The simplest polynomial equations, the inflection point, the recurrence equations up to degree 6, and the method of finite differences

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The simplest polynomial equations, the inflection point, the recurrence equations up to degree 6, and the method of finite differences

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Abstract: This study is an extension and complementary of the Offset in Quadratics study. It determines the simplest equations for any polynomial up to 6^{th}-degree, the inflection points equations, as well as the two possible recurrence equations. Then we describe the behavior of any polynomial under the method of finite differences.
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1 Introduction

This study is an extension and complementary of the Offset in Quadratics study. It determines the simplest equations for any polynomial up to 6^{th}-degree, the inflection points equations, as well as the two possible recurrence equations.

Then we describe the behavior of any polynomial under the method of finite differences.

In the end, we have a general summary.

This study will serve as a background to future studies of the polynomials.

1.1 Previous conventions:

Because our tables will show vertical sequences where the indexes will be on vertical and because on vertical, we have Y-axis in the XY-plane, so the sequences integers elements have to appear in X-axis as a function of the Y-axis. Due to that, in all these studies we will represent any polynomial equation as being in the function of y, or just function $Y[y]$, or $x = Y[y]$.

1.2 Notation for Polynomials In these studies

Generically we will denote any polynomial element as being $Y[y]$. When we want to draw the polynomial in the XY-plane we will make x in the function of y. In the cartesian plane (square lattice grid) we can consider $x = Y[y]$. In different grid other than cartesian plane $x \neq Y[y]$.

When we want to distinguish the d^{th}-degree of the polynomial, we will notate $Yd[y]$ or $x = Yd[y]$.

When we want to make a p^{th}-power operation on an d^{th}-degree polynomial, we will notate: $(Yd[y])^p$.

- Constant (polynomial degree 0) will be noted as

$$Y0[y] = c$$

- Linear (polynomial 1^{st}-degree) will be noted as

$$Y1[y] = by + c$$

- Quadratic (polynomial 2^{nd}-degree) will be noted as

$$Y2[y] = ay^2 + by + c$$

- Cubic (polynomial $3^{\text{rd-}}$degree) will be noted as

$$Y3[y] = a_3y^3 + ay^2 + by + c$$

- Quartic (polynomial 4^{th}-degree) will be noted as

$$Y4[y] = a_4y^4 + a_3y^3 + ay^2 + by + c$$

- Quintic (polynomial 5^{th}-degree) will be noted as

$$Y5[y] = a_5y^5 + a_4y^4 + a_3y^3 + ay^2 + by + c$$

And so on for Sextic, Septic, Octic, Nonic, Decic, etc.

Generic equation of polynomial d^{th}-degree:

$$Yd[y] = a_dy^d + a_{d-1}y^{d-1} + \cdots + a_4y^4 + a_3y^3 + ay^2 + by + c$$

Generically, to be used in any recurrence equation, we will adopt these equalities notation:

$$\begin{aligned} Yd[-3] &= e \\ Yd[-2] &= f \\ Yd[-1] &= g = x_1 \\ Yd[0] &= h = x_2 \\ Yd[1] &= i = x_3 \\ Yd[2] &= j \\ Yd[3] &= k \end{aligned}$$

1.3 Notation for index direction in any polynomial sequence (to be used in recurrence equations)

Any polynomial Integer sequence has 2 directions. This is the reason any polynomial has 2 recurrence equations. So, if the direction is given by

$$Yd[y] \equiv (\dots, e, f, g, h, i, j, k, \dots) = \backslash(\dots, k, j, i, h, g, f, e, \dots)\backslash$$

then, the reversal direction will be given by

$$\backslash Yd[y]\backslash \equiv (\dots, k, j, i, h, g, f, e, \dots) = \backslash(\dots, e, f, g, h, i, j, k, \dots)\backslash$$

1.4 Inflection point vs. vertex nomenclature

Because of the definition of the inflection point is in differential calculus *"an inflection point, point of inflection, flex, or inflection (British English: inflexion[citation needed]) is a point on a continuous plane curve at which the curve changes from being concave (concave downward) to convex (concave upward), or vice versa*"; and

Because of the definition of the vertex in geometry as being "*In geometry, a vertex (plural: vertices or vertexes) is a point where two or more curves, lines, or edges meet. As a consequence of this definition, the point where two lines meet to form an angle and the corners of polygons and polyhedra are vertices*";

Because *"In the geometry of planar curves, a vertex is a point of where the first derivative of curvature is zero*";

And like all studies between polynomials, no feature or phenomenon indicates that there is a difference in behavior between quadratic and other polynomial orders, then, there is no reason to differentiate the inflection point phenomena in quadratics from other polynomials. So, there is no reason to have different names.

In these studies, we will refer to this phenomenon in our tables, text, and figures as being only the inflection point, even in quadratics which usually has the usual vertex name. Moreover, higher degrees of polynomials then quadratics, besides inflection point may have two or more turning points. But, the common phenomenon among all polynomials is the inflection point.

The definition of a single Inflection Point nomenclature in common to all polynomials becomes important when we compare the behavior of the offset at all degrees.

In these studies, the coordinates of an inflection point in XY-plane will be given by x_{ip} and y_{ip}. Also, we will denote an inflection point as being $ip(x_{ip}, y_{ip})$.

2 The simplest equation for 1st-degree polynomials (Linear)

From our definition of notation, the general polynomial equation of degree 1, is

$$Y1[y] = by + c$$

We have to determine the value of 2 coefficients given by b, c. Then, to determine all the coefficients it is easier to choose 2 consecutive elements from $Y1[y]$.

As we learned from the offset study, the simpler equation will be obtained with one of the consecutive elements in the central index $Y1[0]$ in $y = 0$.

So, we have:

$$Y1[0] = h = c$$
$$Y1[1] = i = b + c$$

Using Cramer's Rule:

$$\Delta = \begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} = -1$$
$$\Delta_b = \begin{vmatrix} h & 1 \\ i & 1 \end{vmatrix} = h - i$$
$$\Delta_c = \begin{vmatrix} 0 & h \\ 1 & i \end{vmatrix} = -h$$
$$b = \frac{\Delta_b}{\Delta} = \frac{h - i}{-1} = i - h$$
$$c = \frac{\Delta_c}{\Delta} = \frac{-h}{-1} = h$$

Conclusion: The general most simple equation for polynomial 1st-degree is

$$Y1[y] = (i - h)y + h$$

2.1 Inflection Point in 1st-degree polynomials

The linear polynomial inflection point is defined as being

$$\frac{d^0 Y1[y]}{dy^0} = 0$$
$$\frac{d^0(by + c)}{dy^0} = 0$$
$$by_{ip} + c = 0$$
$$y_{ip_{Y1[y]}}\left[@\frac{d^0 Y1[y]}{dy^0} = 0\right] = -\frac{0!\,c}{1!\,b} = -\left(\frac{h}{i - h}\right)$$
$$x_{ip}[y] = by_{ip} + c$$
$$x_{ip}[y] = b\left(-\frac{c}{b}\right) + c$$
$$x_{ip}[y] = 0$$

$ip_{Y1}(x, y) = \left(0, -\frac{c}{b}\right)$

2.2 Recurrence equation towards increasing index in 1st-degree polynomials

The general simplest equation of a linear polynomial

$$Y1[y] = (i - h)y + h$$

Because of the initial dots

$$Y1[0] = h$$
$$Y1[1] = i$$

Then, the next term will be $Y1[2]$ in the positive direction of the index y:

$$Y1[2] = (i - h)2 + h$$
$$Y1[2] = -h + 2i$$

Then, substituting the letters

$$Y1[2] = -Y1[0] + 2Y1[1]$$

So, the positive index direction recurrence equation of linear polynomials is

$$Y1[y] = -Y1[y - 2] + 2Y1[y - 1]$$

2.3 Recurrence equation towards decreasing index in 1st-degree polynomials

The general simplest equation of a linear polynomial

$$Y1[y] = (i - h)y + h$$

Because of the initial dots

$$Y1[0] = h$$
$$Y1[1] = i$$

Then, the next term will be $Y1[-1]$ in the negative direction of the index y:

$$Y1[-1] = (i - h)(-1) + h$$
$$Y1[-1] = 2h - i$$

Then, substituting the letters

$$Y1[-1] = 2Y1[0] - Y1[1]$$

So, the positive index direction recurrence equation of linear polynomials is

$$\backslash Y1[y]\backslash = 2Y1[y + 1] - Y1[y + 2]$$

3 The simplest equation for 2nd-degree polynomials (Quadratic)

From our definition of notation, the general polynomial equation of degree 2, is

$$Y2[y] = ay^2 + by + c$$

We have to determine the value of 3 coefficients given by a, b, c. Then, to determine all the coefficients it is easier to choose 3 consecutive elements from $Y2[y]$.

As we learned from the offset study, the simpler equation will be obtained with one of the consecutive elements in the central index $Y2[0]$ in $y = 0$.

So, we have:

$$Y2[-1] = g = a(-1)^2 + b(-1) + c$$
$$Y2[0] = h = a(0)^2 + b(0) + c$$
$$Y2[1] = i = a(1)^2 + b(1) + c$$

Or,

$$Y2[-1] = g = a - b + c$$
$$Y2[0] = h = c$$
$$Y2[1] = i = a + b + c$$

Using Cramer's Rule:

$$\Delta = \begin{vmatrix} 1 & -1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{vmatrix} = -2$$
$$\Delta_a = \begin{vmatrix} g & -1 & 1 \\ h & 0 & 1 \\ i & 1 & 1 \end{vmatrix} = -g + 2h - i$$
$$\Delta_b = \begin{vmatrix} 1 & g & 1 \\ 0 & h & 1 \\ 1 & i & 1 \end{vmatrix} = g - i$$
$$\Delta_c = \begin{vmatrix} 1 & -1 & g \\ 0 & 0 & h \\ 1 & 1 & i \end{vmatrix} = -2h$$

Then,

$$a = \frac{\Delta_a}{\Delta} = \frac{-g + 2h - i}{-2} = \frac{g - 2h + i}{2}$$
$$b = \frac{\Delta_b}{\Delta} = \frac{g - i}{-2} = \frac{-g + i}{2}$$
$$c = \frac{\Delta_c}{\Delta} = \frac{-2h}{-2} = h$$

Conclusion: The general most simple equation for polynomial 2nd-degree is

$$Y2[y] = \frac{g - 2h + i}{2} y^2 + \frac{-g + i}{2} y + h$$

3.1 Inflection Point in 2nd-degree polynomials

The quadratic polynomial inflection point (very common vertex) is defined as being

$$\frac{d^1Y2[y]}{dy^1} = 0$$

So,

$$\frac{d(ay^2 + by + c)}{dy} = 0$$

$$2!\,ay_{ip} + 1!\,b = 0$$

$$y_{ip} = -\frac{b}{2a} = -\frac{1}{2}\left(\frac{\frac{-g+i}{2}}{\frac{g-2h+i}{2}}\right)$$

$$y_{ip_{X2[y]}}\left[@\frac{d^1Y2[y]}{dy^1} = 0\right] = -\frac{1!\,b}{2!\,a} = -\frac{1}{2}\left(\frac{\frac{-g+i}{2}}{\frac{g-2h+i}{2}}\right) = -\frac{1}{2}\left(\frac{-g+i}{g-2h+i}\right)$$

Then,

$$x_{ip} = ay_{ip}^2 + by_{ip} + c$$

$$x_{ip} = a\left(-\frac{b}{2a}\right)^2 + b\left(-\frac{b}{2a}\right) + c$$

$$x_{ip} = \frac{b^2}{4a} - \frac{b^2}{2a} + c$$

$$x_{ip} = \frac{b^2 - 2b^2 + 4ac}{4a}$$

$$x_{ip} = -\frac{b^2 - 4ac}{4a}$$

$$ip_{Y2}(x, y) = \left(-\frac{b^2 - 4ac}{4a}, -\frac{b}{2a}\right)$$

3.2 Recurrence equation towards increasing index in 2nd-degree polynomials

The general simplest equation of quadratic polynomial

$$Y2[y] = \frac{g - 2h + i}{2} y^2 + \frac{-g + i}{2} y + h$$

Because the initial dots

$$Y2[-1] = g$$
$$Y2[0] = h$$
$$Y2[1] = i$$

Then, the next term will be $Y2[2]$ in the positive direction of the index y:

$$Y2[2] = \frac{g - 2h + i}{2} 2^2 + \frac{-g + i}{2} 2 + h$$
$$Y2[2] = 2g - 4h + 2i - g + i + h$$
$$Y2[2] = g - 3h + 3i$$

