant, relative universal Horn model subclasses of S are proved constructively to be exactly lower cones of S that covers any four-valued expansion L_4 of Belnap's four-valued logic B_4. Moreover, we find algebraic criteria of the [inferential] paracompleteness of the extension of L_4 relatively axiomatized by the *Resolution* rule. We also find lattices of extensions of L_4 satisfying certain rules (in particular, non-paracomplete extensions) under certain conditions covering many interesting four-valued expansions of B_4 including both itself and its bounded version (as well as their purely implicative expansions).

1. Introduction

The present paper is devoted to the issue of four-valued expansions of Belnap's useful (within both Computer Science and Artificial Intelligence) four-valued logic B_4 [2] going back to [9]. More specifically, any four-valued expansion of B_4 being defined by an expansion of a four-valued disjunctive matrix defining B_4 itself, is disjunctive as well, and so is the expansion of B_4 as such. This makes the problem of finding lattices of disjunctive extensions of four-valued expansions of B_4, to which the present paper is mainly (but not exclusively) devoted, rather acute within Non-Classical Logic. In this connection, recall that disjunctivity is a fundamental feature of the classical logic equally inherited by many non-classical ones. And what is more, any axiomatic extension of any disjunctive finitary logic is disjunctive itself, so the mentioned problem properly subsumes that of finding axiomatic extensions of disjunctive finitary logics (in particular, those defined by finite classes of finite disjunctive matrices like four-valued expansions of B_4).

We start our study of it from elaborating generic tools of exploring the lattices of disjunctive extensions (collectively with their finite relative axiomatizations and finite matrix semantics) of the logics of finite classes of finite disjunctive matrices — especially, those of such single matrices with *equality determinant* in the sense of [11] covering expansions of the four-valued matrix defining B_4, in which case the lattices under consideration are found quite effectively, making the problem involved a part of *Applied* Non-Classical Logic. These tools enable us not merely to find quite effectively the distributive lattices of disjunctive extensions of four-valued expansions of B_4 but also to study special extensions like that which is relatively axiomatized by the notorious *Resolution* rule going back to Automated Reasoning

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(cf., e.g., [14]), whose particular meaning (within the framework involved) consists in the fact that it axiomatizes Kleene's three-valued logic [3] relatively to B_4.

The rest of the paper is as follows. The exposition of the material of the paper is entirely self-contained (of course, modulo very basic issues concerning Set Theory, Lattice Theory, Universal Algebra, Model Theory and Mathematical Logic not specified here explicitly, to be found, e.g., in standard mathematical handbooks like [1] and [5]). Section 2 is a concise summary of mainly specific basic issues underlying the paper including those presented in [13]. Section 3 is devoted to generic tools concerning disjunctivity described in the Abstract. Next, Section 4 deals with main issues concerning four-valued expansions of B_4, Subsection 4.1 being a concise summary of those presented in [13]. Then, in Subsection 4.3/4.2 we find the lattice of disjunctive extensions of four-valued expansions of B_4 together with their finite relative axiomatizations and finite matrix semantics/study their extensions by the Resolution rule. Further, in Subsection 4.4/4.5 we find the lattices of non-paracomplete/Kleene extensions (viz., those of the least proper disjunctive one) of expansions of B_4 /obeying certain conditions.

2. Basic issues

Standard notations like img, dom, ker, hom, π_i, Con, et. al., as well as related notions are supposed to be clear.

2.1. **Set-theoretical background.**

We follow the standard set-theoretical convention, according to which natural numbers (including 0) are treated as finite ordinals (viz., sets of lesser natural numbers), the ordinal of all them being denoted by ω. The proper class of all ordinals is denoted by ∞. Likewise, functions are viewed as binary relations. In addition, singletons are often identified with their unique elements, unless any confusion is possible.

Given a set S, the set of all subsets of S [of cardinality $\in K \subseteq \infty$] is denoted by $\wp_{[K]}(S)$. Next, S-tuples (viz., functions with domain S) are often written in either sequence $\bar{t}$ or vector $\vec{t}$ forms, its s-th component (viz., the value under argument s), where $s \in S$, being written as either t_s or t^s. As usual, given two more sets A and B, any relation between them is identified with the equally-denoted relation between A^S and B^S defined point-wise. Further, elements of $S^* \triangleq (S^0 \cup S^+)$, where $S^+ \triangleq (\bigcup_{i\in(\omega\setminus 1)} S^i)$, are identified with ordinary finite tuples, the binary concatenation operation on which being denoted by $*$, as usual. Then, any binary operation $\diamond$ on S determines the equally-denoted mapping $\diamond : S^+ \to S$ as follows: by induction on the length $l = (\operatorname{dom} \bar{a})$ of any $\bar{a} \in S^+$, put:

$$\diamond\bar{a} \triangleq \begin{cases} a_0 & \text{if } l = 1, \\ (\diamond(\bar{a}{\upharpoonright}(l-1))) \diamond a_{l-1} & \text{otherwise.} \end{cases}$$

Furthermore, given any $f : S \to S$, put $f^1 \triangleq f$ and $f^0 \triangleq \Delta_S \triangleq \{\langle s, s\rangle \mid s \in S\}$, said to be *diagonal*. A subset $T \subseteq S$ is said to be *proper*, if $T \neq S$. After all, given any $T \subseteq S/R \subseteq S^2$, an n-ary operation g on S, where $n \in \omega$, is said to be *T-idempotent/R-[anti-]monotonic*, provided, for all $a \in T/\bar{b}, \bar{c} \in (A^n \cap R)$, it holds that $g(n \times \{a\}) = a/\langle g(\bar{b}), g(\bar{c})\rangle \in R^{[-1]}$, respectively.

Let A be a set. An *anti-chain of* any $S \subseteq \wp(A)$ is any $N \subseteq S$ such that $\max(N) = N$. Likewise, a *lower cone of* S is any $L \subseteq S$ such that, for each $X \in L$, $(\wp(X) \cap S) \subseteq L$. This is said to be *generated by* a $G \subseteq L$, whenever $L = G_S^{\triangledown} \triangleq (S \cap \bigcup\{\wp(X) \mid X \in G\})$. (Clearly, in case A is finite, the mappings $N \mapsto N_S^{\triangledown}$ and $L \mapsto \max(L)$ are inverse to one another bijections between the sets of all antichains and lower cones of S.) A $U \subseteq \wp(A)$ is said to be *upward-directed*, provided, for every $S \in \wp_\omega(U)$, there is some $T \in U$ such that $(\bigcup S) \subseteq T$. A subset of $\wp(A)$ is said to be *inductive*, whenever it is closed under unions of upward-directed subsets. Further, any $X \in T \subseteq \wp(A)$ is said to be *K-meet-irreducible (in/of T)*, where $K \subseteq \infty$, provided it belongs to every $U \in \wp_K(T)$ such that $(A \cap \bigcap U) = X$ (in which case $X \neq A$, whenever $0 \in K$), the set of all them being denoted by $\mathrm{MI}^K(T)$.[1] A *closure system over A* is any $\mathcal{C} \subseteq \wp(A)$ such that, for every $S \subseteq \mathcal{C}$, it holds that $(A \cap \bigcap S) \in \mathcal{C}$, in which case the poset $\langle \mathcal{C}, \subseteq \cap \mathcal{C}^2 \rangle$ to be identified with $\mathcal{C}$ alone is a complete lattice with meet $A \cap \bigcap$. In that case, any $\mathcal{B} \subseteq \mathcal{C}$ is called a *(closure) basis of $\mathcal{C}$*, provided $\mathcal{C} = \{A \cap \bigcap S | S \subseteq \mathcal{B}\}$. An *operator over A* is any unary operation O on $\wp(A)$. This is said to be *(monotonic) [idempotent] {transitive} ⟨inductive/finitary/compact⟩*, provided, for all $(B,)D \in \wp(A)\langle$resp., any upward-directed $U \subseteq \wp(A)\rangle$, it holds that $(O(B))[D]\{O(O(D)\} \subseteq O(D)\langle O(\bigcup U) \subseteq \bigcup O[U]\rangle$. A *closure operator over A* is any monotonic idempotent transitive operator C over A, in which case $\operatorname{img} C$ is a closure system over A, determining C uniquely, because, for every closure basis $\mathcal{B}$ of $\operatorname{img} C$ (including $\operatorname{img} C$ itself) and each $X \subseteq A$, it holds that $C(X) = (A \cap \bigcap\{Y \in \mathcal{B} | X \subseteq Y\})$, called *dual to C* and vice versa. (Clearly, C is inductive iff $\operatorname{img} C$ is so.)

Remark 2.1. As a consequence of Zorn's Lemma, according to which any inductive non-empty set has a maximal element, given any inductive closure system $\mathcal{C}$, $\mathrm{MI}(\mathcal{C})$ is a closure basis of $\mathcal{C}$, and so is $\mathrm{MI}^K(\mathcal{C}) \supseteq \mathrm{MI}(\mathcal{C})$, where $K \subseteq \infty$. □

A *[dual] Galois retraction between* posets $\langle P, \leqq \rangle$ and $\langle Q, \lesssim \rangle$ is any couple $\langle f, g \rangle$ of anti-monotonic [resp., monotonic] mappings $f : P \to Q$ and $g : Q \to P$ such that $(g \circ f) = \Delta_P$ and $(f \circ g) \subseteq \lesssim^{[-1]}$, in which case case the former poset is said to be a *[dual] Galois retract of* the latter. (Galois retractions are exactly Galois connections with injective/surjective left/right component; cf. [10] and [12]. Moreover, dual Galois retractions between $\langle P, \leqq \rangle$ and $\langle Q, \lesssim \rangle$ are exactly Galois retractions between $\langle P, \leqq \rangle$ and $\langle Q, \lesssim^{-1} \rangle$.)

2.2. **Propositional logics and matrices.** Unless otherwise specified, we deal with a fixed but arbitrary signature Σ of connectives of finite arity to be treated as function symbols. Given any $\alpha \in \wp_{\infty \setminus 1}(\omega)$, $\mathfrak{Fm}^\alpha_\Sigma$ denotes the absolutely free Σ-algebra freely-generated by the set $V_\alpha \triangleq \{x_i \mid i \in \alpha\}$ of *variables of rank α*, the corresponding superscript being traditionally omitted in denoting its operations, its endomorphisms/elements of its carrier $\mathrm{Fm}^\alpha_\Sigma$ being called *Σ-substitutions/-formulas of rank α*. (In general, the reservation "of rank α" is normally omitted, whenever $\alpha = \omega$, unless any confusion is possible.) Then, a *[finitary] Σ-rule* is any couple $R = \langle \Gamma, \varphi \rangle$, normally written as $\Gamma \vdash \varphi$, with the set of its *premises* $\Gamma \in \wp_{[\omega]}(\mathrm{Fm}^\omega_\Sigma)$ and its *conclusion* $\varphi \in \mathrm{Fm}^\omega_\Sigma$, any Σ-rule of the form $\sigma(R) \triangleq (\sigma[\Delta] \vdash \sigma(\psi))$, where $\sigma \in \hom(\mathfrak{Fm}^\omega_\Sigma, \mathfrak{Fm}^\omega_\Sigma)$, being called a *substitutional Σ-instance of R*. As usual, Σ-*axioms* are Σ-rules without premises to be identified with their conclusions. Then, an *[axiomatic/finitary] Σ-calculus* is any set of Σ-rules [without/with finitely many premises].

Given a *Σ-logic C* (viz., a *structural* closure operator over $\mathrm{Fm}^\omega_\Sigma$ in the sense that $\operatorname{img} C$ is closed under inverse Σ-substitutions), we sometimes write $X \vdash_C Y$, where $X, Y \subseteq \mathrm{Fm}^\omega_\Sigma$, for $C(X) \supseteq Y$ as well as $\phi \equiv_C \psi$, where $\phi, \psi \in \mathrm{Fm}^\omega_\Sigma$, for $C(\phi) = C(\psi)$, in which case $\equiv_C$ becomes a binary relation on $\mathrm{Fm}^\omega_\Sigma$. Then, C is said to *satisfy* a Σ-rule $\Gamma \vdash \varphi$, provided $\Gamma \vdash_C \varphi$, Σ-axioms satisfied in C being referred to as *theorems of C*. Next, C is said to be *non-pseudo-axiomatic*, provided $(\bigcap_{k \in \omega} C(x_k)) \subseteq C(\varnothing)$ (the converse inclusion always holds by the monotonicity of C). Likewise, it is said to be *theorem-less*, whenever it has no theorem. Further, C

[1]Within any context, any mention of K is normally omitted, whenever $K = \infty$. Likewise, "finitely-/pairwise-" means "ω-/$\{2\}$-", respectively.

is said to be *[inferentially] (in)consistent*, if $x_1 \notin (\in) C(\varnothing[\cup\{x_0\}])$. Furthermore, a Σ-logic C' is said to be a *[proper] extension of* C, if $C \subseteq [\subsetneq] C'$, in which case C is said to be a *[proper] sublogic of* C'. Then, a[n] *[axiomatic]* Σ*-calculus* $\mathcal{C}$ is said to *axiomatize* C' *relatively to* C, if C' is the least extension of C satisfying each rule in $\mathcal{C}$ [in which case C' is said to be an *axiomatic extension of* C]. Finally, C is said to be *[maximally]* $\wr$*-paraconsistent*, where $\wr \in \Sigma$ is unary, provided $x_1 \notin C(\{x_0, \wr x_0\})$ [and C has no proper $\wr$-paraconsistent extension]. Likewise, C is said to be *[inferentially] maximal*, whenever it is [inferentially] consistent and has no proper [inferentially] consistent extension.

A Σ-rule $\Gamma \vdash \varphi$ is said to be *derivable in* a finitary Σ-calculus $\mathcal{C}$ if there is a $\mathcal{C}$*-derivation of* it, i.e., a proof of φ (in the conventional proof-theoretical sense) by means of axioms in Γ and rules in the set $\mathrm{SI}_\Sigma(\mathcal{C})$ of all substitutional Σ-instances of rules in $\mathcal{C}$. The extension of the diagonal Σ-logic relatively axiomatized by $\mathcal{C}$ is said to be *axiomatized by* $\mathcal{C}$, in which case it is inductive and satisfies any Σ-rule iff this is derivable in $\mathcal{C}$. (Conversely, any inductive Σ-logic is axiomatized by the set of all finitary Σ-rules satisfied in it.)

Remark 2.2. Given a Σ-logic C, we have the Σ-logic $C_{+/-0}$, defined by $C_{+/-0}(X) \triangleq C(X)$, for all non-empty $X \subseteq \mathrm{Fm}^\omega_\Sigma$, and $C_{+/-0}(\varnothing) \triangleq (\varnothing/(\bigcap_{k\in\omega} C(x_k)))$, being the greatest/least theorem-less/non-pseudo-axiomatic sublogic/extension of C, called the *theorem-less/non-pseudo-axiomatic version of* C. Then, the mappings $C \mapsto C_{+0}$ and $C \mapsto C_{-0}$ are inverse to one another isomorphisms between the posets of all non-pseudo-axiomatic and of all theorem-less Σ-logics ordered by $\subseteq$. □

As usual, Σ-matrices (cf. [4]) are treated as first-order model structures (viz., algebraic systems; cf. [5]) of the first-order signature $\Sigma \cup \{D\}$ with unary *truth* predicate D, any finitary Σ-rule $\Gamma \vdash \phi$ being viewed as the first-order (either basic or universal, depending upon the context) strict Horn formula $(\bigwedge \Gamma) \to \phi$ under the standard identification of any Σ-formula ψ with the first-order atomic formula $D(\psi)$.[2] In this way, given any class M of Σ-matrices and any [axiomatic] finitary Σ-calculus $\mathcal{C}$, $\mathsf{M} \cap \mathrm{Mod}(\mathcal{C})$ is referred to as the *relative (equality-free first-order) universal [positive] (strict) Horn model subclass of* M *relatively axiomatized by* $\mathcal{C}$.

A Σ-matrix $\mathcal{A}$, traditionally identified with the couple $\langle \mathfrak{A}, D^{\mathcal{A}} \rangle$, is said to be *n-valued/truth[-non]-empty/(in)consistent*, where $n \in \omega$, provided $|A| = n/D^{\mathcal{A}} = [\neq]\varnothing/D^{\mathcal{A}} \neq (=) A$. It is said to be *finite/generated by* a $B \subseteq A$, whenever $\mathfrak{A}$ is so. Then, it is said to be *K-generated*, where $K \subseteq \infty$, whenever it is generated by a $B \in \wp_K(A)$. Given any $\Sigma' \subseteq \Sigma$, $\mathcal{A}$ is said to be a *(Σ-)expansion of* $(\mathcal{A}{\restriction}\Sigma') \triangleq \langle \mathfrak{A}{\restriction}\Sigma', D^{\mathcal{A}} \rangle$. (Any notation being specified for single Σ-matrices is supposed to be extended to their classes member-wise.)

Let $\mathcal{A}$ and $\mathcal{B}$ be two Σ-matrices. A *(strict) [surjective] homomorphism from* $\mathcal{A}$ *[on]to* $\mathcal{B}$ is any $h \in \hom(\mathfrak{A}, \mathfrak{B})$ such that $[h[A] = B$ and$]$ $D^{\mathcal{A}} \subseteq (=) h^{-1}[D^{\mathcal{B}}]$, the set of all them being denoted by $\hom^{[\mathrm{S}]}_{(\mathrm{S})}(\mathcal{A}, \mathcal{B})$. Then, $\mathcal{A}$ is said to be a *submatrix of* $\mathcal{B}$, whenever $\Delta_A \in \hom_{\mathrm{S}}(\mathcal{A}, \mathcal{B})$, in which case we set $(\mathcal{B}{\restriction}A) \triangleq \mathcal{A}$. Injective/bijective strict homomorphisms from $\mathcal{A}$ to $\mathcal{B}$ are referred to as *embeddings/isomorphisms of/from* $\mathcal{A}$ *into/onto* $\mathcal{B}$, in case of existence of which $\mathcal{A}$ is said to be *embeddable/isomorphic into/to* $\mathcal{B}$.

