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STRUCTURAL COMPLETENESS OF THREE-VALUED LOGICS WITH SUBCLASSICAL NEGATION

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Abstract. A propositional logic|calculus is said to be *structurally complete*, whenever it cannot be extended by non-derivable rules without deriving *new* axioms. Here, we study this property within the framework of three-valued logics with subclassical negation (3VLSN) precisely specified and comprehensively marked semantically here. The principal contribution of the paper is then an effective — in case of finitely many connectives — algebraic criterion of the structural completeness of any paraconsistent/"both disjunctive and paracomplete" 3VLSN, according to which it is structurally complete "only if"/iff it is *maximally* paraconsistent/paracomplete, that is, has no proper paraconsistent/paracomplete extension, and "only if"/if it has no classical extension. On the other hand, any [not necessarily] classical logic with[out] theorems is [not] structurally complete. In this connection, we also obtain equally effective algebraic criteria of the mentioned properties within the general framework of 3VLSN.

1. Introduction

Structural completeness of a propositional logic|calculus is one of its most fundamental properties, meaning its factual *deductive maximality* in the sense of absence of any possibility to enhance it by essentially new rules with retaining theorems (viz., derivable axioms).[1] Therefore, studying it — even, for a single logic|calculus (not saying about their generic classes) — is an extremely acute logical problem. This feature is [not] typical of any [not necessarily] classical (more precisely, two-valued classically-negative) logics with[out] theorems. The situation with many-valued (even, three-valued) logics is but much more complicated. While there are structurally complete three-valued logics like both Gödel's one G_3 [3] (as well as its implication-less fragment; cf. [15] for its structural completeness) and the bounded expansion of Kleene's one $\mathbb{K}_3$ [5], there are also structurally incomplete ones with theorems like Łukasiewicz' one $Ł_3$ [7] as well as both the *logic of paradox/antinomies* LP/LA [11]/[1] and HZ [4], the structural incompleteness of all of which has been due to [16], [17] and [20].

On the other hand, a third truth value (apart from the to classical ones — "truth" and "falsehood") is normally invoked to express the incompletenes/inconsistency of information about assertions, in which case resulting logics become *paracomplete/paraconsistent*, respectively, in the sense that they refute the *"Excluded Middle Law"/"Ex Contradictione Quodlibet"* axiom/rule, and so such logics definitely deserve a particular emphasis within the three-valud framework. Properly speaking, the former, as opposed to the latter, first, presumes disjunctivity and, second, holds for logics without theorems, making these just *formally* paracomplete, so we

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[1] As a matter of fact, a more appropriate term for this conception would be something like "deductive/inferential completeness|maximality". However, we follow the traditional terminology originally adopted within the Polish Logic School (cf., e.g., [12]).

naturally garble the native conception of paracompleteness with its more genuine "inferential" version.

It is remarkable that the issue of structural completeness of paraconsistent/"disjunctive paracomplete" three-valued logics appears to be closely related to — more precisely, characterized by — those of their *(axiomatic) [pre]maximal* paraconsistency/paracompleteness — in the sense of having no [more than one] proper paraconsistent/paracomplete (axiomatic) extension — as well as both being {sub}classical {in the sense of having a classical extension} and having theorems. Therefore, we explore these properties within the three-valued framework as well.

As a matter of fact, the issue of structural completeness is a particular case of that of *structural completion of* a logic|calculus as the unique structurally complete extension with same theorems that, in its turn, an instance of the problem of finding the lattice of extensions of a given logic|calculus. Here, we explore (at least, partially) these problems within the three-valued framework too, providing a generic insight into particular results obtained in [16, 17, 20] *ad hoc*.