Then, substituting the letters

$$Y2[2] = Y2[-1] - 3Y2[0] + 3Y2[1]$$

So, the positive index direction recurrence equation of quadratic polynomials is

$$Y2[y] = Y2[y - 3] - 3Y2[y - 2] + 3Y2[y - 1]$$

3.3 Recurrence equation towards decreasing index in 2nd-degree polynomials

The general simplest equation of quadratic polynomial

$$Y2[y] = \frac{g - 2h + i}{2} y^2 + \frac{-g + i}{2} y + h$$

Because the initial dots

$$Y2[-1] = g$$
$$Y2[0] = h$$
$$Y2[1] = i$$

Then, the next term will be $Y2[-2]$ in the negative direction of the index y:

$$Y2[-2] = \frac{g - 2h + i}{2} (-2)^2 + \frac{-g + i}{2} (-2) + h$$
$$Y2[-2] = 2g - 4h + 2i + g - i + h$$
$$Y2[-2] = 3g - 3h + i$$

Then, substituting the letters

$$Y2[-2] = 3Y2[-1] - 3Y2[0] + Y2[1]$$

So, the positive index direction recurrence equation of quadratic polynomials is

$$\backslash Y2[y] \backslash = 3Y2[y + 1] - 3Y2[y + 2] + Y2[y + 3]$$

4 The simplest equation for 3rd-degree polynomials (Cubic)

From our definition of notation, the general polynomial equation of degree 3, is

$$Y3[y] = a_3 y^3 + a y^2 + b y + c$$

We have to determine the value of 4 coefficients given by a_3, a, b, c. Then, to determine all the coefficients it is easier to choose 4 consecutive elements from $Y3[y]$.

As we learned from the offset study, the simpler equation will be obtained with one of the consecutive elements in the central index $Y3[0]$ in $y = 0$.

So, we have:

$$\begin{aligned} Y3[-1] &= g = a_3(-1)^3 + a(-1)^2 + b(-1) + c \\ Y3[0] &= h = c \\ Y3[1] &= i = a_3 + a + b + c \\ Y3[2] &= j = a_3(2)^3 + a(2)^2 + b(2) + c \end{aligned}$$

Then,

$$\begin{aligned} Y3[-1] &= g = -a_3 + a - b + c \\ Y3[0] &= h = c \\ Y3[1] &= i = a_3 + a + b + c \\ Y3[2] &= j = 8a_3 + 4a + 2b + c \end{aligned}$$

Using Cramer's Rule:

$$\Delta = \begin{vmatrix} -1 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 8 & 4 & 2 & 1 \end{vmatrix} = 12$$

$$\Delta_{a_3} = \begin{vmatrix} g & 1 & -1 & 1 \\ h & 0 & 0 & 1 \\ i & 1 & 1 & 1 \\ j & 4 & 2 & 1 \end{vmatrix} = -2g + 6h - 6i + 2j$$

$$\Delta_a = \begin{vmatrix} -1 & g & -1 & 1 \\ 0 & h & 0 & 1 \\ 1 & i & 1 & 1 \\ 8 & j & 2 & 1 \end{vmatrix} = 6g - 12h + 6i$$

$$\Delta_b = \begin{vmatrix} -1 & 1 & g & 1 \\ 0 & 0 & h & 1 \\ 1 & 1 & i & 1 \\ 8 & 4 & j & 1 \end{vmatrix} = -4g - 6h + 12i - 2j$$

$$\Delta_c = \begin{vmatrix} -1 & 1 & -1 & g \\ 0 & 0 & 0 & h \\ 1 & 1 & 1 & i \\ 8 & 4 & 2 & j \end{vmatrix} = 12h$$

Then,

$$a_3 = \frac{\Delta_{a_3}}{\Delta} = \frac{-2g + 6h - 6i + 2j}{12} = \frac{-g + 3h - 3i + j}{6}$$

$$a = \frac{\Delta_a}{\Delta} = \frac{6g - 12h + 6i}{12} = \frac{g - 2h + i}{2}$$

$$b = \frac{\Delta_b}{\Delta} = \frac{-4g - 6h + 12i - 2j}{12} = \frac{-2g - 3h + 6i - j}{6}$$

$$c = \frac{\Delta_c}{\Delta} = \frac{12h}{12} = h$$

Conclusion: The general most simple equation for polynomial 3rd-degree is

$$Y3[y] = \frac{-g + 3h - 3i + j}{6} y^3 + \frac{g - 2h + i}{2} y^2 + \frac{-2g - 3h + 6i - j}{6} y + h$$

4.1 Inflection Point in 3rd-degree polynomials

The cubic polynomial inflection point is defined as being

$$\frac{d^2 Y3[y]}{dy^2} = 0$$

$$\frac{d^2(a_3 y^3 + a y^2 + b y + c)}{dy^2} = 0$$

Then,

$$3!\, y_{ip} + 2!\, a = 0$$

$$y_{ip_{X3[y]}}\left[@\frac{d^2 Y3[y]}{dy^2} = 0\right] = -\frac{2!\, a}{3!\, a_3} = -\frac{1}{3}\left(\frac{\frac{g - 2h + i}{2}}{\frac{-g + 3h - 3i + j}{6}}\right) = -1\left(\frac{g - 2h + i}{-g + 3h - 3i + j}\right)$$

Then,

$$x_{ip} = a_3 y_{ip}^3 + a y_{ip}^2 + b y_{ip} + c$$

$$x_{ip} = a_3\left(-\frac{a}{3a_3}\right)^3 + a\left(-\frac{a}{3a_3}\right)^2 + b\left(-\frac{a}{3a_3}\right) + c$$

$$x_{ip} = -\frac{a_3 a^3}{27a_3^3} + \frac{a^3}{9a_3^2} - \frac{ba}{3a_3} + c$$

$$x_{ip} = -\frac{a^3}{27a_3^2} + \frac{a^3}{9a_3^2} - \frac{ba}{3a_3} + c$$

$$x_{ip} = -\frac{a^3}{27a_3^2} + \frac{3a^3}{27a_3^2} - \frac{9a_3 ab}{27a_3^2} + c$$

$$x_{ip_{Y3[y]}} = \frac{2a^3 - 9a_3 ab}{27a_3^2} + c$$

$$ip_{Y3}(x, y) = \left[\frac{2a^3 - 9a_3 ab}{27a_3^2} + c, -\frac{a}{3a_3} = -1\left(\frac{g - 2h + i}{-g + 3h - 3i + j}\right)\right]$$

4.2 Recurrence equation towards increasing index in 3rd-degree polynomials

The general simplest equation of a cubic polynomial

$$Y3[y] = \frac{-g + 3h - 3i + j}{6} y^3 + \frac{g - 2h + i}{2} y^2 + \frac{-2g - 3h + 6i - j}{6} y + h$$

Because of the initial dots

$$Y3[-1] = g$$
$$Y3[0] = h$$
$$Y3[1] = i$$
$$Y3[2] = j$$

Then, the next term will be $Y3[3]$ in the positive direction of the index y:

$$Y3[3] = \frac{-g + 3h - 3i + j}{6} 3^3 + \frac{g - 2h + i}{2} 3^2 + \frac{-2g - 3h + 6i - j}{6} 3 + h$$

$$Y3[3] = \frac{-9g + 27h - 27i + 9j}{2} + \frac{9g - 18h + 9i}{2} + \frac{-2g - 3h + 6i - j}{2} + \frac{2h}{2}$$

$$Y3[3] = \frac{-2g + 8h - 12i + 8j}{2}$$

$$Y3[3] = -g + 4h - 6i + 4j$$

Then, substituting the letters

$$Y3[3] = -Y3[-1] + 4Y3[0] - 6Y3[1] + 4Y3[2]$$

So, the positive index direction recurrence equation of cubic polynomials is

$$Y3[y] = -Y3[y - 4] + 4Y3[y - 3] - 6Y3[y - 2] + 4Y3[y - 1]$$

4.3 Recurrence equation towards decreasing index in 3rd-degree polynomials

The general simplest equation of a cubic polynomial

$$Y3[y] = \frac{-g + 3h - 3i + j}{6} y^3 + \frac{g - 2h + i}{2} y^2 + \frac{-2g - 3h + 6i - j}{6} y + h$$

Because of the initial dots

$$Y3[-1] = g$$
$$Y3[0] = h$$
$$Y3[1] = i$$
$$Y3[2] = j$$

Then, the next term will be $Y3[-2]$ in the negative direction of the index y:

$$Y3[-2] = \frac{-g + 3h - 3i + j}{6} (-2)^3 + \frac{g - 2h + i}{2} (-2)^2 + \frac{-2g - 3h + 6i - j}{6} (-2) + h$$

$$Y3[-2] = \frac{8g - 24h + 24i - 8j}{6} + \frac{4g - 8h + 4i}{2} + \frac{4g + 6h - 12i + 2j}{6} + h$$

$$Y3[-2] = \frac{8g - 24h + 24i - 8j}{6} + \frac{12g - 24h + 12i}{6} + \frac{4g + 6h - 12i + 2j}{6} + \frac{6h}{6}$$

$$Y3[-2] = \frac{24g - 36h + 24i - 6j}{6}$$

$$Y3[-2] = 4g - 6h + 4i - j$$

Then, substituting the letters

$$Y3[-2] = 4Y3[-1] - 6Y3[0] + 4Y3[1] - Y3[2]$$

So, the negative index direction recurrence equation of cubic polynomials is

$$\backslash Y3[y] \backslash = 4Y3[y+1] - 6Y3[y+2] + 4Y3[y+3] - Y3[y+4]$$

5 The simplest equation for 4th-degree polynomials (Quartic)

From our definition of notation, the general polynomial equation of degree 4, is

$$Y4[y] = a_4y^4 + a_3y^3 + ay^2 + by + c$$

We have to determine the value of 5 coefficients a_4, a_3, a, b, c. Then, to determine all the coefficients it is easier to choose 5 consecutive elements from $Y4[y]$.

As we learned from the offset study, the simpler equation will be obtained with one of the consecutive elements in the central index $Y4[0]$ in $y = 0$.