Let $\mathcal{A}$ be a Σ-matrix. Elements of $\mathrm{Con}(\mathcal{A}) \triangleq \{\theta \in \mathrm{Con}(\mathfrak{A}) | \theta[D^{\mathcal{A}}] \subseteq D^{\mathcal{A}}\} \ni \Delta_A$ are called *congruences of* $\mathcal{A}$. Given any $\theta \in \mathrm{Con}(\mathcal{A})$, we have the *quotient* Σ-matrix $(\mathcal{A}/\theta) \triangleq \langle \mathfrak{A}/\theta, D^{\mathcal{A}}/\theta \rangle$ *by* θ, in which case $\nu_\theta \in \hom^{\mathrm{S}}_{\mathrm{S}}(\mathcal{A}, \mathcal{A}/\theta)$.

[2] In general, unless otherwise specified, [Σ-matrices are denoted by Calligraphic letters (possibly, with indices), their *underlying*] algebras [viz., algebra (i.e., Σ-) reducts] being denoted by [corresponding] Fraktur letters (possibly, with [same] indices [if any]), their carriers (viz., underlying sets) being denoted by corresponding Italic letters (with same indices, if any).

Recall the following well-known useful observations concerning the closure operator $\mathrm{Cn}^{\alpha}_{\mathsf{M}}$ over $\mathrm{Fm}^{\alpha}_{\Sigma}$, where $\alpha \in \wp_{\infty\setminus 1}(\omega)$ and M a class of Σ-matrices, dual to the closure system with basis $\{h^{-1}[D^{\mathcal{A}}] | \mathcal{A} \in \mathsf{M}, h \in \hom(\mathfrak{Fm}^{\alpha}_{\Sigma}, \mathfrak{A})\})\}$, in which case:

$$\mathrm{Cn}^{\alpha}_{\mathsf{M}}(X) = (\mathrm{Fm}^{\alpha}_{\Sigma} \cap \mathrm{Cn}^{\omega}_{\mathsf{M}}(X)), \tag{2.1}$$

$$(\hom^{[\mathrm{S}]}_{\mathrm{S}}(\mathcal{A},\mathcal{B}) \neq \varnothing) \Rightarrow (\mathrm{Cn}^{\alpha}_{\mathcal{B}}(X) \subseteq [=]\, \mathrm{Cn}^{\alpha}_{\mathcal{A}}(X)), \tag{2.2}$$

$$(\hom^{\mathrm{S}}(\mathcal{A},\mathcal{B}) \neq \varnothing) \Rightarrow (\mathrm{Cn}^{\alpha}_{\mathcal{A}}(\varnothing) \subseteq \mathrm{Cn}^{\alpha}_{\mathcal{B}}(\varnothing)), \tag{2.3}$$

for all $X \subseteq \mathrm{Fm}^{\alpha}_{\Sigma}$, where $\mathcal{A}$ and $\mathcal{B}$ are Σ-matrices.

Given a set I and an I-tuple $\overline{\mathcal{A}}$ of Σ-matrices, the Σ-matrix $(\prod_{i\in I} \mathcal{A}_i) \triangleq \langle \prod_{i\in I} \mathfrak{A}_i, (\prod_{i\in I} A_i) \cap \bigcap_{i\in I} \pi_i^{-1}[D^{\mathcal{A}_i}]\rangle$ is called the *direct product of* $\overline{\mathcal{A}}$ (as usual, when $(I = 2)/((\mathrm{img}\,\overline{\mathcal{A}}) \subseteq \{\mathcal{A}\})$, where $\mathcal{A}$ is a Σ-matrix, $(\mathcal{A}_0 \times \mathcal{A}_1)/\mathcal{A}^I$ stands for the direct product involved), any submatrix $\mathcal{B}$ of it being referred to as a *subdirect product of* $\overline{\mathcal{A}}$, whenever, for each $i \in I$, $\pi_i[B] = A_i$.

Lemma 2.3 (Finite Subdirect Product Lemma; cf. Lemma 2.7 of [13]). *Let* M *be a finite class of finite* Σ*-matrices and* $\mathcal{A}$ *a finitely-generated model of the logic of* M*. Then, there are some congruence* θ *of* $\mathcal{A}$ *and some strict surjective homomorphism from a subdirect product of a finite tuple constituted by consistent submatrices of members of* M *onto* $\mathcal{A}/\theta$*.*

A Σ-logic C is said to be K*-defined by* a class of Σ-matrices M, where $K \subseteq \infty$, provided $C(X) = \mathrm{Cn}^{\omega}_{\mathsf{M}}(X)$, for all $X \in \wp_K(\mathrm{Fm}^{\omega}_{\Sigma})$. The Σ-logic defined (viz., ∞-defined) by M is called the one *of* M. (Due to [4], this is well known to be inductive, whenever both M and all members of it are finite.) A Σ-logic is said to be n*-valued*, where $n \in \omega$, whenever it is defined by an n-valued Σ-matrix.

Remark 2.4. Since any rule with[out] premises is [not] true in any truth-empty matrix, given any class M of Σ-matrices and any non-empty class S of truth-empty Σ-matrices, the logic of $\mathsf{S} \cup \mathsf{M}$ is the theorem-less version of the logic of M. □

Remark 2.5. Since formulas contain finitely many variables, the logic of any class of truth-non-empty matrices is non-pseudo-axiomatic. □

A Σ-matrix $\mathcal{A}$ is said to be $\wr$*-paraconsistent/*, where $\wr \in \Sigma$ is a unary, whenever the logic of $\mathcal{A}$ is so. Likewise, $\mathcal{A}$ is said to be $\diamond$*-implicative/-disjunctive*, where $\diamond$ is a (possibly, secondary) binary connective of Σ, whenever, for all $a, b \in A$, it holds that $((a \diamond^{\mathfrak{A}} b) \in D^{\mathcal{A}}) \Leftrightarrow ((a \in / \notin D^{\mathcal{A}}) \Rightarrow (b \in D^{\mathcal{A}}))$, in which case it is $\veebar_{\diamond}$-disjunctive, where $(x_0 \veebar_{\diamond} x_1) \triangleq ((x_0 \diamond x_1) \diamond x_1)$. Next, it is said to be a *model of* a Σ-logic C, whenever the logic of $\mathcal{A}$ is an extension of C, the class of all models of C being denoted by $\mathrm{Mod}(C)$.

Given a class M of Σ-matrices, the class of all (truth-non-empty) [consistent] submatrices of members of M is denoted by $\mathbf{S}^{(*)}_{[*]}(\mathsf{M})$, respectively. Likewise, the class of all [sub]direct products of tuples (of cardinality $\in K \subseteq \infty$) constituted by members of M is denoted by $\mathbf{P}^{[\mathrm{SD}]}_{(K)}(\mathsf{M})$.

Theorem 2.6 (cf. Theorem 2.8 of [13]). *Let* K *and* M *be classes of* Σ*-matrices,* C *the logic of* M *and* C' *an extension of* C*. Suppose both* M *and all members of it are finite and* $\mathbf{P}^{\mathrm{SD}}_{\omega}(\mathbf{S}_*(\mathsf{M})) \subseteq \mathsf{K}$ *(in particular,* $\mathbf{S}(\mathbf{P}_{\omega}(\mathsf{M})) \subseteq \mathsf{K}$*). Then,* C' *is finitely-defined by* $\mathrm{Mod}(C') \cap \mathsf{K}$*.*

Given any Σ-logic C and any $\Sigma' \subseteq \Sigma$, in which case $\mathrm{Fm}^{\alpha}_{\Sigma} \subseteq \mathrm{Fm}^{\alpha}_{\Sigma'}$ and $\hom(\mathfrak{Fm}^{\alpha}_{\Sigma'}, \mathfrak{Fm}^{\alpha}_{\Sigma'}) = \{h \upharpoonright \mathrm{Fm}^{\alpha}_{\Sigma'} | h \in \hom(\mathfrak{Fm}^{\alpha}_{\Sigma}, \mathfrak{Fm}^{\alpha}_{\Sigma}), h[\mathrm{Fm}^{\alpha}_{\Sigma'}] \subseteq \mathrm{Fm}^{\alpha}_{\Sigma'}\}$, for all $\alpha \in \wp_{\infty\setminus 1}(\omega)$, we have the Σ'-logic C', defined by $C'(X) \triangleq (\mathrm{Fm}^{\omega}_{\Sigma'} \cap C(X))$, for all $X \subseteq \mathrm{Fm}^{\omega}_{\Sigma'}$, called the Σ'*-fragment of* C, in which case C is said to be a *(*Σ*-)expansion of* C'.

In that case, given also any class M of Σ-matrices defining C, C' is, in its turn, defined by $\mathsf{M}\restriction\Sigma'$.

Proposition 2.7. *Let C be a Σ-logic and M a finite class of finite Σ-matrices. Suppose C if finitely-defined by M. Then, C is defined by M, that is, inductive.*

Proof. In that case, $C' \triangleq \mathrm{Cn}^{\omega}_{\mathsf{M}} \subseteq C$, for C' is inductive, while $\equiv_C = \equiv_{C'}$. For proving the converse inclusion, it suffices to prove that $\mathsf{M} \subseteq \mathrm{Mod}(C)$. For consider any $\mathcal{A} \in \mathsf{M}$, any $\Gamma \subseteq \mathrm{Fm}^{\omega}_{\Sigma}$, any $\varphi \in C(\Gamma)$ and any $h \in \hom(\mathfrak{Fm}^{\omega}_{\Sigma}, \mathfrak{A})$ such that $h[\Gamma] \subseteq D^{\mathcal{A}}$. Then, $\alpha \triangleq |A| \in (\wp_{\infty\setminus 1}(\omega) \cap \omega)$. Take any bijection $e : V_\alpha \to A$ to be extended to a $g \in \hom(\mathfrak{Fm}^{\alpha}_{\Sigma}, \mathfrak{A})$. Then, $e^{-1} \circ (h\restriction V_\omega)$ is extended to a Σ-substitution σ, in which case $\sigma(\varphi) \in C(\sigma[\Gamma])$, for C is structural, while $\sigma[\Gamma \cup \{\varphi\}] \subseteq \mathrm{Fm}^{\alpha}_{\Sigma}$. For every $\mathcal{B} \in \mathsf{M}$, we have the equivalence relation $\theta^{\mathcal{B}} \triangleq \{\langle a, b\rangle \in B^2 \mid (a \in D^{\mathcal{B}}) \Leftrightarrow (b \in D^{\mathcal{B}})\}$ on B, in which case $B/\theta^{\mathcal{B}}$ is finite, for B is so. Moreover, as both α, M and all members of the latter are finite, we have the finite set $I \triangleq \{\langle h', \mathcal{B}\rangle \mid \mathcal{B} \in \mathsf{M}, h' \in \hom(\mathfrak{Fm}^{\alpha}_{\Sigma}, \mathfrak{B})\}$, in which case, for each $i \in I$, we set $h_i \triangleq \pi_0(i)$, $\mathcal{B}_i \triangleq \pi_1(i)$ and $\theta_i \triangleq \theta^{\mathcal{B}_i}$. Then, by (2.1), we have $\theta \triangleq (\equiv_{C'} \cap \mathrm{Eq}^{\alpha}_{\Sigma}) = (\mathrm{Eq}^{\alpha}_{\Sigma} \cap \bigcap_{i\in I} h_i^{-1}[\theta_i])$, in which case, for every $i \in I$, $\theta \subseteq h_i^{-1}[\theta_i] = \ker(\nu_{\theta_i} \circ h_i)$, and so $g_i \triangleq (\nu_{\theta_i} \circ h_i \circ \nu_\theta^{-1}) : (\mathrm{Fm}^{\alpha}_{\Sigma}/\theta) \to B_i$. In this way, $f : (\mathrm{Fm}^{\alpha}_{\Sigma}/\theta) \to (\prod_{i\in I} B_i), a \mapsto \langle g_i(a)\rangle_{i\in I}$ is injective, for $(\ker f) = ((\mathrm{Fm}^{\alpha}_{\Sigma}/\theta)^2 \cap \bigcap_{i\in I}(\ker g_i))$ is diagonal. Hence, $\mathrm{Fm}^{\alpha}_{\Sigma}/\theta$ is finite, for $\prod_{i\in I} B_i$ is so, and so is $(\sigma[\Gamma]/\theta) \subseteq (\mathrm{Fm}^{\alpha}_{\Sigma}/\theta)$. For each $c \in (\sigma[\Gamma]/\theta)$, choose any $\phi_c \in (\sigma[\Gamma] \cap \nu_\theta^{-1}[\{c\}]) \neq \varnothing$. Put $\Delta \triangleq \{\phi_c \mid c \in (\sigma[\Gamma]/\theta)\} \in \wp_\omega(\sigma[\Gamma])$. Consider any $\psi \in \sigma[\Gamma]$. Then, $\Delta \ni \phi_{[\psi]_\theta} \equiv_C \psi$, in which case $\psi \in C(\Delta)$, and so $\sigma[\Gamma] \subseteq C(\Delta)$. In this way, $\sigma(\varphi) \in C(\Delta) = C'(\Delta)$, for $\Delta \in \wp_\omega(\mathrm{Fm}^{\omega}_{\Sigma})$, so, by (2.1), $\sigma(\varphi) \in \mathrm{Cn}^{\alpha}_{\mathsf{M}}(\Delta) \subseteq \mathrm{Cn}^{\alpha}_{\mathcal{A}}(\Delta)$. Moreover, $g[\Delta] \subseteq g[\sigma[\Gamma]] = h[\Gamma] \subseteq D^{\mathcal{A}}$, and so $h(\varphi) = g(\sigma(\varphi)) \in D^{\mathcal{A}}$, as required. □

2.2.1. *Classical matrices and logics.* A consistent two-valued Σ-matrix $\mathcal{A}$ is said to be $\wr$-*classical*, where $\wr$ is a unary connective of Σ, provided $(a \in D^{\mathcal{A}}) \Leftrightarrow (\wr^{\mathfrak{A}} a \notin D^{\mathcal{A}})$, for all $a \in A$, in which case it is truth-non-empty. Then, a Σ-logic is said to be $\wr$-*(sub)classical*, whenever it is (a sublogic of) the logic of a $\wr$-classical Σ-matrix. Next, a Σ-logic is said to be *inferentially $\wr$-classical*, whenever it is either $\wr$-classical or the theorem-less version of a $\wr$-classical Σ-logic, in which case it is not $\wr$-paraconsistent.

2.2.2. *Equality determinants.* According to [11], an *equality determinant for* a Σ-matrix $\mathcal{A}$ is any $\Upsilon \subseteq \mathrm{Fm}^1_{\Sigma}$ such that any $a, b \in A$ are equal, whenever, for each $\upsilon \in \Upsilon$, $\upsilon^{\mathfrak{A}}(a) \in D^{\mathcal{A}}$ iff $\upsilon^{\mathfrak{A}}(b) \in D^{\mathcal{A}}$.

3. Disjunctivity

Fix any set A and any $\delta : A^2 \to A$. Given any $X, Y \subseteq A$, set $\delta(X, Y) \triangleq \delta[X \times Y]$. Then, a $Z \subseteq A$ is said to be δ-*disjunctive*, provided, for all $a, b \in A$, it holds that $((\{a, b\} \cap Z) \neq \varnothing) \Leftrightarrow (\delta(a, b) \in Z)$, in which case, for all $X, Y \subseteq A$, we have $((X \subseteq Z)|(Y \subseteq Z)) \Leftrightarrow (\delta(X, Y) \subseteq Z)$. Next, a closure operator C over A is said to be δ-*multiplicative*, provided

$$\delta(C(X), a) \subseteq C(\delta(X, a)), \tag{3.1}$$

for all $(X \cup \{a\}) \subseteq A$, and δ-*disjunctive*, provided, for all $a, b \in A$ and every $Z \subseteq A$, it holds that

$$C(Z \cup \{\delta(a, b)\}) = (C(Z \cup \{a\}) \cap C(Z \cup \{b\})), \tag{3.2}$$

in which case the following clearly hold, by (3.2) with $Z = \varnothing$:

$$\delta(a, b) \in C(a), \tag{3.3}$$

$$\delta(a, b) \in C(b), \tag{3.4}$$

$$a \in C(\delta(a,a)), \tag{3.5}$$
$$\delta(b,a) \in C(\delta(a,b)), \tag{3.6}$$
$$C(\delta(\delta(a,b),c)) = C(\delta(a,\delta(b,c))), \tag{3.7}$$

for all $a, b, c \in A$.

Lemma 3.1. *Let C be a closure operator over A and $\mathcal{B}$ a closure basis of* $\operatorname{img} C$. *Suppose each element of $\mathcal{B}$ is δ-disjunctive. Then,*

$$(C(Z \cup X) \cap C(Z \cup Y)) = C(Z \cup \delta(X,Y)), \tag{3.8}$$

for all $X, Y, Z \subseteq A$. In particular, C is both δ-disjunctive and δ-multiplicative.

Proof. First, for all $a \in A$, we have:

$$\begin{aligned} &(a \in C(Z \cup X) \cap C(Z \cup Y)) \\ \Leftrightarrow\ &\forall W \in \mathcal{B} : ((((Z \subseteq W)\&(X \subseteq W)) \Rightarrow (a \in W)) \\ &\&(((Z \subseteq W)\&(Y \subseteq W)) \Rightarrow (a \in W))) \\ \Leftrightarrow\ &\forall W \in \mathcal{B} : (((Z \subseteq W)\&(X \subseteq W|Y \subseteq W)) \Rightarrow (a \in W)) \\ \Leftrightarrow\ &\forall W \in \mathcal{B} : (((Z \subseteq W)\&(\delta(X,Y) \subseteq W)) \Rightarrow (a \in W)) \\ \Leftrightarrow\ &(a \in C(Z \cup \delta(X,Y))), \end{aligned}$$

in which case (3.8) holds, and so immediately does its particular case (3.2). Finally, applying (3.8) with $Z = \varnothing$ twice, we also get, for all $(X \cup \{a\}) \subseteq A$, $\delta(C(X),a) \subseteq C(\delta(C(X),a)) = (C(C(X)) \cap C(a)) = (C(X) \cap C(a)) = C(\delta(X,a))$, in which case (3.1) holds, as required. □

Lemma 3.2. *Let C be a δ-disjunctive closure operator over A and $X \in (\operatorname{img} C)$. Then, X is δ-disjunctive iff it is pair-wise-meet-irreducible in* $\operatorname{img} C$*, and so it is finitely-meet-irreducible in* $\operatorname{img} C$ *iff it is δ-disjunctive and proper.*

Proof. First, assume X is not δ-disjunctive. Then, in view of (3.3) and (3.4), there is some $\vec{a} \in (A \setminus X)^2$, in which case, for each $i \in 2$, it holds that $X \neq C(X \cup \{a_i\}) \in (\operatorname{img} C)$, such that $\delta(\vec{a}) \in X$. Therefore, by (3.2), we have $X = (\bigcap_{i \in 2} C(X \cup \{a_i\}))$. Hence, X is not pair-wise-meet-irreducible in $\operatorname{img} C$.