The rest of the paper is as follows. The exposition of the material of the paper is entirely self-contained (of course, modulo very basic issues concerning Set and Lattice Theory, Universal Algebra and Logic to be found, if necessary, in standard mathematical handbooks like [2, 9]). Section 2 is a concise summary of particular basic issues underlying the paper, most of which, though having become a part of algebraic and logical folklore, are still recalled just for the exposition to be properly self-contained. In Section 3, we then develop/recall certain advanced generic issues concerning both false-singular (viz., having no more than one non-distinguished value) consistent (viz., having a non-distinguished value) weakly conjunctive matrices and equality determinants as well as both classical matrices and logics and structural completions of finitely-valued logics. Next, in Section 4, we mark semantically the framework of 3VLSN. Further, in Section 5, we explore the issue of their paraconsistent extensions (in particular, that of the {axiomatic} [pre]maximal paraconsistency of paraconsistent 3VLSN going back to [14] {resp., [22]}). Likewise, Section 6 is devoted to classical extensions of 3VLSN. Then, in Section 7, we investigate absence of non-subclassical [inferentially] consistent extensions of subclassical 3VLSN in connection with their [not] having theorems [resp. proper paraconsistent extensions]. After all, in Section 8, we study the structural completeness and completions (as well as the lattices of extensions) of paraconsistent/"(implicative) disjunctive paracomplete" 3VLSN (with lattice conjunction and disjunction) /"as well as their {axiomatic} [pre]maximal ⟨inferential⟩ paracompleteness". Finally, Section 9 is a brief summary of principal contributions of the paper.

2. Basic issues

Notations like img, dom, ker, hom, π_i and Con and related notions are supposed to be clear.

2.1. Set-theoretical background.

2.1. Set-theoretical background. We follow the standard set-theoretical convention, according to which natural numbers (including 0) are treated as finite ordinals (viz., sets of lesser natural numbers), the ordinal of all them being denoted by ω. Then, given any $(N \cup \{n\}) \subseteq \omega$, set $(N \div n) \triangleq \{\frac{m}{n} \mid m \in N\}$. The proper class of all ordinals is denoted by ∞. Also, functions are viewed as binary relations, while singletons are identified with their unique elements, unless any confusion is possible.

Given a set S, the set of all subsets of S [of cardinality $\in K \subseteq \infty$] is denoted by $\wp_{[K]}(S)$, respectively. Then, an *enumeration of* S is any bijection from $|S|$ onto S. As usual, given any equivalence relation θ on S, by ν_θ we denote the function with domain S defined by $\nu_\theta(a) \triangleq [a]_\theta \triangleq \theta[\{a\}]$, for all $a \in S$, whereas we set

$(T/\theta) \triangleq \nu_\theta[T]$, for every $T \subseteq S$. Next, S-tuples (viz., functions with domain S) are often written in the either sequence $\bar{t}$ or vector $\vec{t}$ form, its s-th component (viz., the value under argument s), where $s \in S$, being written as either t_s or t^s, respectively. Given two more sets A and B, any relation $R \subseteq (A \times B)$ (in particular, a mapping $R : A \to B$) determines the equally-denoted relation $R \subseteq (A^S \times B^S)$ (resp., mapping $R : A^S \to B^S$) point-wise. Likewise, given a set A, an S-tuple $\overline{B}$ of sets and any $\bar{f} \in (\prod_{s\in S} B_s^A)$, put $(\prod \bar{f}) : A \to (\prod \overline{B}), a \mapsto \langle f_s(a)\rangle_{s\in S}$. (In case $I = 2$, $f_0 \times f_1$ stands for $(\prod \bar{f})$.) Further, set $\Delta_S \triangleq \{\langle a, a\rangle \mid a \in S\}$, functions of such a kind being referred to as *diagonal*, and $S^+ \triangleq \bigcup_{i\in(\omega\setminus 1)} S^i$, elements of $S^* \triangleq (S^0 \cup S^+)$ being identified with ordinary finite tuples/sequences, the binary concatenation operation on which being denoted by $*$, as usual. Then, any binary operation $\diamond$ on S determines the equally-denoted mapping $\diamond : S^+ \to S$ as follows: by induction on the length $l = (\mathrm{dom}\,\bar{a})$ of any $\bar{a} \in S^+$, put:

$$\diamond\bar{a} \triangleq \begin{cases} a_0 & \text{if } l = 1, \\ (\diamond(\bar{a}{\restriction}(l-1))) \diamond a_{l-1} & \text{otherwise.} \end{cases}$$

In particular, given any $f : S \to S$ and any $n \in \omega$, set $f^n \triangleq (\circ\langle n \times \{f\}, \Delta_S\rangle) : S \to S$. Likewise, given a one more set T, any $\diamond : (S \times T) \to T$ determines the equally-denoted mapping $\diamond : (S^* \times T) \to T$ as follows: by induction on the length (viz., domain) l of any $\bar{a} \in S^*$, for all $b \in T$, put:

$$(\bar{a} \diamond b) \triangleq \begin{cases} b & \text{if } l = 0, \\ a_0 \diamond (((\bar{a}{\restriction}(l \setminus 1)) \circ ((+1){\restriction}(l-1))) \diamond b) & \text{otherwise.} \end{cases}$$

Finally, given any $T \subseteq S$, we have the *characteristic function* $\chi_S^T \triangleq ((T \times \{1\}) \cup ((S \setminus T) \times \{0\})) : S \to 2$ *of* T *in* S.