So, we have:

$$\begin{aligned} Y4[-2] &= f = a_4(-2)^4 + a_3(-2)^3 + a(-2)^2 + b(-2) + c \\ Y4[-1] &= g = a_4(-1)^4 + a_3(-1)^3 + a(-1)^2 + b(-1) + c \\ Y4[0] &= h = c \\ Y4[1] &= i = a_4 + a_3 + a + b + c \\ Y4[2] &= j = a_4(2)^4 + a_3(2)^3 + a(2)^2 + b(2) + c \end{aligned}$$

Then,

$$\begin{aligned} f &= 16a_4 - 8a_3 + 4a - 2b + c \\ g &= a_4 - a_3 + a - b + c \\ h &= c \\ i &= a_4 + a_3 + a + b + c \\ j &= 16a_4 + 8a_3 + 4a + 2b + c \end{aligned}$$

Using Cramer's Rule:

$$\Delta = \begin{vmatrix} 16 & -8 & 4 & -2 & 1 \\ 1 & -1 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 16 & 8 & 4 & 2 & 1 \end{vmatrix} = 288$$

$$\Delta_{a_4} = \begin{vmatrix} f & -8 & 4 & -2 & 1 \\ g & -1 & 1 & -1 & 1 \\ h & 0 & 0 & 0 & 1 \\ i & 1 & 1 & 1 & 1 \\ j & 8 & 4 & 2 & 1 \end{vmatrix} = 12f - 48g + 72h - 48i + 12j$$

$$\Delta_{a_3} = \begin{vmatrix} 16 & f & 4 & -2 & 1 \\ 1 & g & 1 & -1 & 1 \\ 0 & h & 0 & 0 & 1 \\ 1 & i & 1 & 1 & 1 \\ 16 & j & 4 & 2 & 1 \end{vmatrix} = -24f + 48g - 48i + 24j$$

$$\Delta_{a} = \begin{vmatrix} 16 & -8 & f & -2 & 1 \\ 1 & -1 & g & -1 & 1 \\ 0 & 0 & h & 0 & 1 \\ 1 & 1 & i & 1 & 1 \\ 16 & 8 & j & 2 & 1 \end{vmatrix} = -12f + 192g - 360h + 192i - 12j$$

$$\Delta_b = \begin{vmatrix} 16 & -8 & 4 & f & 1 \\ 1 & -1 & 1 & g & 1 \\ 0 & 0 & 0 & h & 1 \\ 1 & 1 & 1 & i & 1 \\ 16 & 8 & 4 & j & 1 \end{vmatrix} = 24f - 192g + 192i - 24j$$

$$\Delta_c = \begin{vmatrix} 16 & -8 & 4 & -2 & f \\ 1 & -1 & 1 & -1 & g \\ 0 & 0 & 0 & 0 & h \\ 1 & 1 & 1 & 1 & i \\ 16 & 8 & 4 & 2 & j \end{vmatrix} = 288h$$

Then,

$$a_4 = \frac{\Delta_{a_4}}{\Delta} = \frac{12f - 48g + 72h - 48i + 12j}{288} = \frac{f - 4g + 6h - 4i + j}{24}$$

$$a_3 = \frac{\Delta_{a_3}}{\Delta} = \frac{-24f + 48g - 48i + 24j}{288} = \frac{-f + 2g - 2i + j}{12}$$

$$a = \frac{\Delta_a}{\Delta} = \frac{-12f + 192g - 360h + 192i - 12j}{288} = \frac{-f + 16g - 30h + 16i - j}{24}$$

$$b = \frac{\Delta_b}{\Delta} = \frac{24f - 192g + 192i - 24j}{288} = \frac{f - 8g + 8i - j}{12}$$

$$c = \frac{\Delta_c}{\Delta} = \frac{288h}{288} = h$$

Conclusion: The general most simple equation for polynomial 4th-degree is

$$Y4[y] = \frac{f - 4g + 6h - 4i + j}{24}y^4 + \frac{-f + 2g - 2i + j}{12}y^3 + \frac{-f + 16g - 30h + 16i - j}{24}y^2 + \frac{f - 8g + 8i - j}{12}y + h$$

5.1 Inflection Point in 4th-degree polynomials

The quartic polynomial inflection point is defined as being

$$\frac{d^3Y4[y]}{dy^3} = 0$$

$$\frac{d^3(a_4y^4 + a_3y^3 + ay^2 + by + c)}{dy^3} = 0$$

$$4!\,a_4y_{ip} + 3!\,a_3 = 0$$

$$y_{ip_{Y4[y]}}\left[@\frac{d^3Y4[y]}{dy^3} = 0\right] = -\frac{3!\,a_3}{4!\,a_4} = -\frac{1}{4}\left(\frac{\frac{-f + 2g - 2i + j}{12}}{\frac{f - 4g + 6h - 4i + j}{24}}\right)$$

$$= -\frac{1}{2}\left(\frac{-f + 2g - 2i + j}{f - 4g + 6h - 4i + j}\right)$$

5.2 Recurrence equation towards increasing index in 4th-degree polynomials

The general simplest equation of a quartic polynomial

$$Y4[y]=\frac{f-4g+6h-4i+j}{24}y^4+\frac{-f+2g-2i+j}{12}y^3+\frac{-f+16g-30h+16i-j}{24}y^2+\frac{f-8g+8i-j}{12}y+h$$

Because of the initial dots

$$Y4[-2]=f$$
$$Y4[-1]=g$$
$$Y4[0]=h$$
$$Y4[1]=i$$
$$Y4[2]=j$$

Then, the next term will be $Y4[3]$ in the positive direction of the index y:

$$Y4[3]=\frac{f-4g+6h-4i+j}{24}3^4+\frac{-f+2g-2i+j}{12}3^3+\frac{-f+16g-30h+16i-j}{24}3^2+\frac{f-8g+8i-j}{12}3+h$$

$$Y4[3]=\frac{27f-108g+162h-108i+27j}{8}+\frac{-9f+18g-18i+9j}{4}+\frac{-3f+48g-90h+48i-3j}{8}+\frac{f-8g+8i-j}{4}+h$$

$$Y4[3]=\frac{27f-108g+162h-108i+27j}{8}+\frac{-18f+36g-36i+18j}{8}+\frac{-3f+48g-90h+48i-3j}{8}+\frac{2f-16g+16i-2j}{8}+\frac{8h}{8}$$

$$Y4[3]=\frac{8f-40g+80h-80i+40j}{8}$$

$$Y4[3]=f-5g+10h-10i+5j$$

Then, substituting the letters

$$Y4[3]=Y4[-2]-5Y4[-1]+10Y4[0]-10Y4[1]+5Y4[2]$$

So, the positive index direction recurrence equation of Quartic polynomials is

$$Y4[y]=Y4[y-5]-5Y4[y-4]+10Y4[y-3]-10Y4[y-2]+5Y4[y-1]$$

5.3 Recurrence equation towards decreasing index in 4th-degree polynomials

The general simplest equation of a quartic polynomial

$$Y4[y]=\frac{f-4g+6h-4i+j}{24}y^4+\frac{-f+2g-2i+j}{12}y^3+\frac{-f+16g-30h+16i-j}{24}y^2+\frac{f-8g+8i-j}{12}y+h$$

Because of the initial dots

$$Y4[-2]=f$$
$$Y4[-1]=g$$
$$Y4[0]=h$$

$$Y4[1] = i$$
$$Y4[2] = j$$

Then, the next term will be $Y4[-3]$ in the negative direction of the index y:

$$Y4[-3] = \frac{f - 4g + 6h - 4i + j}{24}(-3)^4 + \frac{-f + 2g - 2i + j}{12}(-3)^3 + \frac{-f + 16g - 30h + 16i - j}{24}(-3)^2 + \frac{f - 8g + 8i - j}{12}(-3) + h$$

$$Y4[-3] = \frac{27f - 108g + 162h - 108i + 27j}{8} + \frac{9f - 18g + 18i - 9j}{4} + \frac{-3f + 48g - 90h + 48i - 3j}{8} + \frac{-f + 8g - 8i + j}{4} + h$$

$$Y4[-3] = \frac{27f - 108g + 162h - 108i + 27j}{8} + \frac{18f - 36g + 36i - 18j}{8} + \frac{-3f + 48g - 90h + 48i - 3j}{8} + \frac{-2f + 16g - 16i + 2j}{8} + \frac{8h}{8}$$

$$Y4[-3] = \frac{40f - 80g + 80h - 40i + 8j}{8}$$

$$Y4[-3] = 5f - 10g + 80h - 5i + j$$

Then, substituting the letters

$$Y4[-3] = 5Y4[-2] - 10Y4[-1] - 10Y4[0] - 5Y4[1] + Y4[2]$$

So, the negative index direction recurrence equation of quartic polynomials is

$$\backslash Y4[y]\backslash = 5Y4[y+1] - 10Y4[y+2] - 10Y4[y+3] - 5Y4[y+4] + Y4[y+5]$$

6 The simplest equation for 5th-degree polynomials (Quintic)

From our definition of notation, the general polynomial equation of degree 5, is

$$Y5[y] = a_5 y^5 + a_4 y^4 + a_3 y^3 + a y^2 + b y + c$$

We have to determine the value of 6 coefficients given by a_5, a_4, a_3, a, b, c. Then, to determine all the coefficients it is easier to choose 6 consecutive elements from $Y5[y]$.

As we learned from the offset study, the simpler equation will be obtained with one of the consecutive elements in the central index $Y5[0]$ in $y = 0$.

So, we have:

$$\begin{aligned}
Y5[-2] &= f = a_5(-2)^5 + a_4(-2)^4 + a_3(-2)^3 + a(-2)^2 + b(-2) + c \\
Y5[-1] &= g = a_5(-1)^5 + a_4(-1)^4 + a_3(-1)^3 + a(-1)^2 + b(-1) + c \\
Y5[0] &= h = c \\
Y5[1] &= i = a_5 + a_4 + a_3 + a + b + c \\
Y5[2] &= j = a_5(2)^5 + a_4(2)^4 + a_3(2)^3 + a(2)^2 + b(2) + c \\
Y5[3] &= k = a_5(3)^5 + a_4(3)^4 + a_3(3)^3 + a(3)^2 + b(3) + c
\end{aligned}$$

We have a linear system:

$$\begin{aligned}
-32a_5 + 16a_4 - 8a_3 + 4a - 2b + c &= f \\
-a_5 + a_4 - a_3 + a - b + c &= g \\
c &= h \\
a_5 + a_4 + a_3 + a + b + c &= i \\
32a_5 + 16a_4 + 8a_3 + 4a + 2b + c &= j \\
243a_5 + 81a_4 + 27a_3 + 9a + 3b + c &= k
\end{aligned}$$

Using Cramer's Rule:

$$\Delta = \begin{vmatrix} -32 & 16 & -8 & 4 & -2 & 1 \\ -1 & 1 & -1 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 32 & 16 & 8 & 4 & 2 & 1 \\ 243 & 81 & 27 & 9 & 3 & 1 \end{vmatrix} = -34560$$

$$\Delta_{a_5} = \begin{vmatrix} f & 16 & -8 & 4 & -2 & 1 \\ g & 1 & -1 & 1 & -1 & 1 \\ h & 0 & 0 & 0 & 0 & 1 \\ i & 1 & 1 & 1 & 1 & 1 \\ j & 16 & 8 & 4 & 2 & 1 \\ k & 81 & 27 & 9 & 3 & 1 \end{vmatrix} = 288f - 1440g + 2880h - 2880i + 1440j - 288k$$

$$\Delta_{a_4} = \begin{vmatrix} -32 & f & -8 & 4 & -2 & 1 \\ -1 & g & -1 & 1 & -1 & 1 \\ 0 & h & 0 & 0 & 0 & 1 \\ 1 & i & 1 & 1 & 1 & 1 \\ 32 & j & 8 & 4 & 2 & 1 \\ 243 & k & 27 & 9 & 3 & 1 \end{vmatrix} = -1440f + 5760g - 8640h + 5760i - 1440j$$

$$\Delta_{a_3} = \begin{vmatrix} -32 & 16 & f & 4 & -2 & 1 \\ -1 & 1 & g & 1 & -1 & 1 \\ 0 & 0 & h & 0 & 0 & 1 \\ 1 & 1 & i & 1 & 1 & 1 \\ 32 & 16 & j & 4 & 2 & 1 \\ 243 & 81 & k & 9 & 3 & 1 \end{vmatrix}$$

$$= 1440f + 1440g - 14400h + 20160i - 10080j + 1440k$$

$$\Delta_a = \begin{vmatrix} -32 & 16 & -8 & f & -2 & 1 \\ -1 & 1 & -1 & g & -1 & 1 \\ 0 & 0 & 0 & h & 0 & 1 \\ 1 & 1 & 1 & i & 1 & 1 \\ 32 & 16 & 8 & j & 2 & 1 \\ 243 & 81 & 27 & k & 3 & 1 \end{vmatrix} = 1440f - 23040g + 43200h - 23040i + 1440j$$

$$\Delta_b = \begin{vmatrix} -32 & 16 & -8 & 4 & f & 1 \\ -1 & 1 & -1 & 1 & g & 1 \\ 0 & 0 & 0 & 0 & h & 1 \\ 1 & 1 & 1 & 1 & i & 1 \\ 32 & 16 & 8 & 4 & j & 1 \\ 243 & 81 & 27 & 9 & k & 1 \end{vmatrix}$$

$$= -1728f + 17280g + 11520h - 34560i + 8640j - 1152k$$

$$\Delta_c = \begin{vmatrix} -32 & 16 & -8 & 4 & -2 & f \\ -1 & 1 & -1 & 1 & -1 & g \\ 0 & 0 & 0 & 0 & 0 & h \\ 1 & 1 & 1 & 1 & 1 & i \\ 32 & 16 & 8 & 4 & 2 & j \\ 243 & 81 & 27 & 9 & 3 & k \end{vmatrix} = -34560h$$