Conversely, assume X is not pair-wise-meet-irreducible in $\operatorname{img} C$. Then, there is some $\vec{Y} \in ((\operatorname{img} C) \setminus \{X\})^2$ such that $X = (\bigcap_{i \in 2} Y_i)$, in which case, for each $i \in 2$, $X \subsetneq Y_i$, so there is some $a_i \in (Y_i \setminus X) \neq \varnothing$. In this way, by (3.2), we have $\delta(\vec{a}) \in C(X \cup \{\delta(\vec{a})\}) = (\bigcap_{i \in 2} C(X \cup \{a_i\})) \subseteq (\bigcap_{i \in 2} Y_i) = X$. Thus, X is not δ-disjunctive, as required. □

3.1. **Disjunctive logics and matrices.** Fix any (possibly, secondary) binary connective $\veebar$ of Σ.

Remark 3.3. Given two Σ-matrices $\mathcal{A}$ and $\mathcal{B}$ with $\hom_{\mathrm{S}}^{[\mathrm{S}]}(\mathcal{A},\mathcal{B}) \neq \varnothing$, $\mathcal{A}$ is $\veebar$-disjunctive if[f] $\mathcal{B}$ is so. □

Corollary 3.4. *Let $\alpha \in \wp_{\infty \setminus 1}(\omega)$ and M a class of $\veebar$-disjunctive Σ-matrices. Then, $\mathrm{Cn}_{\mathsf{M}}^{\alpha}$ is both $\veebar$-disjunctive and $\veebar$-multiplicative.*

Proof. For each $\mathcal{A} \in \mathsf{M}$ and every $h \in \hom(\mathfrak{Fm}_{\Sigma}^{\alpha}, \mathfrak{A})$, $h^{-1}[D^{\mathcal{A}}]$ is $\veebar$-disjunctive, by Remark 3.3. Then, Lemma 3.1 completes the proof. □

Corollary 3.5. *Let C be an inductive Σ-logic. Then, the following are equivalent:*

(i) *C is $\veebar$-disjunctive;*
(ii) $\operatorname{img} C$ *has a basis consisting of $\veebar$-disjunctive sets;*
(iii) (3.3)*,* (3.5)*,* (3.6) *hold and C is $\veebar$-multiplicative;*

(iv) (3.3), (3.5), (3.6) *hold and, for any axiomatization* $\mathcal{C}$ *of* C*, every* $(\Gamma \vdash \phi) \in \mathrm{SI}_\Sigma(\mathcal{C})$ *and each* $\psi \in \mathrm{Fm}^\omega_\Sigma$*, it holds that* $(\phi \veebar \psi) \in C(\Gamma \veebar \psi)$.

Proof. First, (i)⇒(ii) is by Remark 2.1 and Lemma 3.2. Next, (ii)⇒(iii) is by Lemma 3.1. Further, (iv) is a particular case of (iii). Then, the converse is proved by induction on the length of $\mathcal{C}$-derivations. Finally, assume (iii) holds, in which case (3.4) holds by (3.3) and (3.6), and so does the inclusion from left to right in (3.2), by (3.3) and (3.4). Conversely, consider any $\varphi \in (C(Z \cup \{\phi\}) \cap C(Z \cup \{\psi\}))$. Then, by (3.3), (3.6) and (3.1), we have $(\psi \veebar \varphi) \in C(Z \cup \{\phi \veebar \psi\})$. Likewise, by (3.3), (3.5) and (3.1), we also have $\varphi \in C(Z \cup \{\psi \veebar \varphi\})$. Hence, we eventually get $\varphi \in C(Z \cup \{\phi \veebar \psi\})$, in which case (3.2) holds, and so does (i), as required. □

Corollary 3.6. *Any axiomatic extension of an inductive* $\veebar$*-disjunctive* Σ*-logic is* $\veebar$*-disjunctive.*

Proof. By Corollary 3.5(i)⇔(iv) and (3.3). □

3.1.1. *Disjunctive extensions of logics defined by finite classes of finite disjunctive matrices.* Given a Σ-rule $\Gamma \vdash \phi$ and a Σ-formula ψ, put $((\Gamma \vdash \phi) \veebar \psi) \triangleq ((\Gamma \veebar \psi) \vdash (\phi \veebar \psi))$. (This notation is naturally extended to Σ-calculi member-wise.)

Let σ_{+1} be the Σ-substitution extending $[x_i/x_{i+1}]_{i\in\omega}$.

Lemma 3.7. *Let* $\Gamma \vdash \phi$ *be a* Σ*-rule and* $\mathcal{A}$ *a* $\veebar$*-disjunctive* Σ*-matrix. Then,* $\mathcal{A} \in \mathrm{Mod}(\sigma_{+1}(\Gamma \vdash \phi) \veebar x_0)$ *iff* $\mathcal{A} \in \mathrm{Mod}(\Gamma \vdash \phi)$.

Proof. The "if" part is by the strucuturality of $\mathrm{Cn}^\omega_{\mathcal{A}}$ and Corollary 3.4. Conversely, assume $\mathcal{A} \in \mathrm{Mod}(\sigma_{+1}(\Gamma \vdash \phi) \veebar x_0)$. Consider an arbitrary $h \in \hom(\mathfrak{Fm}^\omega, \mathfrak{A})$ such that $h(\phi) \notin D^{\mathcal{A}}$. Let $g \in \hom(\mathfrak{Fm}^\omega, \mathfrak{A})$ extend $[x_0/h(\phi); x_{i+1}/h(x_i)]_{i\in\omega}$, in which case $(g \circ \sigma_{+1}) = h$, and so, by the $\veebar$-disjunctivity of $\mathcal{A}$, we have $g(\sigma_{+1}(\phi) \veebar x_0) = (h(\phi) \veebar^{\mathfrak{A}} h(\phi)) \notin D^{\mathcal{A}}$. Hence, there is some $\psi \in \Gamma$ such that $(h(\psi) \veebar^{\mathfrak{A}} h(\phi)) = g(\sigma_{+1}(\psi) \veebar x_0) \notin D^{\mathcal{A}}$, in which case, by the $\veebar$-disjunctivity of $\mathcal{A}$, we eventually get $h(\psi) \notin D^{\mathcal{A}}$, and so $\mathcal{A} \in \mathrm{Mod}(\Gamma \vdash \phi)$, as required. □

Lemma 3.8. *Let* C *be an inductive* $\veebar$*-disjunctive logic,* $\mathcal{C}$ *a finitary* Σ*-calculus and* $\mathcal{A} \subseteq \mathcal{C}$ *an axiomatic* Σ*-calculus. Then, the extension* C' *of* C *relatively axiomatized by* $\mathcal{C}' \triangleq (\mathcal{A} \cup (\sigma_{+1}[\mathcal{C} \setminus \mathcal{A}] \veebar x_0))$ *is* $\veebar$*-disjunctive.*

Proof. Then, C being inductive, is axiomatized by a finitary Σ-calculus $\mathcal{C}''$, in which case C' is axiomatized by the finitary Σ-calculus $\mathcal{C}'' \cup \mathcal{C}'$, and so is inductive. Moreover, C', being an extension of C, inherits (3.3), (3.5), (3.6) and (3.7) held for C. Then, we prove the $\veebar$-disjunctivity of C' with applying Corollary 3.5(i)⇔(iv) to both C and C'. For consider any Σ-substitution σ and any $\psi \in \mathrm{Fm}^\omega_\Sigma$. First, consider any $\phi \in \mathcal{A}$. Then, by the structurality of C' and (3.3), we have $(\sigma(\phi) \veebar \psi) \in C'(\varnothing)$. Now, consider any $(\Gamma \vdash \phi) \in (\mathcal{C} \setminus \mathcal{A})$. Let ς be the Σ-substitution extending $(\sigma{\upharpoonright}(V_\omega \setminus V_1)) \cup [x_0/(\sigma(x_0) \veebar \psi)]$, in which case $(\varsigma \circ \sigma_{+1}) = (\sigma \circ \sigma_{+1})$, and so, by (3.7) and the structurality of C', we eventually get $(\sigma[\sigma_{+1}[\Gamma] \veebar x_0] \veebar \psi) = ((\varsigma[\sigma_{+1}[\Gamma]] \veebar \sigma(x_0)) \veebar \psi) \vdash_{C'} (\varsigma[\sigma_{+1}[\Gamma]] \veebar (\sigma(x_0) \veebar \psi)) = \varsigma[\sigma_{+1}[\Gamma] \veebar x_0] \vdash_{C'} \varsigma(\sigma_{+1}(\varphi) \veebar x_0) = (\varsigma(\sigma_{+1}(\varphi)) \veebar (\sigma(x_0) \veebar \psi)) \vdash_{C'} ((\varsigma(\sigma_{+1}(\varphi)) \veebar \sigma(x_0)) \veebar \psi) = (\sigma(\sigma_{+1}(\varphi) \veebar x_0) \veebar \psi)$. □

Lemma 3.9. *Let* K *be a finite class of consistent* $\veebar$*-disjunctive* Σ*-matrices. Then, the set of all relative universal [positive] Horn model subclasses of* K *is a closure system over* K *closed under unions, and so forms a finite distributive lattice.*

Proof. Consider any set I of universal [positive] Horn model subclasses of K, in which case it is finite, for K is so, and so there are some bijection $e : n \to I$, where $n \triangleq |I| \in \omega$, some $\overline{\mathcal{C}} : n \to \wp(\wp_{\omega[\cap 1]}(\mathrm{Fm}^\omega_\Sigma) \times \mathrm{Fm}^\omega_\Sigma)$ and some $\bar{\alpha} : n \to \wp_{\omega\setminus 1}(\omega \setminus 1)$ such that, for every $i \in n$, $e(i) = (\mathsf{K} \cap \mathrm{Mod}(\mathcal{C}_i))$, $\mathcal{C}_i \subseteq (\wp_\omega(\mathrm{Fm}^{\alpha_i}_\Sigma) \times \mathrm{Fm}^{\alpha_i}_\Sigma)$ and $(\alpha_i \cap \alpha_j) = \varnothing$, for all $j \in (n \setminus \{i\})$. Then, we clearly have $(\mathsf{K} \cap \mathrm{Mod}(\bigcup_{i\in n} \mathcal{C}_i)) = (\mathsf{K} \cap (\bigcap I))$.

Moreover, every member of $(\bigcup I) \subseteq \mathsf{K}^{[*]}$ is a model of $\mathcal{C} \triangleq \{(\bigcup \mathrm{img}(\pi_0 \circ \bar{R})) \vdash \veebar\langle\pi_1 \circ \bar{R}, x_0\rangle \mid \bar{R} \in \prod \overline{\mathcal{C}}\} \in \wp(\wp_{\omega[\cap 1]}(\mathrm{Fm}^{\omega}_{\Sigma}) \times \mathrm{Fm}^{\omega}_{\Sigma})$. Conversely, consider any $\mathcal{A} \in (\mathsf{K} \setminus (\bigcup I))$. Then, for every $i \in n$, $\mathcal{A} \notin e(i)$, in which case there are some $R_i \in \mathcal{C}_i$ and some $h_i : \alpha_i \to A$ such that $\mathcal{A} \not\models R_i[h_i]$, and so $((\bigcup_{i\in n} \pi_0[R_i]) \vdash \veebar\langle\langle\pi_1(R_i)\rangle_{i\in n}, x_0\rangle) \in \mathcal{C}$ is not true in $\mathcal{A}$ under $[x_0/a] \cup \bigcup_{i\in n} h_i$, where $a \in (A \setminus D^{\mathcal{A}}) \neq \varnothing$, for $\mathcal{A}$ is consistent. Thus, $(\bigcup I) = (\mathsf{K} \cap \mathrm{Mod}(\mathcal{C}))$, as required. □

Theorem 3.10. *Let* M *be a finite class of finite* $\veebar$*-disjunctive matrices,* C *the logic of* M *and* $\mathsf{K}^{[*]} \triangleq \mathbf{S}^{[*]}_{*}(\mathsf{M})$*. Then, the following hold:*

(i) *the mappings* $C' \mapsto (\mathrm{Mod}(C') \cap \mathsf{K}^{[*]})$ *and* $\mathsf{S} \mapsto \mathrm{Cn}^{\omega}_{\mathsf{S}}$ *are inverse to one another dual isomorphisms between the poset of all* $\veebar$*-disjunctive [non-pseudo-axiomatic] extensions of* C *and that of all relative universal Horn model subclasses of* $\mathsf{K}^{[*]}$*, the latter poset forming a finite distributive lattice, and so doing the former one;*

(ii) *for any finitary* Σ*-calculus* $\mathcal{C}$*, the following hold:*
   a) *the extension of* C *relatively axiomatized by* $\mathcal{C}$*, being* $\veebar$*-disjunctive [and non-pseudo-axiomatic], corresponds to the relative universal Horn model subclass of* $\mathsf{K}^{[*]}$ *relatively axiomatized by* $\mathcal{C}$*;*
   b) *[providing* $(\mathcal{C} \cap \mathrm{Fm}^{\omega}_{\Sigma}) \neq \varnothing$*] the relative universal Horn model subclass of* $\mathsf{K}^{[*]}$ *relatively axiomatized by* $\mathcal{C}$ *corresponds to the* $\veebar$*-disjunctive [non-pseudo-axiomatic] extension of* C *relatively axiomatized by* $(\mathcal{C} \cap \mathrm{Fm}^{\omega}_{\Sigma}) \cup (\sigma_{+1}[\mathcal{C} \setminus \mathrm{Fm}^{\omega}_{\Sigma}] \veebar x_0)$*;*

(iii) *[providing every member of* M *is truth-non-empty] relative universal positive Horn model subclasses of* $\mathsf{K}^{[*]}$ *correspond exactly to [non-pseudo-axiomatic] axiomatic extensions of* C*, corresponding objects having same axiomatic relative axiomatizations and forming dual finite distributive lattices;*

(iv) *for any* $\mathsf{C} \subseteq \mathsf{K}^{[*]}$*,* $\mathbf{S}^{[*]}_{*}(\mathsf{C})$*, being a relative universal Horn model subclass of* $\mathsf{K}^{[*]}$*, corresponds to the logic of* C*.*

In particular, $\veebar$*-disjunctive extensions of* C *are inductive.*

Proof. (i) First, the fact that $(\mathrm{Mod}(\mathrm{Cn}^{\omega}_{\mathsf{S}}) \cap \mathsf{K}^{[*]}) = \mathsf{S}$, where S is a relative universal Horn model subclass of $\mathsf{K}^{[*]}$, is immediate, while the fact that $\mathrm{Cn}^{\omega}_{\mathsf{S}}$ is a $\veebar$-disjunctive [and non-pseudo-axiomatic] extension of C is by (2.2), Corollary 3.4 and Remark[s] 3.3 [and 2.5]. Now, consider any $\veebar$-disjunctive [non-pseudo-axiomatic] extension C' of C. Then, we have the inductive $\veebar$-disjunctive [non-pseudo-axiomatic] extension C'' of C (for C is inductive) defined as follows: for every $Z \subseteq \mathrm{Fm}^{\omega}_{\Sigma}$, put $C''(Z) \triangleq (\bigcup C'[\wp_{\omega}(Z)])$. Consider any finitary Σ-rule $\Gamma \vdash \varphi$ such that $\varphi \notin C''(\Gamma)$ [and $\Gamma \neq \varnothing$]. Then, by Corollary 3.5(i)⇒(ii), there is some $\veebar$-disjunctive $X \in (\mathrm{img}\, C'') \subseteq (\mathrm{img}\, C)$ such that $\Gamma \subseteq X \not\ni \varphi$. Moreover, as Γ is finite, there is some $\alpha \in (\omega \setminus 1) \subseteq \wp_{\infty\setminus 1}(\omega)$ such that $(\Gamma \cup \{\varphi\}) \subseteq \mathrm{Fm}^{\alpha}_{\Sigma}$, in which case, in view of (2.1), $\Gamma \subseteq Y \triangleq (X \cap \mathrm{Fm}^{\alpha}_{\Sigma}) \in (\mathrm{img}\, \mathrm{Cn}^{\alpha}_{\mathsf{M}})$ is both $\veebar$-disjunctive [and non-empty] as well as proper, for $\varphi \in (\mathrm{Fm}^{\alpha}_{\Sigma} \setminus Y)$. Furthermore, by the structurality of C'', $\langle \mathfrak{Fm}^{\omega}_{\Sigma}, X\rangle$ is a model of C'', and so is its consistent [truth-non-empty] submatrix $\mathcal{D} \triangleq \langle \mathfrak{Fm}^{\alpha}_{\Sigma}, Y\rangle$, in view of (2.2). On the other hand, by Corollary 3.4, $\mathrm{Cn}^{\alpha}_{\mathsf{M}}$ is $\veebar$-disjunctive. Hence, by Lemma 3.2, Y is finitely-meet-irreducible in $\mathrm{img}\, \mathrm{Cn}^{\alpha}_{\mathsf{M}}$. And what is more, since both α, M and all members of M are finite, $\mathcal{B} \triangleq \{h^{-1}[D^{\mathcal{A}}] \mid \mathcal{A} \in \mathsf{M}, h \in \hom(\mathfrak{Fm}^{\alpha}_{\Sigma}, \mathfrak{A})\}$ is a finite basis of $\mathrm{img}\, \mathrm{Cn}^{\alpha}_{\mathsf{M}}$. Therefore, $Y \in \mathcal{B}$, in which case there are some $\mathcal{A} \in \mathsf{M}$ and some $h \in \hom(\mathfrak{Fm}^{\alpha}_{\Sigma}, \mathfrak{A})$ such that $Y = h^{-1}[D^{\mathcal{A}}]$, and so h is a surjective strict homomorphism from $\mathcal{D}$ onto $\mathcal{B} \triangleq (\mathcal{A} \restriction (\mathrm{img}\, h))$. In this way, by (2.2), $\mathcal{B}$ is a consistent [truth-non-empty] model of C''. Finally, as $\Gamma \subseteq Y = h^{-1}[D^{\mathcal{B}}] \not\ni \varphi$, we conclude that $\Gamma \vdash \varphi$ is not true in

$\mathcal{B} \in \mathsf{S} \triangleq (\mathrm{Mod}(C'') \cap \mathsf{K}^{[*]})$ under h. Thus, since both S and all members of it are finite, in which case $C''' \triangleq \mathrm{Cn}^{\omega}_{\mathsf{S}}$ is inductive [and non-pseudo-axiomatic, by Remark 2.5], and so $C'' = C'''$, by Proposition 2.7, we eventually get $C' = C''' = C''$, as required, for, in that case, C', being inductive, is axiomatized by a finitary Σ-calculus. In this way, Lemma 3.9 completes the argument.