Then,

$$a_5 = \frac{\Delta_{a_5}}{\Delta} = \frac{288f - 1440g + 2880h - 2880i + 1440j - 288k}{-34560}$$

$$= \frac{-f + 5g - 10h + 10i - 5j + k}{120}$$

$$a_4 = \frac{\Delta_{a_4}}{\Delta} = \frac{-1440f + 5760g - 8640h + 5760i - 1440j}{-34560} = \frac{f - 4g + 6h - 4i + j}{24}$$

$$a_3 = \frac{\Delta_{a_3}}{\Delta} = \frac{1440f + 1440g - 14400h + 20160i - 10080j + 1440k}{-34560}$$

$$= \frac{-f - g + 10h - 14i + 7j - k}{24}$$

$$a = \frac{\Delta_a}{\Delta} = \frac{1440f - 23040g + 43200h - 23040i + 1440j}{-34560} = \frac{-f + 16g - 30h + 16i - j}{24}$$

$$b = \frac{\Delta_b}{\Delta} = \frac{-1728f + 17280g + 11520h - 34560i + 8640j - 1152k}{-34560}$$

$$= \frac{3f - 30g - 20h + 60i - 15j + 2k}{60}$$

$$c = \frac{\Delta_c}{\Delta} = \frac{-34560h}{-34560} = h$$

Conclusion: The general most simple equation for polynomial 5th-degree is

$$Y5[y] = \frac{-f+5g-10h+10i-5j+k}{120}y^5 + \frac{f-4g+6h-4i+j}{24}y^4 + \frac{-f-g+10h-14i+7j-k}{24}y^3 + \frac{-f+16g-30h+16i-j}{24}y^2 + \frac{3f-30g-20h+60i-15j+2k}{60}y + h$$

6.1 Inflection Point in 5th-degree polynomials

The quintic polynomial inflection point is defined as being

$$\frac{d^4Y5[y]}{dy^4} = 0$$

$$\frac{d^4(a_5y^5 + a_4y^4 + a_3y^3 + ay^2 + by + c)}{dy^4} = 0$$

$$5!\,a_5y_{ip} + 4!\,a_4 = 0$$

$$y_{ip_{Y5[y]}}\left[@\frac{d^4Y5[y]}{dy^4} = 0\right] = -\frac{4!\,a_4}{5!\,a_5} = -\frac{1}{5}\left(\frac{\frac{f-4g+6h-4i+j}{24}}{\frac{-f+5g-10h+10i-5j+k}{120}}\right) = -1\left(\frac{f-4g+6h-4i+j}{-f+5g-10h+10i-5j+k}\right)$$

6.2 Recurrence equation towards increasing index in 5th-degree polynomials

The general simplest equation of a quintic polynomial

$$Y5[y] = \frac{-f+5g-10h+10i-5j+k}{120}y^5 + \frac{f-4g+6h-4i+j}{24}y^4 + \frac{-f-g+10h-14i+7j-k}{24}y^3 + \frac{-f+16g-30h+16i-j}{24}y^2 + \frac{3f-30g-20h+60i-15j+2k}{60}y + h$$

Because of the initial dots

$$Y5[-2] = f$$
$$Y5[-1] = g$$
$$Y5[0] = h$$
$$Y5[1] = i$$
$$Y5[2] = j$$
$$Y5[3] = k$$

Then, the next term will be $Y5[4]$ in the positive direction of the index y:

$$Y5[4] = \frac{-f+5g-10h+10i-5j+k}{120}4^5 + \frac{f-4g+6h-4i+j}{24}4^4 + \frac{-f-g+10h-14i+7j-k}{24}4^3 + \frac{-f+16g-30h+16i-j}{24}4^2 + \frac{3f-30g-20h+60i-15j+2k}{60}4 + h$$

$$Y5[4] = \frac{-f + 5g - 10h + 10i - 5j + k}{120}1024 + \frac{f - 4g + 6h - 4i + j}{24}256 + \frac{-f - g + 10h - 14i + 7j - k}{24}64 + \frac{-f + 16g - 30h + 16i - j}{24}16 + \frac{3f - 30g - 20h + 60i - 15j + 2k}{60}4 + h$$

$$Y5[4] = \frac{-128f + 640g - 1280h + 1280i - 640j + 128k}{15} + \frac{32f - 128g + 192h - 128i + 32j}{3} + \frac{-8f - 8g + 80h - 112i + 56j - 8k}{3} + \frac{-2f + 32g - 60h + 32i - 2j}{3} + \frac{3f - 30g - 20h + 60i - 15j + 2k}{15} + h$$

$$Y5[4] = \frac{-128f + 640g - 1280h + 1280i - 640j + 128k}{15} + \frac{160f - 640g + 960h - 640i + 160j}{15} + \frac{-40f - 40g + 400h - 560i + 280j - 40k}{15} + \frac{-10f + 160g - 300h + 160i - 10j}{15} + \frac{3f - 30g - 20h + 60i - 15j + 2k}{15} + \frac{15h}{15}$$

$$Y5[4] = \frac{-15f + 90g - 225h + 300i - 225j + 90k}{15} = -f + 6g - 15h + 20i - 15j + 6k$$

Then, substituting the letters

$$Y5[4] = -Y5[-2] + 6Y5[-1] - 15Y5[0] + 20Y5[1] - 15Y5[2] + 6Y5[3]$$

So, the positive index direction recurrence equation of quintic polynomials is

$$Y5[y] = -Y5[y-6] + 6Y5[y-5] - 15Y5[y-4] + 20Y5[y-3] - 15Y5[y-2] + 6Y5[y-1]$$

6.3 Recurrence equation towards decreasing index in 5th-degree polynomials

The general simplest equation of a quintic polynomial

$$Y5[y] = \frac{-f + 5g - 10h + 10i - 5j + k}{120}y^5 + \frac{f - 4g + 6h - 4i + j}{24}y^4 + \frac{-f - g + 10h - 14i + 7j - k}{24}y^3 + \frac{-f + 16g - 30h + 16i - j}{24}y^2 + \frac{3f - 30g - 20h + 60i - 15j + 2k}{60}y + h$$

Because of the initial dots

$$Y5[-2] = f$$
$$Y5[-1] = g$$
$$Y5[0] = h$$
$$Y5[1] = i$$
$$Y5[2] = j$$
$$Y5[3] = k$$

Then, the next term will be $Y5[-3]$ in the positive direction of the index y:

$$Y5[-3] = \frac{-f+5g-10h+10i-5j+k}{120}(-3)^5 + \frac{f-4g+6h-4i+j}{24}(-3)^4 + \frac{-f-g+10h-14i+7j-k}{24}(-3)^3 + \frac{-f+16g-30h+16i-j}{24}(-3)^2 + \frac{3f-30g-20h+60i-15j+2k}{60}(-3) + h$$

$$Y5[-3] = \frac{-f+5g-10h+10i-5j+k}{40}(-81) + \frac{f-4g+6h-4i+j}{8}27 + \frac{-f-g+10h-14i+7j-k}{8}(-9) + \frac{-f+16g-30h+16i-j}{8}3 + \frac{3f-30g-20h+60i-15j+2k}{20}(-1) + h$$

$$Y5[-3] = \frac{-f+5g-10h+10i-5j+k}{40}(-81) + \frac{f-4g+6h-4i+j}{40}135 + \frac{-f-g+10h-14i+7j-k}{40} + \frac{-f+16g-30h+16i-j}{40}15 + \frac{3f-30g-20h+60i-15j+2k}{40}(-2) + h$$

$$Y5[-3] = \frac{81f-405g+810h-810i+405j-81k}{40} + \frac{135f-540g+810h-540i+135j}{40} + \frac{45f+45g-450h+630i-315j+45k}{40} + \frac{-15f+240g-450h+240i-15j}{40} + \frac{-6f+60g+40h-120i+30j-4k}{40} + \frac{40h}{40}$$

$$Y5[-3] = \frac{240f-600g+800h-600i+240j-40k}{40} = 6f-10g+20h-15i+6j-k$$

Then, substituting the letters

$$Y5[-3] = 6Y5[-2] - 10Y5[-1] + 20Y5[0] - 15Y5[1] + 6Y5[2] - Y5[3]$$

So, the negative index direction recurrence equation of quintic polynomials is

$$\backslash Y5[y]\backslash = 6Y5[y+1] - 10Y5[y+2] + 20Y5[y+3] - 15Y5[y+4] + 6Y5[y+5] - Y5[y+6]$$

7 The simplest equation for 6^{th}-degree polynomials (Sextic)

From our definition of notation, the general polynomial equation of degree 6, is

$$Y6[y] = a_6 y^6 + a_5 y^5 + a_4 y^4 + a_3 y^3 + a y^2 + b y + c$$

We have to determine the value of 7 coefficients given by $a_6, a_5, a_4, a_3, a, b, c$. Then, to determine all the coefficients it is easier to choose 7 consecutive elements from $Y6[y]$.

As we learned from the offset study, the simpler equation will be obtained with one of the consecutive elements in the central index $Y6[0]$ in $y = 0$.

So, we have:

$$\begin{aligned}
Y6[-3] &= e = a_6(-3)^6 + a_5(-3)^5 + a_4(-3)^4 + a_3(-3)^3 + a(-3)^2 + b(-3) + c \\
Y6[-2] &= f = a_6(-2)^6 + a_5(-2)^5 + a_4(-2)^4 + a_3(-2)^3 + a(-2)^2 + b(-2) + c \\
Y6[-1] &= g = a_6(-1)^6 + a_5(-1)^5 + a_4(-1)^4 + a_3(-1)^3 + a(-1)^2 + b(-1) + c \\
Y6[0] &= h = c \\
Y6[1] &= i = a_6 + a_5 + a_4 + a_3 + a + b + c \\
Y6[2] &= j = a_6(2)^6 + a_5(2)^5 + a_4(2)^4 + a_3(2)^3 + a(2)^2 + b(2) + c \\
Y6[3] &= k = a_6(3)^6 + a_5(3)^5 + a_4(3)^4 + a_3(3)^3 + a(3)^2 + b(3) + c
\end{aligned}$$

We have a linear system:

$$\begin{aligned}
Y6[-3] &= e = 729a_6 - 243a_5 + 81a_4 - 27a_3 + 9a - 3b + c \\
Y6[-2] &= f = 64a_6 - 32a_5 + 16a_4 - 8a_3 + 4a - 2b + c \\
Y6[-1] &= g = a_6 - a_5 + a_4 - a_3 + a - b + c \\
Y6[0] &= h = c \\
Y6[1] &= i = a_6 + a_5 + a_4 + a_3 + a + b + c \\
Y6[2] &= j = 64a_6 + 32a_5 + 16a_4 + 8a_3 + 4a + 2b + c \\
Y6[3] &= k = 729a_6 + 243a_5 + 81a_4 + 27a_3 + 9a + 3b + c
\end{aligned}$$

Using Cramer's Rule:

$$\Delta = \begin{vmatrix} 729 & -243 & 81 & -27 & 9 & -3 & 1 \\ 64 & -32 & 16 & -8 & 4 & -2 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 64 & 32 & 16 & 8 & 4 & 2 & 1 \\ 729 & 243 & 81 & 27 & 9 & 3 & 1 \end{vmatrix} = -24883200$$

$$\Delta_{a_6} = \begin{vmatrix} e & -243 & 81 & -27 & 9 & -3 & 1 \\ f & -32 & 16 & -8 & 4 & -2 & 1 \\ g & -1 & 1 & -1 & 1 & -1 & 1 \\ h & 0 & 0 & 0 & 0 & 0 & 1 \\ i & 1 & 1 & 1 & 1 & 1 & 1 \\ j & 32 & 16 & 8 & 4 & 2 & 1 \\ k & 243 & 81 & 27 & 9 & 3 & 1 \end{vmatrix}$$

$$= -34560e + 207360f - 518400g + 691200h - 518400i + 207360j - 34560k$$

$$\Delta_{a_5}=\begin{vmatrix}729 & e & 81 & -27 & 9 & -3 & 1\\ 64 & f & 16 & -8 & 4 & -2 & 1\\ 1 & g & 1 & -1 & 1 & -1 & 1\\ 0 & h & 0 & 0 & 0 & 0 & 1\\ 1 & i & 1 & 1 & 1 & 1 & 1\\ 64 & j & 16 & 8 & 4 & 2 & 1\\ 729 & k & 81 & 27 & 9 & 3 & 1\end{vmatrix}$$

$$= 103680e - 414720f + 518400g - 518400i + 414720j - 103680k$$

$$\Delta_{a_4}=\begin{vmatrix}729 & -243 & e & -27 & 9 & -3 & 1\\ 64 & -32 & f & -8 & 4 & -2 & 1\\ 1 & -1 & g & -1 & 1 & -1 & 1\\ 0 & 0 & h & 0 & 0 & 0 & 1\\ 1 & 1 & i & 1 & 1 & 1 & 1\\ 64 & 32 & j & 8 & 4 & 2 & 1\\ 729 & 243 & k & 27 & 9 & 3 & 1\end{vmatrix}$$

$$= 172800e - 2073600f + 6739200g - 9676800h + 6739200i - 2073600j + 172800k$$

$$\Delta_{a_3}=\begin{vmatrix}729 & -243 & 81 & e & 9 & -3 & 1\\ 64 & -32 & 16 & f & 4 & -2 & 1\\ 1 & -1 & 1 & g & 1 & -1 & 1\\ 0 & 0 & 0 & h & 0 & 0 & 1\\ 1 & 1 & 1 & i & 1 & 1 & 1\\ 64 & 32 & 16 & j & 4 & 2 & 1\\ 729 & 243 & 81 & k & 9 & 3 & 1\end{vmatrix}$$

$$= -518400e + 4147200f - 6739200g + 6739200i - 4147200j + 518400k$$

$$\Delta_{a}=\begin{vmatrix}729 & -243 & 81 & -27 & e & -3 & 1\\ 64 & -32 & 16 & -8 & f & -2 & 1\\ 1 & -1 & 1 & -1 & g & -1 & 1\\ 0 & 0 & 0 & 0 & h & 0 & 1\\ 1 & 1 & 1 & 1 & i & 1 & 1\\ 64 & 32 & 16 & 8 & j & 2 & 1\\ 729 & 243 & 81 & 27 & k & 3 & 1\end{vmatrix}$$

$$= -138240e + 1866240f - 18662400g + 33868800h - 18662400i + 1866240j - 138240k$$

$$\Delta_{b}=\begin{vmatrix}729 & -243 & 81 & -27 & 9 & e & 1\\ 64 & -32 & 16 & -8 & 4 & f & 1\\ 1 & -1 & 1 & -1 & 1 & g & 1\\ 0 & 0 & 0 & 0 & 0 & h & 1\\ 1 & 1 & 1 & 1 & 1 & i & 1\\ 64 & 32 & 16 & 8 & 4 & j & 1\\ 729 & 243 & 81 & 27 & 9 & k & 1\end{vmatrix}$$

$$= 414720e - 3732480f + 18662400g - 18662400i + 3732480j - 414720k$$

$$\Delta_c=\begin{vmatrix}729 & -243 & 81 & -27 & 9 & -3 & e\\ 64 & -32 & 16 & -8 & 4 & -2 & f\\ 1 & -1 & 1 & -1 & 1 & -1 & g\\ 0 & 0 & 0 & 0 & 0 & 0 & h\\ 1 & 1 & 1 & 1 & 1 & 1 & i\\ 64 & 32 & 16 & 8 & 4 & 2 & j\\ 729 & 243 & 81 & 27 & 9 & 3 & k\end{vmatrix}=-24883200h$$

$$a_6=\frac{\Delta_{a_6}}{\Delta}$$
$$=\frac{-34560e+207360f-518400g+691200h-518400i+207360j-34560k}{-24883200}$$
$$a_5=\frac{\Delta_{a_5}}{\Delta}=\frac{103680e-414720f+518400g-518400i+414720j-103680k}{-24883200}$$
$$a_4=\frac{\Delta_{a_4}}{\Delta}$$
$$=\frac{172800e-2073600f+6739200g-9676800h+6739200i-2073600j+172800k}{-24883200}$$
$$a_3=\frac{\Delta_{a_3}}{\Delta}=\frac{-518400e+4147200f-6739200g+6739200i-4147200j+518400k}{-34560}$$
$$a=\frac{\Delta_a}{\Delta}$$
$$=\frac{-138240e+1866240f-18662400g+33868800h-18662400i+1866240j-138240k}{-34560}$$
$$b=\frac{\Delta_b}{\Delta}=\frac{414720e-3732480f+18662400g-18662400i+3732480j-414720k}{-34560}$$
$$c=\frac{\Delta_c}{\Delta}=\frac{-24883200h}{-24883200}$$

Or

$$a_6=\frac{\Delta_{a_6}}{\Delta}=\frac{e-6f+15g-20h+15\mathrm{i}-6j+k}{720}$$
$$a_5=\frac{\Delta_{a_5}}{\Delta}=\frac{-e+4f-5g+5\mathrm{i}-4j+k}{240}$$
$$a_4=\frac{\Delta_{a_4}}{\Delta}=\frac{-e+12f-39g+56h-39\mathrm{i}+12j-k}{144}$$
$$a_3=\frac{\Delta_{a_3}}{\Delta}=\frac{e-8f+13g-13\mathrm{i}+8j-k}{48}$$
$$a=\frac{\Delta_a}{\Delta}=\frac{2e-27f+270g-490h+270\mathrm{i}-27j+2k}{360}\ =$$
$$b=\frac{\Delta_b}{\Delta}=\frac{-e+9f-45g+45\mathrm{i}-9j+k}{60}$$
$$c=\frac{\Delta_c}{\Delta}=h$$

Conclusion: The general most simple equation for polynomial 6^{th}-degree is

$$Y6[y] = \frac{e - 6f + 15g - 20h + 15i - 6j + k}{720} y^6 + \frac{-e + 4f - 5g + 5i - 4j + k}{240} y^5 + \frac{-e + 12f - 39g + 56h - 39i + 12j - k}{144} y^4 + \frac{e - 8f + 13g - 13i + 8j - k}{48} y^3 + \frac{2e - 27f + 270g - 490h + 270i - 27j + 2k}{360} y^2 + \frac{-e + 9f - 45g + 45i - 9j + k}{60} y + h$$

7.1 Inflection Point in 6th-degree polynomials

The inflection point is defined as being

$$\frac{d^5 Y6[y]}{dy^5} = 0$$

$$\frac{d^5(a_6 y^6 + a_5 y^5 + a_4 y^4 + a_3 y^3 + a y^2 + by + c)}{dy^5} = 0$$

$$6!\, a_6 y_{ip} + 5!\, a_5 = 0$$

$$y_{ip_{Y6[y]}}\left[@\frac{d^5 Y6[y]}{dy^5} = 0\right] = -\frac{5!\, a_5}{6!\, a_6} = -\frac{1}{6}\left(\frac{\frac{-e + 4f - 5g + 5i - 4j + k}{240}}{\frac{e - 6f + 15g - 20h + 15i - 6j + k}{720}}\right)$$

$$= -\frac{1}{2}\left(\frac{-e + 4f - 5g + 5i - 4j + k}{e - 6f + 15g - 20h + 15i - 6j + k}\right)$$

7.2 Recurrence equation towards increasing index in 6th-degree polynomials

The general simplest equation of a sextic polynomial

$$Y6[y] = \frac{e - 6f + 15g - 20h + 15i - 6j + k}{720} y^6 + \frac{-e + 4f - 5g + 5i - 4j + k}{240} y^5 + \frac{-e + 12f - 39g + 56h - 39i + 12j - k}{144} y^4 + \frac{e - 8f + 13g - 13i + 8j - k}{48} y^3 + \frac{2e - 27f + 270g - 490h + 270i - 27j + 2k}{360} y^2 + \frac{-e + 9f - 45g + 45i - 9j + k}{60} y + h$$

Because of the initial dots

$$Y6[-3] = e$$
$$Y6[-2] = f$$
$$Y6[-1] = g$$
$$Y6[0] = h$$
$$Y6[1] = i$$
$$Y6[2] = j$$

$$Y6[3] = k$$

Then, the next term will be $Y6[4]$ in the positive direction of the index y:

$$Y6[4] = \frac{e - 6f + 15g - 20h + 15i - 6j + k}{720}4^6 + \frac{-e + 4f - 5g + 5i - 4j + k}{240}4^5 + \frac{-e + 12f - 39g + 56h - 39i + 12j - k}{144}4^4 + \frac{e - 8f + 13g - 13i + 8j - k}{48}4^3 + \frac{2e - 27f + 270g - 490h + 270i - 27j + 2k}{360}4^2 + \frac{-e + 9f - 45g + 45i - 9j + k}{60}4 + h$$

$$Y6[4] = \frac{e - 6f + 15g - 20h + 15i - 6j + k}{720}4096 + \frac{-e + 4f - 5g + 5i - 4j + k}{240}1024 + \frac{-e + 12f - 39g + 56h - 39i + 12j - k}{144}256 + \frac{e - 8f + 13g - 13i + 8j - k}{48}64 + \frac{2e - 27f + 270g - 490h + 270i - 27j + 2k}{360}16 + \frac{-e + 9f - 45g + 45i - 9j + k}{60}4 + h$$

$$Y6[4] = \frac{e - 6f + 15g - 20h + 15i - 6j + k}{45}256 + \frac{-e + 4f - 5g + 5i - 4j + k}{15}64 + \frac{-e + 12f - 39g + 56h - 39i + 12j - k}{9}16 + \frac{e - 8f + 13g - 13i + 8j - k}{3}4 + \frac{2e - 27f + 270g - 490h + 270i - 27j + 2k}{45}2 + \frac{-e + 9f - 45g + 45i - 9j + k}{15} + h$$

$$Y6[4] = \frac{e - 6f + 15g - 20h + 15i - 6j + k}{45}256 + \frac{-e + 4f - 5g + 5i - 4j + k}{45}192 + \frac{-e + 12f - 39g + 56h - 39i + 12j - k}{45}80 + \frac{e - 8f + 13g - 13i + 8j - k}{45}60 + \frac{2e - 27f + 270g - 490h + 270i - 27j + 2k}{45}2 + \frac{-e + 9f - 45g + 45i - 9j + k}{45}3 + \frac{45h}{45}$$

$$Y6[4] = \frac{256e - 1536f + 3840g - 5120h + 3840i - 1536j + 256k}{45} + \frac{-192e + 768f - 960g + 960i - 768j + 192k}{45} + \frac{-80e + 960f - 3120g + 4480h - 3120i + 960j - 80k}{45} + \frac{60e - 480f + 780g - 780i + 480j - 60k}{45} + \frac{4e - 54f + 540g - 980h + 540i - 54j + 4k}{45} + \frac{-3e + 27f - 135g + 135i - 27j + 3k}{45} + \frac{45h}{45}$$

$$Y6[4] = \frac{45e - 315f + 945g - 1575h + 1575i - 945j + 315k}{45} = e - 7f + 21g - 35h + 35i - 21j + 7k$$

Then, substituting the letters

$$Y6[4] = Y6[-3] - 7Y6[-2] + 21Y6[-1] - 35Y6[0] + 35Y6[1] - 21Y6[2] + 7Y6[3]$$

So, the positive index direction recurrence equation of sextic polynomials is

$$Y6[y] = Y6[y-7] - 7Y6[y-6] + 21Y6[y-5] - 35Y6[y-4] + 35Y6[y-3] - 21Y6[y-2] + 7Y6[y-1]$$