(ii) Consider any finitary Σ-calculus $\mathcal{C}$. Then:
 a) is immediate, in view of (2.2), due to which $\mathsf{K} \subseteq \mathrm{Mod}(C)$.
 b) Let C' be the extension of C relatively axiomatized by $(\mathcal{C} \cap \mathrm{Fm}^{\omega}_{\Sigma}) \cup (\sigma_{+1}[\mathcal{C} \setminus \mathrm{Fm}^{\omega}_{\Sigma}] \veebar x_0)$. Then, by Lemma 3.8 with $\mathcal{A} = (\mathcal{C} \cap \mathrm{Fm}^{\omega}_{\Sigma})$, C' is $\veebar$-disjunctive. [And what is more, since $\mathcal{A} \neq \varnothing$, C' is not theorem-less, and so is non-pseudo-axiomatic.] Then, **a)** and Lemma 3.7 complete the argument.

(iii) is by (i), (ii), Lemma 3.9, Corollary 3.6 and Remark[s] 3.3 [and 2.5, due to which C, being the axiomatic extension of C relatively axiomatized by the axiomatic Σ-calculus $\varnothing$, is non-pseudo-axiomatic].

(iv) is by (2.2). □

As it is demonstrated by Theorem 4.24 below, $(\mathcal{C} \cap \mathrm{Fm}^{\omega}_{\Sigma}) \cup (\sigma_{+1}[\mathcal{C} \setminus \mathrm{Fm}^{\omega}_{\Sigma}] \veebar x_0)$ cannot be replaced by $\mathcal{C}$ in the item (ii)**b)** of Theorem 3.10, and so the reservations "positive" and "axiomatic" cannot be omitted in its item (iii).

Let $\rhd$ be any (possibly, secondary) binary connective of Σ. By induction on $l = (\mathrm{dom}\,\bar{\phi}) \in \omega$ for any $\bar{\phi} \in (\mathrm{Fm}^{\omega}_{\Sigma})^*$ and any $\psi \in \mathrm{Fm}^{\omega}_{\Sigma}$, put:

$$(\bar{\phi} \rhd \psi) \triangleq \begin{cases} \psi & \text{if } l = 0, \\ \phi_0 \rhd (((\bar{\phi} \upharpoonright (l \setminus 1)) \circ ((+1) \upharpoonright (l-1))) \rhd \psi) & \text{otherwise.} \end{cases}$$

Remark 3.11. Let M be a finite class of finite $\rhd$-implicative as well as $\veebar$-disjunctive (in particular, $\veebar = \veebar_{\rhd}$) Σ-matrices, in which case $x_0 \rhd x_0$ is true in it, and so every member of $\mathsf{K}_{[*]} \triangleq \mathbf{S}_{[*]}(\mathsf{M})$ is truth-non-empty. Then, any finitary Σ-rule $\Gamma \vdash \psi$ is true in any member of K iff $\bar{\phi} \rhd \psi$ is so, where $\bar{\phi} : |\Gamma| \to \Gamma$ is any bijection, in which case any universal Horn model subclass of K_* is positive, and so $\veebar$-disjunctive extensions of the logic of M are exactly axiomatic ones, in view of Theorem 3.10(iii). □

3.1.1.1. Disjunctive extensions of the logics of single finite disjunctive matrices with equality determinant.

Lemma 3.12. *Let $\mathcal{A}$ be a finite $\veebar$-disjunctive Σ-matrix with equality determinant Υ, $\mathsf{S} \subseteq \mathbf{S}(\mathcal{A})$ and $\mathcal{B} \in \mathbf{S}_*(\mathcal{A})$. Suppose $\mathcal{B} \notin \mathbf{S}(\mathsf{S})$. Then, there is some finitary Σ-rule satisfied in S but is not satisfied in $\mathcal{B}$.*

Proof. In case $\mathsf{S} = \varnothing$, the axiom $\vdash x_0$ is satisfied in it but is not satisfied in any consistent Σ-matrix (in particular, in $\mathcal{B}$). Now, assume $\mathsf{S} \neq \varnothing$, in which case $n \triangleq |\mathsf{S}| \in (\omega \setminus 1)$, and so there is a bijection $\overline{\mathcal{C}} : n \to \mathsf{S}$. Consider any $i \in n$, in which case $B \nsubseteq C_i$, and so there is some $a_i \in (B \setminus C_i) \neq \varnothing$. Define a $\Delta_i \in \wp_{\omega}(\mathrm{Fm}^{\omega}_{\Sigma})$ and a $\bar{\psi}^i \in (\mathrm{Fm}^{\omega}_{\Sigma})^*$ as follows. Let $m \triangleq |C_i| \in (\omega \setminus 1)$. Take any bijection $\vec{c} : m \to C_i$. By induction on any $j \in (m+1)$, define a $\Gamma_j \in \wp_{\omega}(\mathrm{Fm}^1_{\Sigma})$ and a $\bar{\phi}^j \in (\mathrm{Fm}^1_{\Sigma})^*$ such that, for all $b \in (A \setminus D^{\mathcal{A}})$, it holds that $\mathcal{A} \not\models (\Gamma_j \vdash (\veebar\langle\bar{\phi}^j, x_n\rangle))[x_0/a_i, x_n/b]$, while, for all $k \in j$ and all $a \in A$, it holds that $\mathcal{A} \models (\Gamma_j \vdash (\veebar\langle\bar{\phi}^j, x_n\rangle))[x_0/c_k, x_n/a]$, as follows. First, put $\Gamma_j \triangleq \varnothing$ and $\bar{\phi}^j \triangleq \varnothing$, in case $j = 0$. Next, assume $j > 0$, in which case $(j-1) \in m$, and so $c_{j-1} \neq a_i$. Therefore, there is some $\upsilon \in \Upsilon$ such that $\upsilon^{\mathfrak{A}}(a_i) \in D^{\mathcal{A}}$ iff $\upsilon^{\mathfrak{A}}(c_{j-1}) \notin D^{\mathcal{A}}$. Then, set:

$$\langle \Gamma_j, \bar{\phi}^j \rangle \triangleq \begin{cases} \langle \Gamma_{j-1}, \langle \bar{\phi}^{j-1}, \upsilon \rangle \rangle & \text{if } \upsilon^{\mathfrak{A}}(a_i) \notin D^{\mathcal{A}}, \upsilon \notin (\mathrm{img}\,\bar{\phi}^{j-1}), \\ \langle \Gamma_{j-1}, \bar{\phi}^{j-1} \rangle & \text{if } \upsilon^{\mathfrak{A}}(a_i) \notin D^{\mathcal{A}}, \upsilon \in (\mathrm{img}\,\bar{\phi}^{j-1}), \\ \langle \Gamma_{j-1} \cup \{\upsilon\}, \bar{\phi}^{j-1} \rangle & \text{otherwise.} \end{cases}$$

Finally, put $\Delta_i \triangleq (\Gamma_m[x_0/x_i])$ and $\bar{\psi}^i \triangleq (\bar{\phi}^m[x_0/x_i])$. Let $\Xi \triangleq (\bigcup_{i\in n}\Delta_i)$, $\bar{\xi} \triangleq (*\langle\bar{\psi}^i\rangle_{i\in n})$ and

$$\varphi \triangleq \begin{cases} x_n & \text{if } \bar{\xi} = \varnothing, \\ \veebar\bar{\xi} & \text{otherwise.} \end{cases}$$

In this way, the finitary Σ-rule $\Xi \vdash \varphi$ is true in S but is not true in $\mathcal{B}$ under $[x_i/a_i; x_n/b]_{i\in n}$, where $b \in (B \setminus D^{\mathcal{A}}) \neq \varnothing$, for $\mathcal{B}$ is consistent, as required. □

As an immediate consequence of (2.2) and Lemma 3.12, we get:

Theorem 3.13. *Let M, C and $\mathsf{K}^{[*]}$ be as in Theorem 3.10. Suppose $\mathsf{M} = \{\mathcal{A}\}$, where $\mathcal{A}$ is a Σ-matrix with equality determinant. Then, relative universal Horn model subclasses of $\mathsf{K}^{[*]}$ are exactly lower cones of it, under identification of its members with the carriers of their underlying algebras.*

In this way, Lemma 3.12 collectively with Theorems 3.10 and 3.13 provide an effective procedure of finding the lattice of disjunctive extensions of the logic of a finite disjunctive matrix with equality determinant collectively with their finite relative axiomatizations and finite anti-chain matrix semantics. Concluding this discussion, we should like to highlight that the effective procedure of finding relative axiomatizations of disjunctive extensions to be extracted from the constructive proof of Lemma 3.12 (actually subsuming that of Lemma 3.4 of [13], in view of Remark 3.11, and in this way, collectively with Theorem 3.10, subsuming Theorem 3.5 therein) is definitely and obviously much less computationally complex than the straightforward one of direct search among all finite sets of finitary rules.

4. Main issues

4.1. **Preliminaries.** We use the following standard notations going back to [2]:

$$\mathsf{t} \triangleq \langle 1,1\rangle, \qquad \mathsf{f} \triangleq \langle 0,0\rangle, \qquad \mathsf{b} \triangleq \langle 1,0\rangle, \qquad \mathsf{n} \triangleq \langle 0,1\rangle.$$

Also, put $\mu : 2^2 \to 2^2, \langle a,b\rangle \mapsto \langle b,a\rangle$. Moreover, by $\sqsubseteq$ we denote the partial ordering on 2^2 defined by $(\vec{a} \sqsubseteq \vec{b}) \overset{\text{def}}{\Longleftrightarrow} ((a_0 \leqslant b_0)\&(b_1 \leqslant a_1))$, for all $\vec{a},\vec{b} \in 2^2$. Then, given any $B \subseteq 2^2$, $(\sqsubseteq \cap B^2)$-[anti-]monotonic n-ary operations on B, where $n \in \omega$, are referred to as *[anti-]regular*. (Clearly, μ is anti-regular.)

Throughout the rest of the paper, fix any signature $\Sigma \supseteq \Sigma_{\sim[01]} \triangleq (\{\sim,\wedge,\vee\}[\cup\{\bot,\top\}])$ such that either $\{\bot,\top\} \subseteq \Sigma$ or $(\{\bot,\top\} \cap \Sigma) = \varnothing$ as well as any Σ-matrix $\mathcal{A}$ being an expansion of the $\Sigma_{\sim[01]}$-matrix $\mathcal{DM}_{4[01]} \triangleq \langle \mathfrak{DM}_{4[01]}, \{\mathsf{b},\mathsf{t}\}\rangle$, whose underlying algebra $\mathfrak{DM}_{4[01]}$ is the diamond [bounded] De Morgan non-Boolean lattice with carrier 2^2 and the partial ordering given by the natural one on 2 point-wise, that defines the [bounded version of] Belnap's logic $B_{4[01]}$, in which case the logic C of $\mathcal{A}$ is a four-valued expansion of B_4 (this exhausts all *four-valued* expansions of B_4, $\mathcal{A}$ being uniquely determined by C; cf. Theorems 4.7 and 4.8 of [13]).

Remark 4.1 (cf. Example 2 of [11]). $\{x_0, \sim x_0\}$ is an equality determinant for the $\vee$-disjunctive $\mathcal{A}$. □

Theorem 4.2 (cf. Theorems 4.20 and 4.21 of [13])**.** *C is $\sim$-subclassical iff $\{\mathsf{f},\mathsf{t}\}$ forms a subalgebra of $\mathfrak{A}$, in which case the logic C^{PC} of $\mathcal{A}_{\not{n}\not{b}} \triangleq (\mathcal{A}\restriction\{\mathsf{f},\mathsf{t}\})$ is the only $\sim$-classical extension of C and is an extension of any inferentially consistent extension of C.*

Proposition 4.3 (cf. Lemma 4.11 of [13])**.** *C is theorem-less iff $\{\mathsf{n}\}$ forms a subalgebra of $\mathfrak{A}$.*

The extension of C relatively axiomatized by the *Excluded Middle law* axiom

$$x_1 \vee \sim x_1 \tag{4.1}$$

is denoted by C^{EM}.

By $\mathcal{A}_{-\mathsf{n}}$ we denote the submatrix of $\mathcal{A}$ generated by $\{\mathsf{f}, \mathsf{b}, \mathsf{t}\}$, the logic of it being denoted by $C^{-\mathsf{n}}$. (Clearly, providing $\{\mathsf{f}, \mathsf{b}, \mathsf{t}\}$ forms a subalgebra of $\mathfrak{A}$, we have $\mathcal{A}_{-\mathsf{n}} = \mathcal{A}_{\not\mathsf{n}} \triangleq (\mathcal{A}\upharpoonright\{\mathsf{f}, \mathsf{b}, \mathsf{t}\})$.)

Theorem 4.4 (cf. Theorem 4.29 of [13])**.** *$\mathcal{A}_{-\mathsf{n}}$ is a model of any $\sim$-paraconsistent extension of C. In particular, $C^{-\mathsf{n}}$ is the greatest $\sim$-paraconsistent extension of C, and so maximally $\sim$-paraconsistent, in which case an extension of C is $\sim$-paraconsistent iff it is a sublogic of $C^{-\mathsf{n}}$.*

Theorem 4.5 (cf. Theorem 4.31 of [13])**.** *The following are equivalent:*
- (i) *C is maximally $\sim$-paraconsistent;*
- (ii) *$C^{\mathrm{EM}} \neq C^{-\mathsf{n}}$;*
- (iii) *$\{\mathsf{f}, \mathsf{b}, \mathsf{t}\}$ does not form a subalgebra of $\mathfrak{A}$;*
- (iv) *C^{EM} is not $\sim$-paraconsistent;*
- (v) *C^{EM} is not maximally $\sim$-paraconsistent.*

Next, a subalgebra $\mathfrak{B}$ of $\mathfrak{A}$ is said to be *regular*, whenever its operations are so. (Clearly, every subalgebra of $\mathfrak{DM}_{4[01]}$ is regular.) Likewise, $\mathfrak{B}$ is said to be *b-idempotent*, where $b \in B$, whenever its operations are so. (Clearly, $\mathfrak{B}$ is b-idempotent iff $\{b\}$ forms a subalgebra of it.) Finally, $\mathfrak{B}$ is said to be *specular*, whenever $(\mu\upharpoonright B) \in \hom(\mathfrak{B}, \mathfrak{A})$. (Clearly, every subalgebra of $\mathfrak{DM}_{4[01]}$ is specular.)

4.2. **The resolutional extension.** An extension C' of C is said to be *(maximally) [inferentially] paracomplete*, provided $(x_0 \vee \sim x_0) \notin C'(\varnothing[\cup\{x_1\}])$ (and C' has no proper [inferentially] paracomplete extension). Then, a model of C is said to be *[inferentially] paracomplete*, whenever the logic of it is so. Clearly, a paracomplete extension/model of C is inferentially so iff it is non-pseudo-axiomatic/truth-non-empty. Moreover, a submatrix $\mathcal{B}$ of $\mathcal{A}$ is paracomplete iff $\mathsf{n} \in B$.

By $C^{[\mathrm{EM}+]\mathrm{R}}$ we denote the *resolutional* extension of $C^{[\mathrm{EM}]}$, viz., the one relatively axiomatized by the *Resolution* rule:

$$\{x_1 \vee x_0, \sim x_1 \vee x_0\} \vdash x_0. \tag{4.2}$$

Put $\mathsf{S}_{[*]\not\mathsf{b}} \triangleq \{\mathcal{B} \in \mathbf{S}_{[*]}(\mathcal{A}) \mid \mathsf{b} \notin B\}$.

Lemma 4.6. *Let $\wr$ and $\veebar$ be (possibly, secondary) unary and binary connectives of Σ, C' a $\veebar$-disjunctive Σ-logic and C'' an extension of C'. Then,*

$$\{x_1 \veebar x_0, \wr x_1 \veebar x_0\} \vdash (x_2 \veebar x_0) \tag{4.3}$$

is satisfied in C'' iff

$$\{x_1 \veebar x_0, \wr x_1 \veebar x_0\} \vdash x_0 \tag{4.4}$$

is so.

Proof. In that case, (3.4) and (3.5), being valid for C', remain so for C''. First, assume (4.3) is satisfied in C'', in which case (4.3)$[x_2/x_0]$ is so, in view of the structurality of C'', and so is (4.4), in view of (3.5) and the transitivity of C''. Conversely, the fact that (4.4) and (3.4) are satisfied in C'' implies the fact that (4.3) is so, in view of the transitivity of C'', as required. □

Theorem 4.7. *$C^{\mathrm{EM}+\mathrm{R}}$ is equal to C^{PC}, if C is $\sim$-subclassical, and inconsistent, otherwise.*

Proof. With using Remark 4.1, Theorems 3.10, 4.2 and Lemma 4.6. Then, $C^{\mathrm{EM+R}}$ is defined by the set S of all non-paracomplete members of $\mathsf{S}_{*,\not{\mathsf{b}}}$. In that case, $\mathsf{S} = \{\mathcal{A}_{\not{\mathsf{n}}\not{\mathsf{b}}}\}$, if $\{\mathsf{f},\mathsf{t}\}$ forms a subalgebra of $\mathfrak{A}$, and $\mathsf{S} = \varnothing$, otherwise, as required. □

By (2.2), Remarks 3.3, 4.1, Lemmas 3.7, 4.6 and Theorem 3.10, we also have:

Lemma 4.8. *C^{R} is defined by $\mathsf{S}_{[*]\not{\mathsf{b}}}$.*

In addition, we also get:

Corollary 4.9. *Suppose $\{\mathsf{f},\mathsf{n},\mathsf{t}\}$ forms a subalgebra of $\mathfrak{A}$. Then, C^{R} is defined by $\mathcal{A}_{\not{\mathsf{b}}} \triangleq (\mathcal{A}\restriction\{\mathsf{f},\mathsf{n},\mathsf{t}\})$.*

Proof. Then, $\mathsf{S}_{\not{\mathsf{b}}} = \mathbf{S}(\mathcal{A}_{\not{\mathsf{b}}})$, and so (2.2) and Lemma 4.8 complete the argument. □

Theorem 4.10. *The following are equivalent:*

(i) *C^{R} is paracomplete;*
(ii) *there is some subalgebra $\mathfrak{B}$ of $\mathfrak{A}$ such that $\mathsf{b} \notin B \ni \mathsf{n}$;*
(iii) *the carrier of the subalgebra of $\mathfrak{A}$ generated by $\{\mathsf{n}\}$ does not contain b;*
(iv) *there is no $\varphi \in \mathrm{Fm}^1_\Sigma$ such that $\varphi^{\mathfrak{A}}(\mathsf{n}) = \mathsf{b}$.*

Proof. In view of Lemma 4.8, C^{R} is paracomplete iff $\mathsf{S}_{\not{\mathsf{b}}}$ contains a paracomplete matrix. Thus, (i)⇔(ii) holds. Finally, (ii)⇔(iii)⇔(iv) are immediate. □

Lemma 4.11. *Let $a \in \{\mathsf{b},\mathsf{n}\}$. Suppose $\{\mathsf{f},\mathsf{t}\}[\cup\{a\}]$ forms a [regular] subalgebra of $\mathfrak{A}$. Then, $K^a_4 \triangleq \{\langle\mathsf{f},\mathsf{f}\rangle, \langle a,\mathsf{f}\rangle, \langle a,\mathsf{t}\rangle, \langle\mathsf{t},\mathsf{t}\rangle\}$ forms a subalgebra of $(\mathfrak{A}\restriction\{\mathsf{f},a,\mathsf{t}\}) \times (\mathfrak{A}\restriction\{\mathsf{f},\mathsf{t}\})$.*

Proof. Let $\mathfrak{B}$ be the subalgebra of $(\mathfrak{A}\restriction\{\mathsf{f},a,\mathsf{t}\}) \times (\mathfrak{A}\restriction\{\mathsf{f},\mathsf{t}\})$ generated by K^a_4. If $\langle\mathsf{t},\mathsf{f}\rangle$ was in B, there would be some $\varphi \in \mathrm{Fm}^4_\Sigma$ such that both $\varphi^{\mathfrak{A}}(\mathsf{f},a,a,\mathsf{t}) = \mathsf{t}$ and $\varphi^{\mathfrak{A}}(\mathsf{f},\mathsf{f},\mathsf{t},\mathsf{t}) = \mathsf{f}$, in which case, since $(\mathsf{n}/\mathsf{b}) \sqsubseteq / \sqsupseteq b$, for every $b \in \{\mathsf{f},\mathsf{t}\}$, by the regularity of $\mathfrak{A}\restriction\{\mathsf{f},a,\mathsf{t}\}$, we would get $\mathsf{t} \sqsubseteq / \sqsupseteq \mathsf{f}$. Therefore, as $\sim^{\mathfrak{A}}(\mathsf{f}/\mathsf{t}) = (\mathsf{t}/\mathsf{f})$, we conclude that $B = K^a_4$, as required. □

Lemma 4.12. *Let $B \subseteq \{\mathsf{b},\mathsf{n}\}$. Suppose $\{\mathsf{f},\mathsf{t}\} \cup B$ forms a specular subalgebra of $\mathfrak{A}$. Then, $\{\mathsf{f},\mathsf{t}\}$ forms a subalgebra of $\mathfrak{A}$.*

Proof. By contradiction. For suppose $\{\mathsf{f},\mathsf{t}\}$ does not form a subalgebra of $\mathfrak{A}$. In that case, there are some $\varsigma \in \Sigma$ of some arity $n \in \omega$ and some $\bar{a} \in \{\mathsf{f},\mathsf{t}\}^n$ such that $\varsigma^{\mathfrak{A}}(\bar{a}) \in B$. Then, $(\mu \circ \bar{a}) = \bar{a}$, while $\mu(\varsigma^{\mathfrak{A}}(\bar{a})) \neq \varsigma^{\mathfrak{A}}(\bar{a})$, in which case $\mu \notin \hom(\mathfrak{A}\restriction(\{\mathsf{f},\mathsf{t}\} \cup B), \mathfrak{A})$, and so this contradiction completes the argument. □

Theorem 4.13. *Suppose $\{\mathsf{f},\mathsf{n},\mathsf{t}\}$ forms a regular specular subalgebra of $\mathfrak{A}$, in which case $\{\mathsf{f},\mathsf{t}\}$ forms a subalgebra of $\mathfrak{A}_{\not{\mathsf{b}}}$, in view of Lemma 4.12 with $B = \{\mathsf{n}\}$ (in particular, $\Sigma = \Sigma_{\sim[01]}$). Then, an extension of C is inferentially paracomplete iff it is a sublogic of C^{R}. In particular, C^{R} is maximally inferentially paracomplete.*

Proof. In that case, by Corollary 4.9, C^{R} is defined by the truth-non-empty paracomplete (and so inferentially paracomplete) Σ-matrix $\mathcal{A}_{\not{\mathsf{b}}}$, in which case, in particular, any extension of C, being a sublogic of C^{R}, is inferentially paracomplete. Conversely, consider any inferentially paracomplete extension C' of C, in which case $(x_0 \vee \sim x_0) \notin T \triangleq C'(x_1)$, while, by the structurality of C', $\langle \mathfrak{Fm}^\omega_\Sigma, T\rangle$ is a model of C' (in particular, of C), and so is its finitely-generated inferentially paracomplete submatrix $\mathcal{B} \triangleq \langle \mathfrak{Fm}^2_\Sigma, T \cap \mathrm{Fm}^2_\Sigma\rangle$, in view of (2.2). Hence, by Lemma 2.3, there are some set I, some I-tuple $\overline{\mathcal{C}}$ constituted by submatrices of $\mathcal{A}$, some subdirect product $\mathcal{D}$ of $\overline{\mathcal{C}}$, in which case $(\mathfrak{D}\restriction\Sigma_\sim)$ is a De Morgan lattice, some $\theta \in \mathrm{Con}(\mathcal{B})$ and some $g \in \hom^{\mathrm{S}}_{\mathrm{S}}(\mathcal{D}, \mathcal{B}/\theta)$, in which case, by (2.2), $\mathcal{D}$ is an inferentially paracomplete model of C', and so there are some $a \in D^{\mathcal{D}} \subseteq \{\mathsf{b},\mathsf{t}\}^I$ and

$b = (\sim^{\mathfrak{D}} b \vee^{\mathfrak{D}} b) \in (D \setminus D^{\mathcal{D}})$ in which case $\{\mathsf{n}, \mathsf{b}, \mathsf{t}\}^I \ni b \neq c \triangleq (a \vee^{\mathfrak{D}} b) \in D^{\mathcal{D}}$, in view of (3.3) and Remark 4.1. Put $J \triangleq \{i \in I \mid \pi_i(b) = \mathsf{t}\}$, $K \triangleq \{i \in I \mid \pi_i(b) = \mathsf{n}\} \neq \varnothing$, for $b \notin D^{\mathcal{D}}$, and $L \triangleq \{i \in I \mid \pi_i(b) = \mathsf{b} \neq \pi_i(c)\}$. Given any $\bar{a} \in A^4$, put $(a_0|a_1|a_2|a_3) \triangleq ((J \times \{a_0\}) \cup (K \times \{a_1\}) \cup (L \times \{a_2\}) \cup ((I \setminus (J \cup K \cup L)) \times \{a_3\})) \in A^I$. Then, we have:

$$D \ni b = (\mathsf{t}|\mathsf{n}|\mathsf{b}|\mathsf{b}), \tag{4.5}$$

$$D \ni \sim^{\mathfrak{D}} b = (\mathsf{f}|\mathsf{n}|\mathsf{b}|\mathsf{b}), \tag{4.6}$$

$$D \ni c = (\mathsf{t}|\mathsf{t}|\mathsf{t}|\mathsf{b}), \tag{4.7}$$

$$D \ni \sim^{\mathfrak{D}} c = (\mathsf{f}|\mathsf{f}|\mathsf{f}|\mathsf{b}) \tag{4.8}$$

Consider the following complementary cases:

(1) $\mathfrak{A}$ is b-idempotent.
Then, we have the following complementary subcases:
  (a) $J = \varnothing$.
  Then, since $K \neq \varnothing = J$, $\mathfrak{A}_{\not\mathsf{b}}$ is specular and $\{\mathsf{b}\}$ forms a subalgebra of $\mathfrak{A}$, by (4.5), (4.7) and (4.8), we see that $\{\langle x, (x|x|\mu(x)|\mathsf{b})\rangle \mid x \in A_{\not\mathsf{b}}\}$ is an embedding of $\mathcal{A}_{\not\mathsf{b}}$ into $\mathcal{D}$. Hence, by (2.2), $\mathcal{A}_{\not\mathsf{b}}$ is a model of C', for $\mathcal{D}$ is so.
  (b) $J \neq \varnothing$.
  Then, taking Lemma 4.11 into account, since $K \neq \varnothing \neq J$, $\mathfrak{A}_{\not\mathsf{b}}$ is specular and $\{\mathsf{b}\}$ forms a subalgebra of $\mathfrak{A}$, by (4.5), (4.6), (4.7) and (4.8), we see that $\{\langle\langle x, y\rangle, (y|x|\mu(x)|\mathsf{b})\rangle \mid \langle x, y\rangle \in K_4^{\mathsf{n}}\}$ is an embedding of $\mathcal{B} \triangleq ((\mathcal{A}_{\not\mathsf{b}} \times (\mathcal{A} \upharpoonright \{\mathsf{f}, \mathsf{t}\})) \upharpoonright K_4^{\mathsf{n}})$ into $\mathcal{D}$. Moreover, $(\pi_0 \upharpoonright K_4^{\mathsf{n}}) \in \hom_{\mathrm{S}}^{\mathrm{S}}(\mathcal{B}, \mathcal{A}_{\not\mathsf{b}})$. Hence, by (2.2), $\mathcal{A}_{\not\mathsf{b}}$ is a model of C', for $\mathcal{D}$ is so.

(2) $\mathfrak{A}$ is not b-idempotent.
Then, there is some $\varphi \in \mathrm{Fm}_\Sigma^1$ such that $\varphi^{\mathfrak{A}}(\mathsf{b}) \neq \mathsf{b}$, in which case $\phi^{\mathfrak{A}}[\{\mathsf{b}, \mathsf{t}\}] = \{\mathsf{t}\}$ and $\psi^{\mathfrak{A}}[\{\mathsf{b}, \mathsf{t}\}] = \{\mathsf{f}\}$, where $\phi \triangleq (x_0 \vee (\varphi \vee \sim\varphi))$ and $\psi \triangleq \sim\phi$, and so, by (4.7), we get:

$$D \ni \psi^{\mathfrak{D}}(c) = (\mathsf{f}|\mathsf{f}|\mathsf{f}|\mathsf{f}), \tag{4.9}$$

$$D \ni \phi^{\mathfrak{D}}(c) = (\mathsf{t}|\mathsf{t}|\mathsf{t}|\mathsf{t}). \tag{4.10}$$

Consider the following complementary subcases:
  (a) $J = \varnothing$,
  Then, since $K \neq \varnothing = J$ and $\mathfrak{A}_{\not\mathsf{b}}$ is specular, by (4.5), (4.9) and (4.10), we see that $\{\langle x, (x|x|\mu(x)|\mu(x))\rangle \mid x \in A_{\not\mathsf{b}}\}$ is an embedding of $\mathcal{A}_{\not\mathsf{b}}$ into $\mathcal{D}$. Hence, by (2.2), $\mathcal{A}_{\not\mathsf{b}}$ is a model of C', for $\mathcal{D}$ is so.
  (b) $J \neq \varnothing$.
  Then, taking Lemma 4.11 into account, since $K \neq \varnothing \neq J$ and $\mathfrak{A}_{\not\mathsf{b}}$ is specular, by (4.5), (4.6), (4.9) and (4.10), we see that $\{\langle\langle x, y\rangle, (y|x|\mu(x)|\mu(x))\rangle \mid \langle x, y\rangle \in K_4^{\mathsf{n}}\}$ is an embedding of $\mathcal{B} \triangleq ((\mathcal{A}_{\not\mathsf{b}} \times (\mathcal{A} \upharpoonright \{\mathsf{f}, \mathsf{t}\})) \upharpoonright K_4^{\mathsf{n}})$ into $\mathcal{D}$. Moreover, $(\pi_0 \upharpoonright K_4^{\mathsf{n}}) \in \hom_{\mathrm{S}}^{\mathrm{S}}(\mathcal{B}, \mathcal{A}_{\not\mathsf{b}})$. Hence, by (2.2), $\mathcal{A}_{\not\mathsf{b}}$ is a model of C', for $\mathcal{D}$ is so.

Thus, in any case, $\mathcal{A}_{\not\mathsf{b}}$ is a model of C', and so $C' \subseteq C^{\mathrm{R}}$, as required. □

The logic of $\mathcal{DM}_{4[01],\not\mathsf{b}/\not\mathsf{n}}$ is known as [the bounded version of] *Kleene's three-valued logic/Priest's logic of paradox* $K_{3[01]}/LP_{[01]}$ (cf. [3]/both [6] and [8]).

Theorem 4.14. *The following are equivalent:*

(i) $\{\mathsf{f}, \mathsf{n}, \mathsf{t}\}$ *does not form a subalgebra of* $\mathfrak{A}$*;*

(ii) C^{R} *is inferentially either* $\sim$*-classical, if* C *is* $\sim$*-subclassical, or inconsistent, otherwise;*

(iii) C^{R} *is not inferentially paracomplete;*
(iv) *the* $\Sigma_{\sim}$*-fragment of* C^{R} *is not inferentially paracomplete;*
(v) *no* Σ*-expansion of* K_3 *is an extension of* C*;*
(vi) C^{R} *is not an expansion of* K_3.

Proof. First, (vi)⇒(i) is by Corollary 4.9.

Moreover, (vi) is a particular case of (v).

Next, assume (i) holds. We use Remark 2.4, Theorem 4.2, Proposition 4.3 and Lemma 4.8 tacitly. Consider the following four exhaustive cases:

(1) C is both $\sim$-subclassical and not theorem-less.
Then, $\mathsf{S}_{*,\not\mathsf{b}} = \{\mathcal{A}\restriction\{\mathsf{f},\mathsf{t}\}\}$, in which case C^{R} is $\sim$-classical, and so inferentially so.
(2) C is both theorem-less and $\sim$-subclassical.
Then, $\mathsf{S}_{*,\not\mathsf{b}} = \{\mathcal{A}\restriction\{\mathsf{f},\mathsf{t}\}, \mathcal{A}\restriction\{\mathsf{n}\}\}$, in which case C^{R} is inferentially $\sim$-classical.
(3) C is neither $\sim$-subclassical nor theorem-less.
Then, $\mathsf{S}_{*,\not\mathsf{b}} = \varnothing$, in which case C^{R} is inconsistent, and so inferentially so.
(4) C is both theorem-less and not $\sim$-subclassical.
Then, $\mathsf{S}_{*,\not\mathsf{b}} = \{\mathcal{A}\restriction\{\mathsf{n}\}\}$, in which case C^{R} is inferentially inconsistent.

Thus, (ii) holds.

Further, in view of Theorem 4.2, any inferentially $\sim$-classical extension of C is not inferentially paracomplete. And what is more, any inferentially paracomplete extension of C is clearly inferentially consistent. Hence, (ii)⇒(iii) holds.

Furthermore, (iii)⇒(iv) is by the fact that $x_0 \vee \sim x_0$ is a $\Sigma_{\sim}$-formula.

Finally, by Remark 2.5, K_3 is non-pseudo-axiomatic. Moreover, it is paracomplete, and so inferentially so. And what is more, (4.2), being satisfied in K_3, is so in any Σ-expansion of it. In this way, (iv)⇒(v) holds, as required. □

In this connection, it is remarkable that paracomplete analogues of the "maximality" items (i) and (v) of Theorem 4.5 do not hold, generally speaking, as it ensues from the following generic counterexamples:

Example 4.15. Suppose C is $\sim$-subclassical, i.e., $\{\mathsf{f},\mathsf{t}\}$ forms a subalgebra of $\mathfrak{A}$ (cf. Theorem 4.2). Then, $\mathcal{B} \triangleq (\mathcal{A}\times(\mathcal{A}\restriction\{\mathsf{f},\mathsf{t}\}))$ is truth-non-empty, non-$\sim$-paraconsistent and, by (2.3), paracomplete, for $\mathcal{A}$ is so, and so inferentially paracomplete, in which case the logic of $\mathcal{B}$ is a proper inferentially paracomplete extension of C, in view of (2.2). □

Example 4.16. Let $\sqsupset$ be a (possibly, secondary) binary connective of Σ. Suppose both $\{\mathsf{f},\mathsf{t}\}$ and $\{\mathsf{f},\mathsf{n}[/\mathsf{b}],\mathsf{t}\}$ form subalgebras of $\mathfrak{A}$, in which case $\mathcal{A}\restriction\{\mathsf{f},\mathsf{t}\}$ is a submatrix of $\mathcal{A}_{\not\mathsf{b}}$, $\{\mathcal{A}_{\not\mathsf{b}}\}[\cup\{\mathcal{A}_{\not\mathsf{n}}\}]$ defining $C^{\mathrm{R}}[\cap C^{\mathrm{EM}}]$, in view of Corollary 4.9 [and Theorem 4.5(ii)⇒(iii)], while C^{R} satisfies $x_0 \sqsupset x_0$, whereas $\{x_0, x_0 \sqsupset x_1\} \vdash x_1$ is true in $\mathcal{A}\restriction\{\mathsf{f},\mathsf{t}\}$, in which case $\mathcal{B} \triangleq (\mathcal{A}_{\not\mathsf{b}} \times (\mathcal{A}\restriction\{\mathsf{f},\mathsf{t}\}))$ is truth-non-empty, paracomplete, in view of (2.3), for $\mathcal{A}_{\not\mathsf{b}}$ is so, and so inferentially paracomplete, and a model of the rule $\{\sim^i x_0 \sqsupset \sim^{1-i} x_0 \mid i \in 2\} \vdash (x_0 \vee \sim x_0)$, in its turn, [being also true in $\mathcal{A}_{\not\mathsf{n}}$ but] not being true in $\mathcal{A}_{\not\mathsf{b}}$ under $[x_0/\mathsf{n}]$, and so, by (2.2), the logic of $\{\mathcal{B}\}[\cup\{\mathcal{A}_{\not\mathsf{n}}\}]$ is a proper [both $\sim$-paraconsistent and] inferentially paracomplete extension of $C^{\mathrm{R}}[\cap C^{\mathrm{EM}}]$. □

Example 4.16 shows that the preconditions in the formulations of Theorem 4.13 and Corollary 4.26 cannot be omitted. Conversely, as it follows from Theorem 4.13 and Corollary 4.26, the condition of existence of implication $\sqsupset$ satisfying both the Reflexivity axiom in $\mathcal{A}_{\not\mathsf{b}}$ and the Modus Ponens rule in $\mathcal{A}\restriction\{\mathsf{f},\mathsf{t}\}$ is essential within Example 4.16 covering implicative expansions of B_4 (cf. Subsection 5.3 of [13]).