7.3 Recurrence equation towards decreasing index in 6th-degree polynomials

The general simplest equation of a sextic polynomial

$$Y6[y] = \frac{e - 6f + 15g - 20h + 15i - 6j + k}{720}y^6 + \frac{-e + 4f - 5g + 5i - 4j + k}{240}y^5 + \frac{-e + 12f - 39g + 56h - 39i + 12j - k}{144}y^4 + \frac{e - 8f + 13g - 13i + 8j - k}{48}y^3 + \frac{2e - 27f + 270g - 490h + 270i - 27j + 2k}{360}y^2 + \frac{-e + 9f - 45g + 45i - 9j + k}{60}y + h$$

Because of the initial dots

$$Y6[-3] = e$$
$$Y6[-2] = f$$
$$Y6[-1] = g$$
$$Y6[0] = h$$
$$Y6[1] = i$$
$$Y6[2] = j$$
$$Y6[3] = k$$

Then, the next term will be $Y6[-4]$ in the negative direction of the index y:

$$Y6[-4] = \frac{e - 6f + 15g - 20h + 15i - 6j + k}{720}(-4)^6 + \frac{-e + 4f - 5g + 5i - 4j + k}{240}(-4)^5$$
$$+ \frac{-e + 12f - 39g + 56h - 39i + 12j - k}{144}(-4)^4$$
$$+ \frac{e - 8f + 13g - 13i + 8j - k}{48}(-4)^3$$
$$+ \frac{2e - 27f + 270g - 490h + 270i - 27j + 2k}{360}(-4)^2$$
$$+ \frac{-e + 9f - 45g + 45i - 9j + k}{60}(-4) + h$$

$$Y6[-4] = \frac{e - 6f + 15g - 20h + 15i - 6j + k}{720}4096$$
$$+ \frac{-e + 4f - 5g + 5i - 4j + k}{240}(-1024)$$
$$+ \frac{-e + 12f - 39g + 56h - 39i + 12j - k}{144}256$$
$$+ \frac{e - 8f + 13g - 13i + 8j - k}{48}(-64)$$
$$+ \frac{2e - 27f + 270g - 490h + 270i - 27j + 2k}{360}16$$
$$+ \frac{-e + 9f - 45g + 45i - 9j + k}{60}(-4) + h$$

$$Y6[-4] = \frac{e - 6f + 15g - 20h + 15i - 6j + k}{45}256 + \frac{-e + 4f - 5g + 5i - 4j + k}{15}(-64)$$
$$+ \frac{-e + 12f - 39g + 56h - 39i + 12j - k}{9}16$$
$$+ \frac{e - 8f + 13g - 13i + 8j - k}{3}(-4)$$
$$+ \frac{2e - 27f + 270g - 490h + 270i - 27j + 2k}{45}2$$
$$+ \frac{-e + 9f - 45g + 45i - 9j + k}{15}(-1) + h$$

$$Y6[-4] = \frac{e - 6f + 15g - 20h + 15i - 6j + k}{45}256 + \frac{-e + 4f - 5g + 5i - 4j + k}{45}(-192)$$
$$+ \frac{-e + 12f - 39g + 56h - 39i + 12j - k}{45}80$$
$$+ \frac{e - 8f + 13g - 13i + 8j - k}{45}(-60)$$
$$+ \frac{2e - 27f + 270g - 490h + 270i - 27j + 2k}{45}2$$
$$+ \frac{-e + 9f - 45g + 45i - 9j + k}{45}(-3) + \frac{45h}{45}$$

$$Y6[-4] = \frac{256e - 1536f + 3840g - 5120h + 3840i - 1536j + 256k}{45} + \frac{192e - 768f + 960g - 960i + 768j - 192k}{45} + \frac{-80e + 960f - 3120g + 4480h - 3120i + 960j - 80k}{45} + \frac{-60e + 480f - 780g + 780i - 480j + 60k}{45} + \frac{4e - 54f + 540g - 980h + 540i - 54j + 4k}{45} + \frac{3e - 27f + 135g - 135i + 27j - 3k}{45} + \frac{45h}{45}$$

$$Y6[-4] = \frac{315e - 945f + 1575g - 1575h + 945i - 315j + 45k}{45} = 7e - 21f + 35g - 35h + 21i - 7j + k$$

Then, substituting the letters

$$Y6[-4] = 7Y6[-3] - 21Y6[-2] + 35Y6[-1] - 35Y6[0] + 21Y6[1] - 7Y6[2] + Y6[3]$$

So, the negative index direction recurrence equation of sextic polynomials is

$$\backslash Y6[y]\backslash = 7Y6[y+1] - 21Y6[y+2] + 35Y6[y+3] - 35Y6[y+4] + 21Y6[y+5] - 7Y6[y+6] + Y6[y+7]$$

8 The simplest equation for 0th-degree polynomials (Constant)

We will start with constant polynomials or polynomials 0th-degree.

From our definition of notation, the general polynomial equation of degree 0, is

$$Y0[y] = c$$

We have to determine the value of only one coefficient c. Then, to determine all the coefficients it is easier to choose the only one element from $Y0[y]$.

As we learned from the offset study, the simpler equation will be obtained with one of the consecutive elements in the central index $Y0[0]$ in $y = 0$.

So, we have:

$$Y0[0] = h = c$$

Using Cramer's Rule:

$$\Delta = |1| = 1$$

$$\Delta_c = |h| = h$$

$$c = \frac{\Delta_c}{\Delta} = \frac{h}{1} = h$$

Conclusion: The general most simple equation for polynomial 0th-degree is

$$X0[y] = h$$

8.1 Inflection Point in 0th-degree polynomials

The constant polynomial inflection point is defined as being

$$\frac{d^{-1}Y0[y]}{dy^{-1}} = 0$$

$$y_{vertex_{X0[y]}}\left[@\frac{d^{-1}Y0[y]}{dy^{-1}} = 0\right] = 0$$

8.2 Recurrence equation towards increasing index in 0th-degree polynomials

The general simplest equation

$$Y0[y] = c$$

Because of the initial dots

$$Y0[0] = h = c$$

Then, the next term will be $Y0[1]$ in the positive direction of the index y:

$$Y0[1] = h$$

Then, substituting the letter h

$$Y0[1] = Y0[0]$$

So, the positive index direction recurrence equation of linear polynomials is

$$Y0[y] = Y0[y - 1]$$

8.3 Recurrence equation towards decreasing index in 0^{th}-degree polynomials

The general simplest equation

$$Y0[y] = c$$

Because of the initial dots

$$Y0[0] = h = c$$

Then, the next term will be $Y0[-1]$ in the negative direction of the index y:

$$Y0[-1] = h$$

Then, substituting the letter h

$$Y0[-1] = Y0[0]$$

So, the negative index direction recurrence equation of linear polynomials is

$$\backslash Y0[y]\backslash= Y0[y + 1]$$

9 Fibonacci Recurrence Equation

The Fibonacci sequence has a recurrence equation given by $x_n = x_{n-1} + x_{n-2}$. But this equation can generate infinite many different infinite sequences depending only on the two sequential terms chosen as the starting ones x_{n-1} and x_{n-2}. If we say that $x_n = x_{n-1} + x_{n-2}$ where somewhere in the sequence will occur the appearance of two elements in the sequence given by for example (..., 5,8, ...), then we have the Fibonacci sequence. See some examples of sequences with the same recurrence equation given by $x_n = x_{n-1} + x_{n-2}$.

10 Recurrence Equations of any Polynomial

Because the same recurrence equation can generate infinite many different sequences with equivalent or similar properties, then all recurrences equations should be written in such a way that considers at least one smallest possible portion of the sequence (in any index). The smallest possible portion of the sequence can be considered the generator elements which are the minimum elements in any index sequence necessary and sufficient to determine the infinite sequence.

10.1 Polynomials recurrence equations from Pascal's triangle

Row	Pascal's Triangle - vertical 1's to the left											Σ
1	1											1
2	1	1										2
3	1	2	1									4
4	1	3	3	1								8
5	1	4	6	4	1							16
6	1	5	10	10	5	1						32
7	1	6	15	20	15	6	1					64
8	1	7	21	35	35	21	7	1				128
9	1	8	28	56	70	56	28	8	1			256
10	1	9	36	84	126	126	84	36	9	1		512
11	1	10	45	120	210	252	210	120	45	10	1	1024

Row	Pascal's Triangle - vertical 1's to the right										
1											1
2										1	1
3									1	2	1
4								1	3	3	1
5							1	4	6	4	1
6						1	5	10	10	5	1
7					1	6	15	20	15	6	1
8				1	7	21	35	35	21	7	1
9			1	8	28	56	70	56	28	8	1
10		1	9	36	84	126	126	84	36	9	1
11	1	10	45	120	210	252	210	120	45	10	1

Row	Pascal's Triangle - vertical 1's to the left and signalized											Σ
1	-1											-1
2	-1	1										0
3	-1	2	-1									0
4	-1	3	-3	1								0
5	-1	4	-6	4	-1							0
6	-1	5	-10	10	-5	1						0
7	-1	6	-15	20	-15	6	-1					0
8	-1	7	-21	35	-35	21	-7	1				0
9	-1	8	-28	56	-70	56	-28	8	-1			0
10	-1	9	-36	84	-126	126	-84	36	-9	1		0
11	-1	10	-45	120	-210	252	-210	120	-45	10	-1	0

Row	Pascal's Triangle - vertical 1's to the right and signalized										
1											-1
2										1	-1
3									-1	2	-1
4								1	-3	3	-1
5							-1	4	-6	4	-1
6						1	-5	10	-10	5	-1
7					-1	6	-15	20	-15	6	-1
8				1	-7	21	-35	35	-21	7	-1
9			-1	8	-28	56	-70	56	-28	8	-1
10		1	-9	36	-84	126	-126	84	-36	9	-1
11	-1	10	-45	120	-210	252	-210	120	-45	10	-1

Row	Recurrence coefficients towards decreasing index											Σ
1	degree											
2	0	1										1
3	1	2	-1									1
4	2	3	-3	1								1
5	3	4	-6	4	-1							1
6	4	5	-10	10	-5	1						1
7	5	6	-15	20	-15	6	-1					1
8	6	7	-21	35	-35	21	-7	1				1
9	7	8	-28	56	-70	56	-28	8	-1			1
10	8	9	-36	84	-126	126	-84	36	-9	1		1
11	9	10	-45	120	-210	252	-210	120	-45	10	-1	1

Row	Recurrence coefficients towards increasing index										
1											degree
2										1	0
3									-1	2	1
4								1	-3	3	2
5							-1	4	-6	4	3
6						1	-5	10	-10	5	4
7					-1	6	-15	20	-15	6	5
8				1	-7	21	-35	35	-21	7	6
9			-1	8	-28	56	-70	56	-28	8	7
10		1	-9	36	-84	126	-126	84	-36	9	8
11	-1	10	-45	120	-210	252	-210	120	-45	10	9

Perceive that the sum of the coefficients in each row of the last tables is always 1. Note also that the coefficient signals alternate between positive and negative. These properties reflect the behavior, or the action, of the method of differences which is the principle of any polynomial sequence.

Recurrence coefficients sequence towards increasing index: Axxxxxx {1, -1, 2, 1, -3, 3, -1, 4, -6, 4, 1, -5, 10, -10, 5, -1, 6, -15, 20, -15, 6, 1, -7, 21, -35, 35, -21, 7, -1, 8, -28, 56, -70, 56, -28, 8, 1, -9, 36, -84, 126, -126, 84, -36, 9, -1, 10, -45, 120, -210, 252, -210, 120, -45, 10, 1, -11, 55, -165, 330, -462, 462, -330, 165, -55, 11, …}.

Based on the sequence:

A074909 Running sum of Pascal's triangle (A007318), or beheaded Pascal's triangle read by beheaded rows. 1, 1, 2, 1, 3, 3, 1, 4, 6, 4, 1, 5, 10, 10, 5, 1, 6, 15, 20, 15, 6, 1, 7, 21, 35, 35, 21, 7, 1, 8, 28, 56, 70, 56, 28, 8, 1, 9, 36, 84, 126, 126, 84, 36, 9, 1, 10, 45, 120, 210, 252, 210, 120, 45, 10, 1, 11, 55, 165, 330, 462, 462, 330, 165, 55, 11, ...