4.3. **Disjunctive extensions.** Next, C is said to be *hereditary*, provided (under identification of submatrices of $\mathcal{A}$ with the underlying algebras of their carriers)

$$\mathbf{S}^*_*(\mathcal{A}) \supseteq \mathsf{S}_{01} \triangleq \mathbf{S}(\mathcal{DM}_{4,01}) = \{\{\mathsf{f},\mathsf{t},\mathsf{b},\mathsf{n}\},\{\mathsf{f},\mathsf{t},\mathsf{n}\},\{\mathsf{f},\mathsf{t},\mathsf{b}\},\{\mathsf{f},\mathsf{t}\}\}$$

(the inverse inclusion always holds), in which case $C^{\mathrm{EM}[+\mathrm{R}]}[= C^{\mathrm{PC}}]$ is defined by $\mathcal{A}_{\not\mathsf{n}[\not\mathsf{b}]}$, in view of Theorem[s] 4.5 [resp., 4.2 and 4.7], while C^{R} is defined by $\mathcal{A}_{\not\mathsf{b}}$, in view of Corollary 4.9. In particular, (the purely-implicative expansion of) $B_{4[01]}$ is hereditary (cf. Subsection 5.3 of [13]). In this connection, note that, in view of Theorem 4.1 of [7], $\vee$-disjunctive extensions of B_4 are exactly *De Morgan logics* in the sense of the reference [Pyn 95a] of [8]. In this way, the present subsection incorporates the material announced therein advancing it much towards (mainly but not exclusively, hereditary) expansions. Set $\mathsf{S} \triangleq \mathbf{S}_*(\mathcal{DM}_4) = (\mathsf{S}_{01} \cup \{\{\mathsf{n}\}\})$.

Remark 4.17. The mappings $\mathsf{C} \mapsto \mathsf{C}^{\triangledown}_{\mathsf{S}_{[01]}}$ and $\mathsf{C} \mapsto (\mathsf{C} \cap \mathbf{S}^{[*]}_*(\mathcal{A}))$ form a dual Galois retraction between the posets of all lower cones of $\mathbf{S}^{[*]}_*(\mathcal{A})$ and those of $\mathsf{S}_{[01]}$, the former/latter mapping preserving generating subsets/relative axiomatizations. □

There are exactly nine [six] lower cones of $\mathsf{S}_{[01]}$ [but those containing $\{\mathsf{n}\}$, viz., including C_1, i.e., the last three ones]:

$$\mathsf{C}_{4[01]} \triangleq \{\{\mathsf{f},\mathsf{t},\mathsf{b},\mathsf{n}\}\}^{\triangledown}_{\mathsf{S}_{[01]}}, \quad \mathsf{C}^{\mathsf{b}}_{3[01]} \triangleq \{\{\mathsf{f},\mathsf{t},\mathsf{b}\}\}^{\triangledown}_{\mathsf{S}_{[01]}}, \quad \mathsf{C}^{\mathsf{n}}_{3[01]} \triangleq \{\{\mathsf{f},\mathsf{t},\mathsf{n}\}\}^{\triangledown}_{\mathsf{S}_{[01]}},$$
$$\mathsf{C}_{3[01]} \triangleq (\mathsf{C}^{\mathsf{b}}_{3[01]} \cup \mathsf{C}^{\mathsf{n}}_{3[01]}), \quad \mathsf{C}_2 \triangleq \{\{\mathsf{f},\mathsf{t}\}\}, \quad \mathsf{C}_0 \triangleq \varnothing,$$
$$\mathsf{C}_1 \triangleq \{\{\mathsf{n}\}\}, \quad \mathsf{C}^{\mathsf{b}}_{3\Cup 1} \triangleq (\mathsf{C}^{\mathsf{b}}_3 \cup \mathsf{C}_1), \quad \mathsf{C}_{2\Cup 1} \triangleq (\mathsf{C}_2 \cup \mathsf{C}_1).$$

Those eight [five] ones, which are proper (viz., distinct from $\mathsf{S}_{[01]} = \mathsf{C}_{4[01]}$) are relatively axiomatized by the following $\Sigma_\sim$-calculi (actually arisen according to the constructive proof of Lemma 3.12, and so demonstrating its practical applicability), respectively:

$$(4.11)\qquad (4.1),$$
$$(4.12)\qquad \{x_0, \sim x_0\} \vdash x_1,$$
$$(4.13)\qquad \{x_0, \sim x_0\} \vdash (x_1 \vee \sim x_1),$$
$$(4.14)\qquad \{(4.1), (4.12)\},$$
$$(4.15)\qquad x_0,$$
$$(4.16)\qquad x_0 \vdash x_1,$$
$$(4.17)\qquad x_0 \vdash (x_1 \vee \sim x_1),$$
$$(4.18)\qquad \{(4.12), (4.17)\}.$$

And what is more, $\sigma_{+1}(4.16) \vee x_0$ is equivalent to (4.16) under (3.3) and (3.5). Likewise, $\sigma_{+1}(4.17) \vee x_0$ is equivalent to (4.17) under (3.3), (3.4) and (3.5). By IC we denote the inconsistent Σ-logic. Moreover, put $PC_{[01]} \triangleq B^{\mathrm{PC}}_{4[01]}$. In this way, taking Remarks 2.2, 2.4, 2.5, 4.17, Proposition 4.3, Theorems 3.10, 3.13 and Lemma 4.6 into account, we eventually get:

Theorem 4.18. *Suppose C is (not) hereditary and has a/no theorem. Then, $\vee$-disjunctive [non-pseudo-axiomatic] extensions of C form (a Galois retract of) the six/nine[six]-element non-chain distributive lattice depicted at Figure 1 (with not necessarily distinct nodes) with solely solid circles/[with solely solid circles]. Moreover, those of them, whose relative axiomatizations are not given by upper indices, are axiomatized relatively to C by the following calculi:*

$$(4.19)\qquad C^{\mathrm{EM}} \cap C^{\mathrm{R}} \;:\; \{x_1 \vee x_0, \sim x_1 \vee x_0\} \vdash ((x_2 \vee \sim x_2) \vee x_0),$$
$$(4.20)\qquad IC \;:\; (4.15),$$

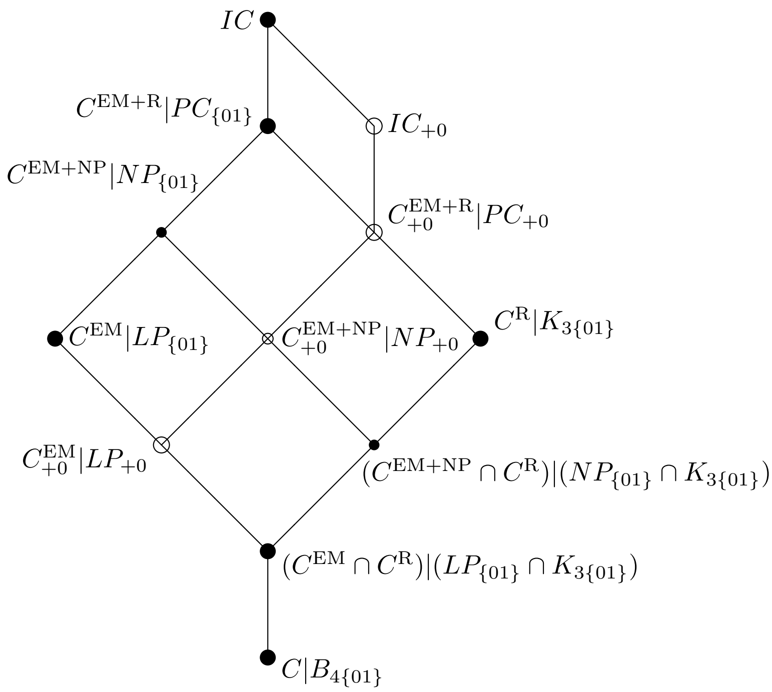

FIGURE 1. The lattice of $\vee$-disjunctive/Kleene extensions of hereditary/strongly hereditary $C|B_{4\{01\}}$ with solely large/non-lowest circles.

(4.21) IC_{+0} : (4.16),

(4.22) C^{EM}_{+0} : (4.17),

(4.23) $C^{\mathrm{EM+R}}_{+0}$: $\{(4.17), (4.2)\}$.

In view of Remark 3.11, Theorem 4.18 subsumes Corollary 5.4 of [13]. And what is more, in view of Theorems 3.10, 3.13 and Remark 4.17, Theorem 4.18, being immediately applicable to hereditary four-valued expansions of B_4, is equally well-applicable to non-hereditary ones, in which case the lattice depicted at Figure 1 is properly degenerated under the corresponding dual Galois retraction. For instance, when dealing with any classically-negative (viz., Boolean) expansion CB_4 (cf. Subsection 5.1 of [13]), $\mathbf{S}_*(\mathcal{A})$ becomes equal to $\{A\}[\cup\{\{\mathsf{f},\mathsf{t}\}\}]$, in which case $\vee$-disjunctive (viz., axiomatic; cf. Remark 3.11) extensions of CB_4 form the two[three]-element chain $CB_4 \subsetneq CB_4^{\mathrm{EM(+R)}} = CB_4^{\mathrm{R}} = [CB_4^{\mathrm{PC}} \subsetneq]IC$. Likewise, given any bilattice expansion BL_4 (cf. Subsection 5.2 of [13]), $\mathbf{S}_*(\mathcal{A})$ becomes equal to $\{A\}[\cup\{\{\mathsf{n}\}\}]$, in which case $\vee$-disjunctive extensions of BL_4 form the two-[three-]element chain $BL_4[\subsetneq IC_{+0}] \subsetneq IC = BL_4^{\mathrm{EM}}$ with $IC_{[+0]} = BL_4^{\mathrm{R}}$, exhausting *all* extensions of BL_4, in view of its inferential maximality proved in Corollary 5.2 of [13].

It is remarkable that, in view of Theorem 5.2 of [7] providing an axiomatization of B_4 given by Definition 5.1 therein, Theorem 4.18 yields axiomatizations of all $\vee$-disjunctive extensions of B_4 (in particular, of K_3 relatively axiomatized by the Resolution rule (4.2)).

4.4. **Non-paracomplete extensions.** By $C^{[\mathrm{EM+}]\mathrm{NP}}$ we denote the least non-$\sim$-paraconsistent extension of $C^{[\mathrm{EM}]}$, viz., that which is relatively axiomatized by the *Ex Contradictione Quodlibet* rule (4.12). Likewise, by $C^{[\mathrm{EM+}]\mathrm{MP}}$ we denote the extension of $C^{[\mathrm{EM}]}$ relatively axiomatized by the rule:

$$\{x_0, \sim x_0 \vee x_1\} \vdash x_1, \tag{4.24}$$

being nothing but *Modus Ponens* for the *material* implication $\sim x_0 \vee x_1$. (Clearly, it is a/an sublogic/extension of $C^{[\mathrm{EM}+](\mathrm{R}/\mathrm{NP})}$, in view of (3.3) held in C by its $\vee$-disjunctivity; cf. Remark 4.1 and Corollary 3.4.) An extension of C is said to be *Kleene*, whenever it satisfies the rule (4.19).

Remark 4.19. Let C' be a Kleene extension of C (in particular, a non-paracomplete one, in view of (3.3)). Then, we have $\{x_0 \vee x_1, \sim x_0 \vee x_1\} \vdash_{C'} (\sim(x_0 \vee x_1) \vee x_1)$. Therefore, in view of (3.3), C' satisfies (4.2) iff it satisfies (4.24). In particular, $C^{\mathrm{EM+R}} = C^{\mathrm{EM+MP}}$. □

Theorem 4.20 (cf. Theorem 4.35 of [13]). *Suppose C is [not] maximally $\sim$-paraconsistent. Then, $C^{\mathrm{EM+NP}}$ is consistent iff C is $\sim$-subclassical, in which case $C^{\mathrm{EM+NP}}$ is defined by $[\mathcal{A}_{\not{\mathsf{n}}}\times]\mathcal{A}_{\not{\mathsf{n}}\not{\mathsf{b}}}$.*

Lemma 4.21 (cf. Lemma 4.1 of [13]). *Let I be a set, $\overline{\mathcal{C}} \in \mathbf{S}(\mathcal{A})^I$, $\mathcal{B}$ a submatrix of $\prod_{i\in I} \mathcal{C}_i$. Suppose $\{\mathsf{f},\mathsf{b},\mathsf{t}\}$ forms a regular subalgebra of $\mathfrak{A}$, $\{I\times\{d\} \mid d \in \{\mathsf{f},\mathsf{t}\}\} \subseteq B$ and, for each $i \in I$, $C_i \subseteq \{\mathsf{f},\mathsf{b},\mathsf{t}\}$. Then, $(B \dot{+} 2) \triangleq ((B \times \{\mathsf{b}\}) \cup \{\langle I \times \{d\}, d\rangle \mid d \in \{\mathsf{f},\mathsf{t}\}\})$ forms a subalgebra of $\mathfrak{B} \times (\mathfrak{A}{\restriction}\{\mathsf{f},\mathsf{b},\mathsf{t}\})$, in which case $\pi_0{\restriction}(B \dot{+} 2)$ is a surjective strict homomorphism from $(\mathcal{B} \dot{+} 2) \triangleq ((\mathcal{B} \times (\mathcal{A}{\restriction}\{\mathsf{f},\mathsf{b},\mathsf{t}\})){\restriction}(B \dot{+} 2))$ onto $\mathcal{B}$.*

Lemma 4.22. *Let C' be a Kleene (in particular, non-paracomplete, in view of* (3.3)*) extension of C. Suppose C is not maximally $\sim$-paraconsistent,* (4.24) *is not satisfied in C' and, for every $\varsigma \in \Sigma$, $\varsigma^{\mathfrak{A}_{-\mathsf{n}}}$ is either regular or both* b*-idempotent and no more than binary. Then, C' is a sublogic of $C^{\mathrm{EM+NP}}$.*

Proof. The case, when $C^{\mathrm{EM+NP}}$ is inconsistent, is evident. Otherwise, by Theorems 4.2, 4.5 and 4.20, $A_{-\mathsf{n}} = \{\mathsf{f},\mathsf{b},\mathsf{t}\}$ and $\{\mathsf{f},\mathsf{t}\}$ form subalgebras of $\mathfrak{A}$, $C^{\mathrm{EM+NP}}$ being defined by the submatrix $\mathcal{B} \triangleq (\mathcal{A}_{-\mathsf{n}} \times (\mathcal{A}{\restriction}\{\mathsf{f},\mathsf{t}\}))$ of $\mathcal{A}^2$, and so it suffices to prove that $\mathcal{B} \in \mathrm{Mod}(C')$. Then, by Theorem 2.6, there are some set I, some $\overline{\mathcal{C}} \in \mathbf{S}(\mathcal{A})^I$ and some subdirect product $\mathcal{D} \in \mathrm{Mod}(C') \subseteq \mathrm{Mod}(C)$ of it not being a model of (4.24), in which case $(\mathfrak{D}{\restriction}\Sigma_{\sim})$ is a De Morgan lattice. Therefore, there are some $a \in D^{\mathcal{D}} \subseteq \{\mathsf{b},\mathsf{t}\}^I$, in which case $\sim^{\mathfrak{D}} a \leqslant a$, and some $b \in (D \setminus D^{\mathcal{A}})$ such that $(\sim^{\mathfrak{D}} a \vee^{\mathfrak{D}} b) \in D^{\mathcal{A}}$, in which case $(\sim^{\mathfrak{D}} a \vee^{\mathfrak{D}} b) \leqslant (a \vee^{\mathfrak{D}} b)$, and so $(a \vee^{\mathfrak{D}} b) \in D^{\mathcal{A}}$. Hence, by (4.19), $(b \vee^{\mathfrak{D}} \sim^{\mathfrak{D}} b) = ((b \vee^{\mathfrak{D}} \sim^{\mathfrak{D}} b) \vee^{\mathfrak{D}} b) \in D^{\mathcal{A}}$, in which case $b \in \{\mathsf{f},\mathsf{b},\mathsf{t}\}^I$. Put $J \triangleq \{i \in I \mid \pi_i(a) = \mathsf{b}\} \supseteq K \triangleq \{i \in I \mid \pi_i(b) = \mathsf{f}\} \neq \varnothing$, for $(\sim^{\mathfrak{D}} a \vee^{\mathfrak{D}} b) \in D^{\mathcal{A}}$ and $b \notin D^{\mathcal{A}}$, and $L \triangleq \{i \in I \mid \pi_i(b) = \mathsf{t}\}$, Then, given any $\vec{a} \in A^5$, set $(a_0|a_1|a_2|a_3|a_4) \triangleq ((((I \setminus (L \cup K)) \cap J) \times \{a_0\}) \cup ((I \setminus (L \cup J)) \times \{a_1\}) \cup ((L \setminus J) \times \{a_2\}) \cup ((L \cap J) \times \{a_3\}) \cup (K \times \{a_4\})) \in A^I$. In this way, $a = (\mathsf{b}|\mathsf{t}|\mathsf{t}|\mathsf{b}|\mathsf{b})$ and $b = (\mathsf{b}|\mathsf{b}|\mathsf{t}|\mathsf{t}|\mathsf{f})$. Therefore, we have:

$$D \ni e \triangleq (a \wedge^{\mathfrak{D}} b) = (\mathsf{b}|\mathsf{b}|\mathsf{t}|\mathsf{b}|\mathsf{f}), \tag{4.25}$$

$$D \ni \sim^{\mathfrak{D}} e = (\mathsf{b}|\mathsf{b}|\mathsf{f}|\mathsf{b}|\mathsf{t}), \tag{4.26}$$

$$D \ni c \triangleq (e \vee^{\mathfrak{D}} \sim^{\mathfrak{D}} b) = (\mathsf{b}|\mathsf{b}|\mathsf{t}|\mathsf{b}|\mathsf{t}), \tag{4.27}$$

$$D \ni \sim^{\mathfrak{D}} c = (\mathsf{b}|\mathsf{b}|\mathsf{f}|\mathsf{b}|\mathsf{f}), \tag{4.28}$$

$$D \ni d \triangleq (e \vee^{\mathfrak{D}} \sim^{\mathfrak{D}} a) = (\mathsf{b}|\mathsf{b}|\mathsf{t}|\mathsf{b}|\mathsf{b}), \tag{4.29}$$

$$D \ni \sim^{\mathfrak{D}} d = (\mathsf{b}|\mathsf{b}|\mathsf{f}|\mathsf{b}|\mathsf{b}). \tag{4.30}$$

Consider the following complementary cases:

(1) $L \subseteq J$.

Then, given any $\vec{a} \in A^4$, set $(a_0|a_1|a_2|a_3) \triangleq ((((I \setminus (L \cup K)) \cap J) \times \{a_0\}) \cup ((I \setminus$

$J) \times \{a_1\}) \cup (L \times \{a_2\}) \cup (K \times \{a_3\})) \in A^I$. In this way, by (4.25), (4.27) and (4.29), we have $e = (\mathsf{b}|\mathsf{b}|\mathsf{b}|\mathsf{f}) \in D$, $c = (\mathsf{b}|\mathsf{b}|\mathsf{b}|\mathsf{t}) \in D$ and $d = (\mathsf{b}|\mathsf{b}|\mathsf{b}|\mathsf{b}) \in D$, respectively. Consider the following complementary subcases:

(a) $\{\mathsf{b}\}$ forms a subalgebra of $\mathfrak{A}_{-\mathsf{n}}$.
Then, as $K \neq \varnothing$, $\{\langle x, (\mathsf{b}|\mathsf{b}|\mathsf{b}|x)\rangle \mid x \in A_{-\mathsf{n}}\}$ is an embedding of $\mathcal{A}_{-\mathsf{n}}$ into $\mathcal{D}$.

(b) $\{\mathsf{b}\}$ does not form a subalgebra of $\mathfrak{A}_{-\mathsf{n}}$.
Then, there is some $\varphi \in \mathrm{Fm}^1_\Sigma$ such that $\varphi^{\mathfrak{A}}(\mathsf{b}) \in \{\mathsf{f}, \mathsf{t}\}$, in which case $\phi^{\mathfrak{A}}(\mathsf{b}) = \mathsf{f}$ and $\psi^{\mathfrak{A}}(\mathsf{b}) = \mathsf{t}$, where $\phi \triangleq (\varphi \wedge {\sim}\varphi)$ and $\psi \triangleq (\varphi \vee {\sim}\varphi)$, and so both $D \ni \phi^{\mathfrak{D}}(d) = (\mathsf{f}|\mathsf{f}|\mathsf{f}|\mathsf{f})$ and $D \ni \psi^{\mathfrak{D}}(d) = (\mathsf{t}|\mathsf{t}|\mathsf{t}|\mathsf{t})$. Hence, as $I \supseteq K \neq \varnothing$, $\{\langle x, (x|x|x|x)\rangle \mid x \in A_{-\mathsf{n}}\}$ is an embedding of $\mathcal{A}_{-\mathsf{n}}$ into $\mathcal{D}$.

Thus, anyway, $\mathcal{A}_{-\mathsf{n}}$ is embeddable into $\mathcal{D}$, in which case, by (2.2), it is a model of C', and so is $\mathcal{B}$, for $\{\mathsf{f}, \mathsf{t}\}$ forms a subalgebra of $\mathfrak{A}_{-\mathsf{n}}$.

(2) $L \nsubseteq J$.
Consider the following complementary subcases:

(a) either $\{\mathsf{b}\}$ forms a subalgebra of $\mathfrak{A}_{-\mathsf{n}}$ or $(((I \setminus (L \cup K)) \cap J) \cup (I \setminus (L \cup J)) \cup (L \cap J)) = \varnothing$.
Then, taking (4.25), (4.26), (4.27), (4.28), (4.29) and (4.30) into account, as $K \neq \varnothing \neq (L \setminus J)$, $\{\langle\langle x, y\rangle, (\mathsf{b}|\mathsf{b}|y|\mathsf{b}|x)\rangle \mid \langle x, y\rangle \in B\}$ is an embedding of $\mathcal{B}$ into $\mathcal{D}$, and so, by (2.2), $\mathcal{B}$ is a model of C'.

(b) $\{\mathsf{b}\}$ does not form a subalgebra of $\mathfrak{A}_{-\mathsf{n}}$ and $(((I \setminus (L \cup K)) \cap J) \cup (I \setminus (L \cup J)) \cup (L \cap J)) \neq \varnothing$.
Then, there is some $\varphi \in \mathrm{Fm}^1_\Sigma$ such that $\varphi^{\mathfrak{A}}(\mathsf{b}) \in \{\mathsf{f}, \mathsf{t}\}$, in which case $\varphi^{\mathfrak{A}}[A_{-\mathsf{n}}] \subseteq \{\mathsf{f}, \mathsf{t}\}$, for $\{\mathsf{f}, \mathsf{t}\}$ forms a subalgebra of $\mathfrak{A}$, and so $\phi^{\mathfrak{A}}[A_{-\mathsf{n}}] = \{\mathsf{f}\}$ and $\psi^{\mathfrak{A}}[A_{-\mathsf{n}}] = \{\mathsf{t}\}$, where $\phi \triangleq (\varphi \wedge {\sim}\varphi)$ and $\psi \triangleq (\varphi \vee {\sim}\varphi)$. In this way,

$$D \ni \phi^{\mathfrak{D}}(a) = (\mathsf{f}|\mathsf{f}|\mathsf{f}|\mathsf{f}|\mathsf{f}), \tag{4.31}$$

$$D \ni \psi^{\mathfrak{D}}(a) = (\mathsf{t}|\mathsf{t}|\mathsf{t}|\mathsf{t}|\mathsf{t}). \tag{4.32}$$

Consider the following complementary subsubcases:

(i) $\mathfrak{A}_{-\mathsf{n}}$ is not regular.
Then, there are some $\varsigma \in \Sigma$ of arity $n \in \omega$, some $\vec{\bar{g}} \in (A^n_{-\mathsf{n}})^2$ and some $i \in 2$ such that $g^i_j \sqsubseteq g^{1-i}_j$, for all $j \in n$, but $\varsigma^{\mathfrak{A}}(\bar{g}^i) \not\sqsubseteq \varsigma^{\mathfrak{A}}(\bar{g}^{1-i})$, in which case $w \triangleq \varsigma^{\mathfrak{A}}(\bar{g}^i) \neq x \triangleq \varsigma^{\mathfrak{A}}(\bar{g}^{1-i}) \in \{\mathsf{f}, \mathsf{t}\}$, and so $\bar{g}^i \neq \bar{g}^{1-i}$, in which case $y \triangleq g^i_j \in \{\mathsf{f}, \mathsf{t}\}$ and $g^{1-i}_j = \mathsf{b}$, for some $j \in n$, in which case $n \neq 0$. Moreover, as $\varsigma^{\mathfrak{A}}$ is not regular, it is b-idempotent, in which case $\bar{g}^{1-i} \neq (n \times \{\mathsf{b}\})$, while $n \leqslant 2$, and so $n = 2$ and $z \triangleq g^{1-i}_{1-j} \neq \mathsf{b}$. Therefore, $g^i_{1-j} = z \in \{\mathsf{f}, \mathsf{t}\}$, in which case $(z|z|z|z|z) \in D$, in view of (4.31) and (4.32). Moreover, by (4.29) and (4.30), we also have $(\mathsf{b}|\mathsf{b}|y|\mathsf{b}|\mathsf{b}) \in D$. In this way, $D \ni f \triangleq \varsigma^{\mathfrak{D}}(\{\langle j, (\mathsf{b}|\mathsf{b}|y|\mathsf{b}|\mathsf{b})\rangle, \langle 1-j, (z|z|z|z|z)\rangle\}) = (x|x|w|x|x)$. Consider the following complementary subsubsubcases:

(A) $w = \mathsf{b}$.
Then, taking (4.30) into account, we have $D \ni ((f \wedge^{\mathfrak{D}} {\sim}^{\mathfrak{D}} f) \vee^{\mathfrak{D}} {\sim}^{\mathfrak{D}} d) = (\mathsf{b}|\mathsf{b}|\mathsf{b}|\mathsf{b}|\mathsf{b})$. Hence, as $I \supseteq K \neq \varnothing$, by (4.31) and (4.32), we see that $\{\langle u, (u|u|u|u|u)\rangle \mid u \in A_{-\mathsf{n}}\}$ is an embedding of $\mathcal{A}_{-\mathsf{n}}$ into $\mathcal{D}$. Therefore, by (2.2), $\mathcal{A}_{-\mathsf{n}}$ is a model of C', and so is $\mathcal{B}$, for $\{\mathsf{f}, \mathsf{t}\}$ forms a subalgebra of $\mathfrak{A}_{-\mathsf{n}}$.

(B) $w \neq \mathsf{b}$.
Then, $w \in \{\mathsf{f}, \mathsf{t}\} \ni x$, so $D \supseteq \{f, \sim^{\mathcal{D}} f\} = \{(\mathsf{f}|\mathsf{f}|\mathsf{t}|\mathsf{f}|\mathsf{f}), (\mathsf{t}|\mathsf{t}|\mathsf{f}|\mathsf{t}|\mathsf{t})\}$. Hence, as $K \neq \varnothing \neq (L \setminus J)$, by (4.29), (4.30), (4.31) and (4.32), we see that $\{\langle\langle u, v\rangle, (u|u|v|u|u)\rangle \mid \langle u, v\rangle \in B\}$ is an embedding of $\mathcal{B}$ into $\mathcal{D}$. Therefore, by (2.2), $\mathcal{B}$ is a model of C'.

(ii) $\mathfrak{A}_{-\mathsf{n}}$ is regular.
Then, Lemma 4.21, used tacitly throughout the rest of the proof, is well-applicable to $\mathcal{B}$. In this way, as $(((I \setminus (L \cup K)) \cap J) \cup (I \setminus (L \cup J)) \cup (L \cap J)) \neq \varnothing \notin \{K, L \setminus J\}$, by (4.25), (4.26), (4.27), (4.28), (4.29), (4.30), (4.31) and (4.32), we see that $\{\langle\langle t, u, v\rangle, (v|v|u|v|t)\rangle \mid \langle t, u, v\rangle \in (B \dot{+} 2)\}$ is an embedding of $\mathcal{B} \dot{+} 2$ into $\mathcal{D}$, in which case, by (2.2), it is a model of C', and so is its strict surjective homomorphic image $\mathcal{B}$. □

Lemma 4.23 (cf. Lemma 4.24 of [13])**.** *Let $\mathcal{B} \in \mathbf{S}(\mathcal{A})$. Suppose $B \cup \{\mathsf{b}\}$ forms a regular subalgebra of $\mathfrak{A}$. Then, any Σ-axiom, being true in $\mathcal{B}$, is so in $\mathcal{A} \restriction (B \cup \{\mathsf{b}\})$.*

Ii is remarkable that it is the gentle operation-wise condition that makes Lemma 4.22 well-applicable to the purely-implicative expansion of $B_{4,01}$ despite of the fact that, in that case, $\mathfrak{A}$ is neither regular nor b-idempotent. This equally concerns the following quite important definitive result:

Theorem 4.24 (cf. [10] for the case $\Sigma = \Sigma_{\sim}$)**.** *Suppose C is both $\sim$-subclassical and not maximally $\sim$-paraconsistent, while, for every $\varsigma \in \Sigma$, $\varsigma^{\mathfrak{A}_{-\mathsf{n}}}$ is either regular or both b-idempotent and no more than binary (in particular, $\Sigma = \Sigma_{\sim[01]}$). Then, proper consistent extensions of $C^{\mathrm{EM}} = C^{-\mathsf{n}}$, that is defined by $\mathcal{A}_{\not{\mathsf{n}}}$, form the two-element chain $C^{\mathrm{EM+NP}} \subsetneq C^{\mathrm{PC}} = C^{\mathrm{EM+(R/MP)}}$ and, providing $\mathfrak{A}_{-\mathsf{n}}$ is regular (in particular, $\Sigma = \Sigma_{\sim[01]}$), have same theorems as C^{EM} has, and so are not axiomatic.*

Proof. With using Theorems 4.2, 4.5, 4.7, 4.20, Remark 4.19 and Lemma 4.22. For just notice that (4.24) is not true in the consistent truth-non-empty Σ-matrix $\mathcal{A}_{-\mathsf{n}} \times (\mathcal{A} \restriction \{\mathsf{f}, \mathsf{t}\})$ under $[x_0/\langle \mathsf{b}, \mathsf{t}\rangle, x_1/\langle \mathsf{f}, \mathsf{t}\rangle]$. Finally, Lemma 4.23 with $B = \{\mathsf{f}, \mathsf{t}\}$ completes the argument. □

In view of Lemma 4.6, Theorem 4.24 shows that $(\mathcal{C} \cap \mathrm{Fm}_{\Sigma}^{\omega}) \cup (\sigma_{+1}[\mathcal{C} \setminus \mathrm{Fm}_{\Sigma}^{\omega}] \veebar x_0)$ cannot be replaced by $\mathcal{C}$ in the item (ii)**b)** of Theorem 3.10, when taking $\mathsf{M} = \{\mathcal{A}_{-\mathsf{n}}\}$ and $\mathcal{C} = \{(4.12)\}$. After all, Theorem 4.24 subsumes some results of [12] concerning purely implicative expansions of $B_{4[01]}$.

4.5. **Kleene extensions.** Next, C is said to be *strongly hereditary*, provided $\{\mathsf{f}, \mathsf{n}, \mathsf{t}\}$ forms a regular specular subalgebra of $\mathfrak{A}$, in which case, since $\mu \circ \mu$ is diagonal, $\{\mathsf{f}, \mathsf{b}, \mathsf{t}\} = \mu[\{\mathsf{f}, \mathsf{n}, \mathsf{t}\}]$ forms a specular subalgebra of $\mathfrak{A}$ as well, and so a regular one, for μ is anti-regular, while $\{\mathsf{f}, \mathsf{t}\}$ forms a subalgebra of $\mathfrak{A}$, in view of Lemma 4.12 with $B = \{\mathsf{n}\}$, and so C is hereditary. By symmetry between n and b, C is strongly hereditary iff $\{\mathsf{f}, \mathsf{b}, \mathsf{t}\}$ forms a regular specular subalgebra of $\mathfrak{A}$, whenever $\mathfrak{A}$ is both regular and specular, while $\{\mathsf{f}, (\mathsf{b}/\mathsf{n}), \mathsf{t}\}$ forms a subalgebra of $\mathfrak{A}$ (in particular, $\Sigma = \Sigma_{\sim[01]}$). According to the following example, equally showing that the framework of strongly hereditary expansions of B_4 is not at all exhausted by solely definitional copies of $B_{4[01]}$, "whenever" cannot be replaced with "iff" above:

Example 4.25. If $\Sigma \triangleq (\Sigma_{\sim[01]} \cup \{\uplus\})$, where $\uplus$ is binary, and $\uplus^{\mathfrak{A}} \triangleq ((\vee^{\mathfrak{A}} \restriction (A_{\not{\mathsf{n}}}^2 \cup A_{\not{\mathsf{b}}}^2)) \cup \{\langle\langle \mathsf{n}, \mathsf{b}\rangle, \mathsf{t}\rangle, \langle\langle \mathsf{b}, \mathsf{n}\rangle, \mathsf{f}/\mathsf{b}\rangle\})$, C is strongly hereditary, and $\mathfrak{A}$ is not specular, as opposed to $\mathfrak{DM}_{4[01]}$, and non-regular/regular. □

From now on, C is supposed to be strongly hereditary. First, as an immediate consequence of Theorems 4.4, 4.5 and 4.13, we have:

Corollary 4.26. *$C^{\mathrm{EM}} \cap C^{\mathrm{R}}$ is the greatest both inferentially paracomplete and $\sim$-paraconsistent extension of C.*

Lemma 4.27. $(\mathcal{A}_{\not{\mathsf{n}}} \times \mathcal{A}_{\not{\mathsf{b}}}) \in \mathrm{Mod}(C^{\mathrm{EM+NP}} \cap C^{\mathrm{R}})$.