Recurrence coefficients sequence towards decreasing index: Axxxxxx {1, 2, -1, 3, -3, 1, 4, -6, 4, -1, 5, -10, 10, -5, 1, 6, -15, 20, -15, 6, -1, 7, -21, 35, -35, 21, -7, 1, 8, -28, 56, -70, 56, -28, 8, -1, 9, -36, 84, -126, 126, -84, 36, -9, 1, 10, -45, 120, -210, 252, -210, 120, -45, 10, -1, 11, -55, 165, -330, 462, -462, 330, -165, 55, -11, 1, …}.

Based on the sequence:

A135278 Triangle read by rows, giving the numbers T(n,m) = binomial(n+1, m+1); or, Pascal's triangle A007318 with its left-hand edge removed. {1, 2, 1, 3, 3, 1, 4, 6, 4, 1, 5, 10, 10, 5, 1, 6, 15, 20, 15, 6, 1, 7, 21, 35, 35, 21, 7, 1, 8, 28, 56, 70, 56, 28, 8, 1, 9, 36, 84, 126, 126, 84, 36, 9, 1, 10, 45, 120, 210, 252, 210, 120, 45, 10, 1, 11, 55, 165, 330, 462, 462, 330, 165, 55, 11, 1, 12, 66, 220, 495, 792, 924, 792, …}.

11 Method of differences in any polynomial

Let's denote the differences between consecutive elements in any polynomial as

$$dif_1[y] = dif = Yd[y+1] - Yd[y]$$

Then, the difference between the consecutive differences is

$$dif_2[y] = difdif = dif_1[y+1] - dif_1[y]$$

Then,

$$dif_3[y] = difdifdif = dif_2[y+1] - dif_2[y]$$
$$dif_4[y] = difdifdifdif = dif_3[y+1] - dif_3[y]$$
$$\ldots$$
$$dif_h[y] = difdif \ldots h \ldots dif = dif_{h-1}[y+1] - dif_{h-1}[y]$$

This procedure in polynomials always gets in a result where $dif_h[y] = 0$ for any y.

Then, we know the polynomial has degree $d = h - 1$.

11.1 Method of differences in 1st-degree polynomials (linear equations)

Given

$$Y1[y] = by + c$$

Then,

$$Y1[y+1] = b(y+1) + c$$
$$Y1[y+1] = by + b + c$$

So,

$$dif_1[y] = (by + b + c) - (by + c)$$

Generically,

$$dif_1[y] = dif = b$$

11.2 Method of differences in 2nd-degree polynomials (quadratic equations)

Given

$$Y2[y] = ay^2 + by + c$$

Then,

$$Y2[y+1] = a(y+1)^2 + b(y+1) + c$$
$$Y2[y+1] = a(y^2 + 2y + 1) + by + b + c$$
$$Y2[y+1] = ay^2 + 2ay + a + by + b + c$$
$$Y2[y+1] = ay^2 + (2a+b)y + a + b + c$$

$$Y2[y+2] = a(y+2)^2 + b(y+2) + c$$
$$Y2[y+2] = a(y^2 + 4y + 4) + by + 2b + c$$
$$Y2[y+2] = ay^2 + 4ay + 4a + by + 2b + c$$
$$Y2[y+2] = ay^2 + (4a+b)y + 4a + 2b + c$$

So,

$$dif_1[y] = (ay^2 + (2a+b)y + a + b + c) - (ay^2 + by + c)$$
$$dif_1[y] = (ay^2 + 2ay + by + a + b + c) - (ay^2 + by + c)$$

$$dif_1[y] = (ay^2 + by + c + 2ay + a + b) - (ay^2 + by + c)$$
$$dif_1[y] = 2ay + a + b$$

$$dif_1[y+1] = (ay^2 + (4a + b)y + 4a + 2b + c) - (ay^2 + (2a + b)y + a + b + c)$$
$$dif_1[y+1] = (ay^2 + 4ay + by + 4a + 2b + c) - (ay^2 + 2ay + by + a + b + c)$$
$$dif_1[y+1] = 2ay + 3a + b$$

Then,

$$dif_2[y] = dif_1[y+1] - dif_1[y]$$
$$dif_2[y] = (2ay + 3a + b) - (2ay + a + b)$$

Generically,

$$dif_2[y] = difdif = 2a$$

11.3 Method of differences in 3rd-degree polynomials (cubic equations)

Given

$$Y3[y] = a_3y^3 + ay^2 + by + c$$

Then,

$$Y3[y+1] = a_3(y+1)^3 + a(y+1)^2 + b(y+1) + c$$
$$Y3[y+1] = a_3(y^3 + 3y^2 + 3y + 1) + a(y^2 + 2y + 1) + by + b + c$$
$$Y3[y+1] = a_3y^3 + 3a_3y^2 + 3a_3y + a_3 + ay^2 + 2ay + a + by + b + c$$
$$Y3[y+1] = a_3y^3 + (3a_3 + a)y^2 + (3a_3 + 2a + b)y + a_3 + a + b + c$$

$$\mathrm{Y3}[y+2] = a_3(y+2)^3 + a(y+2)^2 + b(y+2) + c$$
$$\mathrm{Y3}[y+2] = a_3(y^3 + 6y^2 + 12y + 8) + a(\mathrm{y}^2 + 4y + 4) + b(y+2) + c$$
$$\mathrm{Y3}[y+2] = a_3y^3 + 6a_3y^2 + 12a_3y + 8a_3 + ay^2 + 4ay + 4a + by + 2b + c$$
$$Y3[y+2] = a_3y^3 + (6a_3 + a)y^2 + (12a_3 + 4a + b)y + 8a_3 + 4a + 2b + c$$

$$Y3[y+3] = a_3(y+3)^3 + a(y+3)^2 + b(y+3) + c$$
$$Y3[y+3] = a_3(y^3 + 9y^2 + 27y + 27) + a(\mathrm{y}^2 + 6y + 9) + b(y+3) + c$$
$$\mathrm{Y3}[y+3] = a_3y^3 + 9a_3y^2 + 27a_3y + 27a_3 + ay^2 + 6ay + 9a + by + 3b + c$$
$$Y3[y+3] = a_3y^3 + (9a_3 + a)y^2 + (27a_3 + 6a + b)y + 27a_3 + 9a + 3b + c$$

So,

$$dif_1[y] = Y3[y+1] - Y3[y]$$
$$dif_1[y] = a_3y^3 + (3a_3 + a)y^2 + (3a_3 + 2a + b)y + a_3 + a + b + c - (a_3y^3 + ay^2 + by + c)$$
$$dif_1[y] = 3a_3y^2 + (3a_3 + 2a)y + a_3 + a + b$$

$$dif_1[y+1] = Y3[y+2] - Y3[y+3]$$
$$dif_1[y+1] = a_3y^3 + (6a_3 + a)y^2 + (12a_3 + 4a + b)y + 8a_3 + 4a + 2b + c - (a_3y^3 + (3a_3 + a)y^2 + (3a_3 + 2a + b)y + a_3 + a + b + c)$$
$$dif_1[y+1] = 3a_3y^2 + (9a_3 + 2a)y + 7a_3 + 3a + b$$

$$dif_1[y+2] = Y3[y+3] - Y3[y+2]$$
$$dif_1[y+2] = a_3y^3 + (9a_3 + a)y^2 + (27a_3 + 6a + b)y + 27a_3 + 9a + 3b + c - (a_3y^3 + (6a_3 + a)y^2 + (12a_3 + 4a + b)y + 8a_3 + 4a + 2b + c)$$
$$dif_1[y+2] = 3a_3y^2 + (15a_3 + 2a)y + 19a_3 + 5a + b$$

Now,

$$dif_2[y] = dif_1[y+1] - dif_1[y]$$
$$dif_2[y] = 3a_3y^2 + (9a_3 + 2a)y + 7a_3 + 3a + b - (3a_3y^2 + (3a_3 + 2a)y + a_3 + a + b)$$
$$dif_2[y] = 6a_3y + 6a_3 + 2a$$

$$dif_2[y+1] = dif_1[y+2] - dif_1[y+1]$$
$$dif_2[y+1] = 3a_3y^2 + (15a_3 + 2a)y + 19a_3 + 5a + b - (3a_3y^2 + (9a_3 + 2a)y + 7a_3 + 3a + b)$$
$$dif_2[y+1] = 6a_3y + 12a_3 + 2a$$

Then,

$$dif_3[y] = dif_2[y+1] - dif_2[y]$$
$$dif_3[y] = 6a_3y + 12a_3 + 2a - (6a_3y + 6a_3 + 2a)$$

Generically,

$$dif_3[y] = difdifdif = 6a_3$$

11.4 Method of differences in d^th-degree polynomials

From the generic equation of polynomial d-degree

$$Yd[y] = a_dy^d + a_{d-1}y^{d-1} + \cdots + a_4y^4 + a_3y^3 + ay^2 + by + c$$

We have,

$$dif_d[y] = d!\,a_d$$

12 Understanding the Method of Differences in polynomials

Addition increase the polynomial degree. Subtraction decrease the polynomial degree. This idea comes from The Babbage Engine.

We know that, if $F[y] = y^n$, then,

$$F[y+z] = (y+z)^n$$
$$= \binom{n}{0} y^n + \binom{n}{1} y^{n-1} z + \binom{n}{2} y^{n-2} z^2 + \ldots + \binom{n}{n-2} y^2 z^{n-2} + \binom{n}{n-1} y z^{n-1} + \binom{n}{n} z^n$$

Let's be $z = 1$:

$$F[y+1] = (y+1)^n = \binom{n}{0} y^n + \binom{n}{1} y^{n-1} + \binom{n}{2} y^{n-2} + \ldots + \binom{n}{n-2} y^2 + \binom{n}{n-1} y + \binom{n}{n}$$
$$F[y+1] = (y+1)^n = y^n + \binom{n}{1} y^{n-1} + \binom{n}{2} y^{n-2} + \ldots + \binom{n}{2} y^2 + \binom{n}{1} y + 1$$
$$F[y+1] = F[y] + \binom{n}{1} y^{n-1} + \binom{n}{2} y^{n-2} + \ldots + \binom{n}{2} y^2 + \binom{n}{1} y + 1$$

Being,

$$G[y] = F[y+1] - F[y]$$

Then,

$$G[y] = \binom{n}{1} y^{n-1} + \binom{n}{2} y^{n-2} + \ldots + \binom{n}{2} y^2 + \binom{n}{1} y + 1$$
$$degree\big[G[y]\big] = degree\big[F[y]\big] - 1$$
$$highest\ order\ coefficient[F[y]] = 1$$
$$highest\ order\ coeffcient[G[y]] = \binom{n}{1}$$