Proof. Since, by Theorem 4.20 and Corollary 4.9, $C^{\mathrm{EM+NP}} \cap C^{\mathrm{R}}$ is defined by $\{\mathcal{A}_{\not{\mathsf{b}}}, \mathcal{A}_{\not{\mathsf{n}}} \times \mathcal{A}_{\not{\mathsf{n}}\not{\mathsf{b}}}\}$, $\mathcal{A}_{\not{\mathsf{n}}} \times (\mathcal{A}_{\not{\mathsf{b}}} \times \mathcal{A}_{\not{\mathsf{n}}\not{\mathsf{b}}})$, being isomorphic to $\mathcal{A}_{\not{\mathsf{b}}} \times (\mathcal{A}_{\not{\mathsf{n}}} \times \mathcal{A}_{\not{\mathsf{n}}\not{\mathsf{b}}})$, is a model of $C^{\mathrm{EM+NP}} \cap C^{\mathrm{R}}$, in view of (2.2). Moreover, by Lemma 4.11, $(\mathcal{A}_{\not{\mathsf{b}}} \times \mathcal{A}_{\not{\mathsf{n}}\not{\mathsf{b}}}) \restriction K_4^{\mathsf{n}}$ is a submatrix of $\mathcal{A}_{\not{\mathsf{b}}} \times \mathcal{A}_{\not{\mathsf{n}}\not{\mathsf{b}}}$, in which case $\mathcal{A}_{\not{\mathsf{n}}} \times ((\mathcal{A}_{\not{\mathsf{b}}} \times \mathcal{A}_{\not{\mathsf{n}}\not{\mathsf{b}}}) \restriction K_4^{\mathsf{n}})$ is a submatrix of $\mathcal{A}_{\not{\mathsf{n}}} \times (\mathcal{A}_{\not{\mathsf{b}}} \times \mathcal{A}_{\not{\mathsf{n}}\not{\mathsf{b}}})$, and so it is a model of $C^{\mathrm{EM+NP}} \cap C^{\mathrm{R}}$, in view of (2.2). And what is more, $h \triangleq (\pi_0 \restriction K_4^{\mathsf{n}}) \in \mathrm{hom}_{\mathrm{S}}((\mathcal{A}_{\not{\mathsf{b}}} \times \mathcal{A}_{\not{\mathsf{n}}\not{\mathsf{b}}}) \restriction K_4^{\mathsf{n}}, \mathcal{A}_{\not{\mathsf{b}}})$ is surjective, and so is $g : (A_{\not{\mathsf{n}}} \times K_4^{\mathsf{n}}) \to (A_{\not{\mathsf{n}}} \times A_{\not{\mathsf{b}}}), \langle a, b\rangle \mapsto \langle a, h(b)\rangle$, belonging to $\mathrm{hom}_{\mathrm{S}}(\mathcal{A}_{\not{\mathsf{n}}} \times ((\mathcal{A}_{\not{\mathsf{b}}} \times \mathcal{A}_{\not{\mathsf{n}}\not{\mathsf{b}}}) \restriction K_4^{\mathsf{n}}), \mathcal{A}_{\not{\mathsf{n}}} \times \mathcal{A}_{\not{\mathsf{b}}})$, as required, by (2.2). □

Lemma 4.28 (cf. Lemma 4.33 of [13])**.** *Let I be a finite set and $\mathcal{B}$ a consistent non-$\sim$-paraconsistent submatrix of $\mathcal{A}^I$. Then,* $\mathrm{hom}(\mathcal{B}, \langle \mathfrak{A}, \{\mathsf{t}\}\rangle) \neq \varnothing$.

Corollary 4.29. *Let I be a finite set, $\overline{\mathcal{C}} \in \{\mathcal{A}_{\not{\mathsf{b}}}, \mathcal{A}_{\not{\mathsf{n}}}\}^I$, and $\mathcal{B}$ a consistent non-$\sim$-paraconsistent submatrix of $\prod_{i \in I} \mathcal{C}_i$. Then,* $\mathrm{hom}(\mathcal{B}, \mathcal{A}_{\not{\mathsf{b}}}) \neq \varnothing$.

Proof. In that case, by Lemma 4.28, there is some $h \in \mathrm{hom}(\mathcal{B}, \langle \mathfrak{A}, \{\mathsf{t}\}\rangle) \neq \varnothing$, in which case $\mathfrak{D} \triangleq (\mathfrak{A} \restriction (\mathrm{img}\, h))$ satisfies the *Kleene lattice* identity $((x_0 \wedge \sim x_0) \wedge (x_1 \vee \sim x_1)) \approx (x_0 \wedge \sim x_0)$, for $\mathfrak{B}$ does so, because both $\mathfrak{A}_{\not{\mathsf{b}}}$ and $\mathfrak{A}_{\not{\mathsf{n}}}$ do so, while $h \in \mathrm{hom}(\mathfrak{B}, \mathfrak{D})$ is surjective. Hence, $\{\mathsf{n}, \mathsf{b}\} \nsubseteq D$, for otherwise, the Kleene lattice identity would not be true in $\mathfrak{D}$ under $[x_0/\mathsf{n}, x_1/\mathsf{b}]$. Thus, $\mathcal{D} \triangleq (\langle \mathfrak{A}, \{\mathsf{t}\}\rangle \restriction D)$ is a submatrix of $\langle \mathfrak{A}, \{\mathsf{t}\}\rangle \restriction A_{\not{a}}$, for some $a \in \{\mathsf{n}, \mathsf{b}\}$, in which case $h \in \mathrm{hom}(\mathcal{B}, \langle \mathfrak{A}, \{\mathsf{t}\}\rangle \restriction A_{\not{a}})$, and so the fact that $\mu \restriction A_{\not{\mathsf{n}}}$ is an isomorphism from $\langle \mathfrak{A}, \{\mathsf{t}\}\rangle \restriction A_{\not{\mathsf{n}}}$ onto $(\langle \mathfrak{A}, \{\mathsf{t}\}\rangle \restriction A_{\not{\mathsf{b}}}) = \mathcal{A}_{\not{\mathsf{b}}}$ completes the argument. □

Corollary 4.30. *$C^{\mathrm{EM+NP}} \cap C^{\mathrm{R}}$ is axiomatized by* (4.12) *relatively to $C^{\mathrm{EM}} \cap C^{\mathrm{R}}$.*

Proof. By Corollary 4.9 and Theorem 4.5 [resp., 4.20], $C^{\mathrm{EM[+NP]}} \cap C^{\mathrm{R}}$ is defined by $\{\mathcal{A}_{\not{\mathsf{n}}}[\times \mathcal{A}_{\not{\mathsf{n}}\not{\mathsf{b}}}], \mathcal{A}_{\not{\mathsf{b}}}\}$. Consider any model $\mathcal{B} \in \mathbf{S}(\mathbf{P}_\omega(\{\mathcal{A}_{\not{\mathsf{b}}}, \mathcal{A}_{\not{\mathsf{n}}}\}))$ of (4.12), in which case there is some finite set I, some $\overline{\mathcal{C}} \in \{\mathcal{A}_{\not{\mathsf{b}}}, \mathcal{A}_{\not{\mathsf{n}}}\}^I$ such that $\mathcal{B}$ is a submatrix of $\prod_{i \in I} \mathcal{C}_i$. Put $J \triangleq \mathrm{hom}(\mathcal{B}, \mathcal{A}_{\not{\mathsf{n}}} \times \mathcal{A}_{\not{\mathsf{b}}})$ and $K \triangleq \mathrm{hom}(\mathcal{B}, \mathcal{A}_{\not{\mathsf{b}}})$. Consider any $a \in (B \setminus D^{\mathcal{B}})$, in which case $\mathcal{B}$ is consistent and there is some $i \in I$ such that $\pi_i(a) \notin D^{\mathcal{C}_i}$. Consider the following complementary cases:

(1) $\mathfrak{C}_i = \mathcal{A}_{\not{\mathsf{n}}}$.
Then, by Corollary 4.29, there is some $h \in \mathrm{hom}(\mathcal{B}, \mathcal{A}_{\not{\mathsf{b}}}) \neq \varnothing$, in which case $g : B \to (A_{\not{\mathsf{n}}} \times A_{\not{\mathsf{b}}}), a \mapsto \langle \pi_i(a), h(a)\rangle$ is in J and $g(a) \notin D^{\mathcal{A}_{\not{\mathsf{n}}} \times \mathcal{A}_{\not{\mathsf{b}}}}$.

(2) $\mathfrak{C}_i \neq \mathcal{A}_{\not{\mathsf{n}}}$, in which case $\mathfrak{C}_i = \mathcal{A}_{\not{\mathsf{b}}}$, and so $(\pi_i \restriction B) \in K$.

In this way, $f : B \to ((A_{\not{\mathsf{n}}} \times A_{\not{\mathsf{b}}})^J \times A_{\not{\mathsf{b}}}^K), a \mapsto \langle \langle j(a)\rangle_{j \in J}, \langle k(a)\rangle_{k \in K}\rangle$ is in $\mathrm{hom}_{\mathrm{S}}(\mathcal{B}, (\mathcal{A}_{\not{\mathsf{n}}} \times \mathcal{A}_{\not{\mathsf{b}}})^J \times \mathcal{A}_{\not{\mathsf{b}}}^K)$, and so (2.2), Theorem 2.6, Lemma 4.27 and the finiteness of A complete the argument. □

By $NP_{[01]}$ we denote the extension of $LP_{[01]}$ relatively axiomatized by (4.12).

Theorem 4.31. *Suppose C has a/no theorem. Then, Kleene [non-pseudo-axiomatic] extensions of C form the seven/eleven[seven]-element non-chain distributive lattice depicted at Figure 1 with solely solid circles/[with solely solid circles], both $C^{\mathrm{EM+(NP|R)}}$ and $\{C^{\mathrm{EM+NP}} \cap\} C^{\mathrm{R}}/$ as well as theorem-less proper ones being non-axiomatic extensions of both $C^{\mathrm{EM}} \cap C^{\mathrm{R}}$ and C, and so C^{EM} is the only proper axiomatic extension of $C^{\mathrm{EM}} \cap C^{\mathrm{R}}$ and, providing either $\mathfrak{A}$ is regular or C has no theorem, of C. Moreover, those of them, which are neither $\vee$-disjunctive nor equal to $C^{\mathrm{EM+NP}}$, are relatively axiomatized as follows:*

$$C^{\mathrm{EM+NP}} \cap C^{\mathrm{R}} \quad by \quad (4.12),$$

$$C^{\mathrm{EM+NP}}_{+0} \quad by \quad \{(4.12), (4.17)\},$$

others inheriting the above axiomatizations relatively to C *with possible relacing* (4.2) *by* (4.24).

Proof. We use (2.2), Theorems 4.5, 4.7, 4.18, 4.20, 4.24, Proposition 4.3, Corollaries 3.6, 4.9, 4.30, Lemma 4.23 with $\mathcal{B} = \mathcal{A}_{\not{b}}$ and Remarks 2.2, 2.5 and 4.19 tacitly. First, as C^{EM} is $\sim$-paraconsistent, $(C^{\mathrm{EM+NP}} \cap C^{\mathrm{R}})/C^{\mathrm{EM+NP}}_{+0}/C^{\mathrm{EM+NP}}$ is distinct from $(C^{\mathrm{EM}} \cap C^{\mathrm{R}})/C^{\mathrm{EM}}_{+0}/C^{\mathrm{EM}}$, respectively. Likewise, since (4.24) is not true in $\mathcal{A}_{\not{n}} \times \mathcal{A}_{\not{n}\not{b}}$ under $[x_0/\langle \mathsf{b}, \mathsf{t}\rangle, x_1/\langle \mathsf{f}, \mathsf{t}\rangle]$, $(C^{\mathrm{EM+NP}} \cap C^{\mathrm{R}})/C^{\mathrm{EM+NP}}_{+0}/C^{\mathrm{EM+NP}}$ is distinct from $C^{\mathrm{R}}/C^{\mathrm{EM+R}}_{+0}/C^{\mathrm{EM+R}}$, respectively. Finally, consider any [non-pseudo-axiomatic] extension C' of $C^{\mathrm{EM}} \cap C^{\mathrm{R}}$ and the following exhaustive cases [but (3) and (4)]:

(1) $IC \subseteq C'$.
Then, $C' = IC$.

(2) $C^{\mathrm{PC}} \subseteq C'$ but $IC \not\subseteq C'$.
Then, C' is consistent, and so inferentially consistent, for (4.1), being satisfied in C^{PC}, is so in its extension C', in which case, by Theorem 4.2, $C' = C^{\mathrm{PC}}$.

(3) $IC_{+0} \subseteq C'$ but $C^{\mathrm{PC}} \not\subseteq C'$.
Then, IC, being an extension of C^{PC}, is not a sublogic of C', so, by the following claim, C' has no theorem:

Claim 4.32. *Let* C'' *and* C''' *be* Σ*-logics. Suppose* $C'' \not\subseteq C'''$ *is non-pseudo-axiomatic and* $C''_{+0} \subseteq C'''$. *Then,* C''' *has no theorem.*

Proof. By contradiction. For suppose C''' has a theorem, in which case it is non-pseudo-axiomatic, and so, by Remark 2.2, we get $C'' = (C''_{+0})_{-0} \subseteq C'''_{-0} = C'''$. This contradiction completes the proof. ☐

In this way, as $C'_{-0} \subseteq IC$, we have $C' = (C'_{-0})_{+0} \subseteq IC_{+0}$, and so we get $C' = IC_{+0}$.

(4) $C^{\mathrm{PC}}_{+0} \subseteq C'$ but both $C^{\mathrm{PC}} \not\subseteq C'$ and $IC_{+0} \not\subseteq C'$.
Then, by Claim 4.32, C' has no theorem. Moreover, (4.17), being satisfied in C^{PC}_{+0}, is so in its extension C', in which case, by the structurality of C', $(x_1 \vee \sim x_1) \in (\bigcap_{k \in \omega} C'(x_k)) = C'_{-0}(\varnothing)$, and so $C^{\mathrm{PC}} \subseteq C'_{-0}$. On the other hand, $IC = (IC_{+0})_{-0} \not\subseteq C'_{-0}$, so C'_{-0} is consistent, and so inferentially consistent, for it satisfies (4.1). Hence, by Theorem 4.2, $C'_{-0} = C^{\mathrm{PC}}$. In this way, $C' = (C'_{-0})_{+0} = C^{\mathrm{PC}}_{+0}$.

(5) $(C^{\mathrm{PC}}_{+0}[\cup C^{\mathrm{PC}}]) \not\subseteq C'$ but $C^{\mathrm{R}} \subseteq C'$.
Then, [(4.1), and so, in view of the non-pseudo-axiomaticity of C'] (4.17) is not satisfied in C', in which case, by Theorem 4.13, $C' = C^{\mathrm{R}}$.

(6) $C^{\mathrm{R}} \not\subseteq C'$.
Then, (4.24) is not satisfied in C', in which case, by Lemma 4.22, $C' \subseteq C^{\mathrm{EM+NP}}$, and so we have the following exhaustive subcases [but (c) and (d)]:

(a) $C^{\mathrm{EM+NP}} \subseteq C'$.
Then, $C' = C^{\mathrm{EM+NP}}$.

(b) $C^{\mathrm{EM+NP}} \not\subseteq C'$ but $C^{\mathrm{EM}} \subseteq C'$.
Then, C' is $\sim$-paraconsistent, so, by Theorem 4.4, $C' = C^{\mathrm{EM}}$.

(c) $C^{\mathrm{EM+NP}}_{+0} \subseteq C'$ but $C^{\mathrm{EM}} \not\subseteq C'$.
Then, $C^{\mathrm{EM+NP}} \not\subseteq C'$, so, by Claim 4.32, C' has no theorem. Therefore, $C^{\mathrm{EM+NP}} = (C^{\mathrm{EM+NP}}_{+0})_{-0} \subseteq C'_{-0}$, $(C^{\mathrm{EM}} \cap C^{\mathrm{R}}) = (C^{\mathrm{EM}} \cap C^{\mathrm{R}})_{-0} \subseteq C'_{-0}$ and $C^{\mathrm{R}} \not\subseteq C'_{-0}$, for, otherwise, we would have $C^{\mathrm{R}} = (C^{\mathrm{R}})_{+0} \subseteq$

$(C'_{-0})_{+0} = C'$. Hence, by Lemma 4.22, we have $C'_{-0} \subseteq C^{\mathrm{EM+NP}}$, in which case we get $C' = (C'_{-0})_{+0} \subseteq C^{\mathrm{EM+NP}}_{+0}$, and so $C' = C^{\mathrm{EM+NP}}_{+0}$.

(d) $C^{\mathrm{EM}}_{+0} \subseteq C'$ but both $C^{\mathrm{EM}} \nsubseteq C$ and $C^{\mathrm{EM+NP}}_{+0} \nsubseteq C'$.
Then, by Claim 4.32, C' has no theorem. Moreover, (4.17), being satisfied in C^{EM}_{+0}, is so in C', in which case, by the structurality of C', $(x_1 \vee \sim x_1) \in (\bigcap_{k\in\omega} C'(x_k)) = C'_{-0}(\varnothing)$, and so $C^{\mathrm{EM}} \subseteq C'_{-0}$, while $(C^{\mathrm{EM}} \cap C^{\mathrm{R}}) = (C^{\mathrm{EM}} \cap C^{\mathrm{R}})_{-0} \subseteq C'_{-0}$. Also, $C^{\mathrm{EM+NP}} = (C^{\mathrm{EM+NP}}_{+0})_{-0} \nsubseteq C'_{-0}$, so C'_{-0} is $\sim$-paraconsistent. Hence, by Theorem 4.4, $C'_{-0} = C^{\mathrm{EM}}$. In this way, $C' = (C'_{-0})_{+0} = C^{\mathrm{EM}}_{+0}$.

(e) $(C^{\mathrm{EM+NP}} \cap C^{\mathrm{R}}) \subseteq C'$ but $(C^{\mathrm{EM+NP}}_{+0}[\cup C^{\mathrm{EM+NP}}]) \nsubseteq C'$.
Then, [(4.1), and so, in view of the non-pseudo-axiomaticity of C'] (4.17) is not satisfied in C', in which case, by Theorem 4.13, $C' = (C^{\mathrm{EM+NP}} \cap C^{\mathrm{R}})$.

(f) $(C^{\mathrm{EM+NP}} \cap C^{\mathrm{R}}) \nsubseteq C'$ and $(C^{\mathrm{EM}}_{+0}[\cup C^{\mathrm{EM}}]) \nsubseteq C'$.
Then, C' is both $\sim$-paraconsistent and inferentially paracomplete [in view of the non-pseudo-axiomaticity of C'], and so, by Corollary 4.26, $C' = (C^{\mathrm{EM}} \cap C^{\mathrm{R}})$. □

As an immediate consequence of Theorems 4.18 and 4.31, as opposed to both $C^{\mathrm{EM}}[\cap C^{\mathrm{R}}]$ and C, we have:

Corollary 4.33. *All extensions of C^{R} are $\vee$-disjunctive.*

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