Now, from $G[y] = F[y+1] - F[y] = \binom{n}{1} y^{n-1} + \binom{n}{2} y^{n-2} + \ldots + \binom{n}{2} y^2 + \binom{n}{1} y + 1$, we have:

$$G[y+1] = \binom{n}{1} (y+1)^{n-1} + \binom{n}{2} (y+1)^{n-2} + \ldots + \binom{n}{n-2} (y+1)^2 + \binom{n}{n-1} (y+1) + 1$$
$$G[y+1] = \binom{n}{1} \left[y^{n-1} + \binom{n-1}{1} y^{n-2} + \binom{n-1}{2} y^{n-3} + \ldots + \binom{n-1}{2} y^2 + \binom{n-1}{1} y + 1\right] + \binom{n}{2} \left[y^{n-2} + \binom{n-2}{1} y^{n-3} + \binom{n-2}{2} y^{n-4} + \ldots + \binom{n-2}{2} y^2 + \binom{n-2}{1} y + 1\right] + \ldots + \binom{n}{2} [y^2 + 2y + 1] + \binom{n}{1} (y+1) + 1$$
$$G[y+1] = \left\{\binom{n}{1} y^{n-1}\right\} + \binom{n}{1} [\binom{n-1}{1} y^{n-2} + \binom{n-1}{2} y^{n-3} + \ldots + \binom{n-1}{2} y^2 + \binom{n-1}{1} y + 1] + \left\{\binom{n}{2} y^{n-2}\right\} + \binom{n}{2} [\binom{n-2}{1} y^{n-3} + \binom{n-2}{2} y^{n-4} + \ldots + \binom{n-2}{2} y^2 + \binom{n-2}{1} y + 1] + \ldots + \left\{\binom{n}{2} y^2\right\} + \binom{n}{2} [2y + 1] + \left\{\binom{n}{1} y\right\} + \binom{n}{1} (1) + 1$$
$$G[y+1] = \left\{\binom{n}{1} y^{n-1} + \binom{n}{2} y^{n-2} + \ldots + \binom{n}{2} y^2 + \binom{n}{1} y + 1\right\} + \binom{n}{1} \left[\binom{n-1}{1} y^{n-2} + \binom{n-1}{2} y^{n-3} + \ldots + \binom{n-1}{2} y^2 + \binom{n-1}{1} y + 1\right] + \binom{n}{2} \left[\binom{n-2}{1} y^{n-3} + \binom{n-2}{2} y^{n-4} + \ldots + \binom{n-2}{2} y^2 + \binom{n-2}{1} y + 1\right] + \ldots + \binom{n}{2} [2y + 1] + \binom{n}{1}$$

$$G[y+1] = G[y] + \binom{n}{1}\left[\binom{n-1}{1}y^{n-2} + \binom{n-1}{2}y^{n-3} + \ldots + \binom{n-1}{2}y^2 + \binom{n-1}{1}y + 1\right] + \binom{n}{2}\left[\binom{n-2}{1}y^{n-3} + \binom{n-2}{2}y^{n-4} + \ldots + \binom{n-2}{2}y^2 + \binom{n-2}{1}y + 1\right] + \ldots + \binom{n}{2}[2y+1] + \binom{n}{1}$$

Finally,

$$H[y] = G[y+1] - G[y]$$
$$= \binom{n}{1}\left[\binom{n-1}{1}y^{n-2} + \binom{n-1}{2}y^{n-3} + \ldots + \binom{n-1}{2}y^2 + \binom{n-1}{1}y + 1\right] + \binom{n}{2}\left[\binom{n-2}{1}y^{n-3} + \binom{n-2}{2}y^{n-4} + \ldots + \binom{n-2}{2}y^2 + \binom{n-2}{1}y + 1\right] + \ldots + \binom{n}{2}[2y+1] + \binom{n}{1}$$
$$degree\big[H[y]\big] = degree\big[G[y]\big] - 1 = degree\big[F[y]\big] - 2$$
$$highest\ order\ coefficient\big[H[y]\big] = \binom{n}{1}\binom{n-1}{1}$$

12.1 Conclusion

Reflecting the method of differences in any polynomial degree n, if we recursively continue to do differences of its consecutive elements, we will get a decrease in the power until degree zero. The result always arrives in a constant in the value $n!$.

This is the property of the method of differences likewise derivatives. Each time we do a difference between two consecutive elements in a polynomial n-power, it will result in a decrease in the degree of power in the amount of a unit. If we continue to do differences between consecutive elements recursively until degree zero, it will result in a coefficient equal to a constant n!.

$$recursivelly\big[F[y+1] - F[y]\big] = recursivelly[(y+1)^n - y^n] = \frac{d^n}{dy}(y^n) = n! \equiv A000142$$

12.2 Example in 4th degree:

$$F[y] = y^4$$

$$G[y] = F[y+1] - F[y]$$
$$G[y] = (y+1)^4 - y^4$$
$$G[y] = 4y^3 + 6y^2 + 4y + 1$$

$$G[y+1] = 4(y+1)^3 + 6(y+1)^2 + 4(y+1) + 1$$
$$G[y+1] = 4y^3 + 12y^2 + 12y + 4 + 6y^2 + 12y + 6 + 4y + 4 + 1$$
$$G[y+1] - G[y] = H[y] = 12y^2 + 12y + 4 + 12y + 6 + 4$$
$$H[y] = 12y^2 + 24y + 14$$

$$H[y+1] = 12(y+1)^2 + 24(y+1) + 14$$
$$H[y+1] = 12y^2 + 24y + 12 + 24y + 24 + 14$$
$$H[y+1] - H[y] = I[y] = 12y^2 + 24y + 12 + 24y + 24 + 14$$

$I[y] = 24y + 36$

$$I[y+1] = 24(y+1) + 36$$
$$I[y+1] = 24y + 24 + 36$$
$$I[y+1] - I[y] = 24 = 4!$$

13 Summary

Considering

$$Yd[y] = a_d y^d + a_{d-1} y^{d-1} + \cdots + a_4 y^4 + a_3 y^3 + a y^2 + b y + c$$
$$Yd[-3] = e$$
$$Yd[-2] = f$$
$$Yd[-1] = g$$
$$Yd[0] = h$$
$$Yd[1] = i$$
$$Yd[2] = j$$
$$Yd[3] = k$$

13.1 The simplest equation for d^th-degree polynomials summary

$$Y0[y] = h$$
$$Y1[y] = (i-h)y + h$$
$$Y2[y] = \frac{g-2h+i}{2}y^2 + \frac{-g+i}{2}y + h$$
$$Y3[y] = \frac{-g+3h-3i+j}{6}y^3 + \frac{g-2h+i}{2}y^2 + \frac{-2g-3h+6i-j}{6}y + h$$
$$Y4[y] = \frac{f-4g+6h-4i+j}{24}y^4 + \frac{-f+2g-2i+j}{12}y^3 + \frac{-f+16g-30h+16i-j}{24}y^2 + \frac{f-8g+8i-j}{12}y + h$$
$$Y5[y] = \frac{-f+5g-10h+10i-5j+k}{120}y^5 + \frac{f-4g+6h-4i+j}{24}y^4 + \frac{-f-g+10h-14i+7j-k}{24}y^3 + \frac{-f+16g-30h+16i-j}{24}y^2 + \frac{3f-30g-20h+60i-15j+2k}{60}y + h$$
$$Y6[y] = \frac{e-6f+15g-20h+15i-6j+k}{720}y^6 + \frac{-e+4f-5g+5i-4j+k}{240}y^5 + \frac{-e+12f-39g+56h-39i+12j-k}{144}y^4 + \frac{e-8f+13g-13i+8j-k}{48}y^3 + \frac{2e-27f+270g-490h+270i-27j+2k}{360}y^2 + \frac{-e+9f-45g+45i-9j+k}{60}y + h$$

Sequences of the denominators of y^d: $\{1,1,2,6,24,120,720,\ldots\}$ == A000142 Factorial numbers
Sequences of the denominators of y^{d-1}: $\{1,2,2,12,24,240,\ldots\}$ == Axxxxxx

13.2 Inflection Point Summary for any polynomial

$$y_{ip_{Y0[y]}}\left[@\frac{d^{-1}Y0[y]}{dy^{-1}}=0\right]=0$$
$$y_{ip_{Y1[y]}}\left[@\frac{d^{0}Y1[y]}{dy^{0}}=0\right]=-\frac{0!\,c}{1!\,b}=-1\left(\frac{h}{i-h}\right)$$
$$y_{ip_{Y2[y]}}\left[@\frac{d^{1}Y2[y]}{dy^{1}}=0\right]=-\frac{1!\,b}{2!\,a}=-\frac{1}{2}\left(\frac{-g+i}{g-2h+i}\right)$$
$$y_{ip_{Y3[y]}}\left[@\frac{d^{2}Y3[y]}{dy^{2}}=0\right]=-\frac{2!\,a}{3!\,a_3}=-1\left(\frac{g-2h+i}{-g+3h-3i+j}\right)$$
$$y_{ip_{Y4[y]}}\left[@\frac{d^{3}Y4[y]}{dy^{3}}=0\right]=-\frac{3!\,a_3}{4!\,a_4}=-\frac{1}{2}\left(\frac{-f+2g-2i+j}{f-4g+6h-4i+j}\right)$$
$$y_{ip_{Y5[y]}}\left[@\frac{d^{4}Y5[y]}{dy^{4}}=0\right]=-\frac{4!\,a_4}{5!\,a_5}=-1\left(\frac{f-4g+6h-4i+j}{-f+5g-10h+10i-5j+k}\right)$$
$$y_{ip_{Y6[y]}}\left[@\frac{d^{5}Y6[y]}{dy^{5}}=0\right]=-\frac{5!\,a_5}{6!\,a_6}=-\frac{1}{2}\left(\frac{-e+4f-5g+5i-4j+k}{e-6f+15g-20h+15i-6j+k}\right)$$

…

$$y_{ip_{Yd[y]}}\left[@\frac{d^{d-1}Yd[y]}{dy^{d-1}}=0\right]=-\frac{(d-1)!\,a_{d-1}}{d!\,a_d}=-\frac{1}{d}\left(\frac{a_{d-1}}{a_d}\right)$$

$n=0$, we have $y_{ip}=-\left(\frac{1}{0}\right)\frac{(0)}{(0)}=-\frac{1}{2}$?

$n=1$, we have $y_{ip}=-\left(\frac{1}{1}\right)\frac{(1)}{(1)}=-1$

$n=2$, we have $y_{ip}=-\left(\frac{1}{2}\right)\frac{\left(\frac{1}{2}\right)}{\left(\frac{1}{2}\right)}=-\frac{1}{2}$

$n=3$, we have $y_{ip}=-\left(\frac{1}{3}\right)\frac{\left(\frac{1}{2}\right)}{\left(\frac{1}{6}\right)}=-1$

$n=4$, we have $y_{ip}=-\left(\frac{1}{4}\right)\frac{\left(\frac{1}{22}\right)}{\left(\frac{1}{24}\right)}=-\frac{1}{2}$

$n=5$, we have $y_{ip}=-\left(\frac{1}{5}\right)\frac{\left(\frac{1}{24}\right)}{\left(\frac{1}{120}\right)}=-1$

$n=6$, we have $y_{ip}=-\left(\frac{1}{6}\right)\frac{\left(\frac{1}{240}\right)}{\left(\frac{1}{720}\right)}=-\frac{1}{2}$

13.3 Recurrence equations towards increasing index summary

Y0[y]=+1Y0[y-1]
Y1[y]=-1Y1[y-2]+2Y1[y-1]
Y2[y]=+1Y2[y-3]-3Y2[y-2]+3Y2[y-1]
Y3[y]=-1Y3[y-4]+4Y3[y-3]-6Y3[y-2]+4Y3[y-1]
Y4[y]=+1Y4[y-5]-5Y4[y-4]+10Y4[y-3]-10Y4[y-2]+5Y4[y-1]
Y5[y]=-1Y5[y-6]+6Y5[y-5]-15Y5[y-4]+20Y5[y-3]-15Y5[y-2]+6Y5[y-1]
Y6[y]=+1Y6[y-7]-7Y6[y-6]+21Y6[y-5]-35Y6[y-4]+35Y6[y-3]-21Y6[y-2]+7Y6[y-1]

13.4 Recurrence equations towards decreasing index summary

\Y0 [y]\=1Y0[y+1]
\Y1 [y]\=2Y1 [y+1]-1Y1[y+2]
\Y2[y]\=3Y2[y+1]-3Y2[y+2]+1Y2[y+3]
\Y3 [y]\=4Y3 [y+1]-6Y3 [y+2]+4Y3 [y+3]-1Y3[y+4]
\Y4 [y]\=5Y4 [y+1]-10Y4 [y+2]-10Y4 [y+3]-5Y4 [y+4]+1Y4[y+5]
\Y5 [y]\=6Y5 [y+1]-10Y5 [y+2]+20Y5 [y+3]-15Y5 [y+4]+6Y5[y+5]-1Y5[y+6]
\Y6 [y]\=7Y6 [y+1]-21Y6 [y+2]+35Y6 [y+3]-35Y6 [y+4]+21Y6 [y+5]-7Y6 [y+6]+1Y6 [y+7]

13.4 The method of differences in polynomials summary

$$recursivelly\big[F[y+1]-F[y]\big] = recursivelly[(y+1)^n - y^n] = \frac{d^n}{dy}(y^n) = n!$$
$$\equiv A000142$$

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