Let A be a set. Then, a $U \subseteq \wp(A)$ is said to be *upward-directed*, provided, for every $S \in \wp_\omega(U)$, there is some $T \in U$ such that $(\bigcup S) \subseteq T$, in which case $U \neq \varnothing$, when taking $S = \varnothing$. Next, a subset of $\wp(A)$ is said to be *inductive*, whenever it is closed under unions of upward-directed subsets. Further, a *closure system over* A is any $\mathcal{C} \subseteq \wp(A)$ such that, for every $S \subseteq \mathcal{C}$, it holds that $(A \cap \bigcap S) \in \mathcal{C}$. In that case, any $\mathcal{B} \subseteq \mathcal{C}$ is called a *(closure) basis of* $\mathcal{C}$, provided $\mathcal{C} = \{A \cap \bigcap S | S \subseteq \mathcal{B}\}$. Furthermore, an *operator over* A is any unary operation O on $\wp(A)$. This is said to be *(monotonic) [idempotent]* $\{$*transitive*$\}$ $\langle$*inductive/finitary/compact*$\rangle$, provided, for all $(B,)D \in \wp(A)$ $\langle$resp., any upward-directed $U \subseteq \wp(A)\rangle$, it holds that $(O(B))[D]\{O(O(D)\} \subseteq O(D)\langle O(\bigcup U) \subseteq \bigcup O[U]\rangle$. Finally, a *closure operator over* A is any monotonic idempotent transitive operator over A, in which case $\mathrm{img}\,C$ is a closure system over A, determining C uniquely, because, for every closure basis $\mathcal{B}$ of $\mathrm{img}\,C$ (including $\mathrm{img}\,C$ itself) and each $X \subseteq A$, it holds that $C(X) = (A \cap \bigcap\{Y \in \mathcal{B} | X \subseteq Y\})$, called *dual to* C and vice versa.

2.2. **Algebraic background.** Unless otherwise specified, abstract algebras are denoted by Fraktur letters [possibly, with indices], their carriers (viz., underlying sets) being denoted by corresponding Italic letters [with same indices, if any].

A *(propositional/sentential) language/signature* is any algebraic (viz., functional) signature Σ (to be dealt with throughout the paper by default) constituted by function (viz., operation) symbols of finite arity to be treated as *(propositional/sentential) connectives*. Then, Σ is said to be *constant-free*, whenever it has no nullary connective.

Given a Σ-algebra $\mathfrak{A}$, $\mathrm{Con}(\mathfrak{A})$ is an inductive closure system over A^2 forming a bounded lattice with meet $\theta \cap \vartheta$ of any $\theta, \theta \in \mathrm{Con}(\mathfrak{A})$, their join $\theta \amalg \vartheta$, being the transitive closure of $\theta \cup \vartheta$, zero Δ_A and unit A^2, the dual closure operator being denoted by $\mathrm{Cg}^{\mathfrak{A}}$. Then, $\mathfrak{A}$ is said to be *simple*, provided the lattice involved is two-element, in which case $|A| > 1$. Next, a $B \subseteq A$ is said to *"form a subalgebra of $\mathfrak{A}$"/"be $\mathfrak{A}$-closed"*, whenever it is closed under operations of $\mathfrak{A}$. Furthermore, given a class K of Σ-algebras, set $\hom(\mathfrak{A},\mathsf{K}) \triangleq (\bigcup\{\hom(\mathfrak{A},\mathfrak{B}) \mid \mathfrak{B} \in \mathsf{K}\})$, in which case $\ker[\hom(\mathfrak{A},\mathsf{K})] \subseteq \mathrm{Con}(\mathfrak{A})$, and so $(A^2 \cap \bigcap \ker[\hom(\mathfrak{A},\mathsf{K})]) \in \mathrm{Con}(\mathfrak{A})$.

Given any $\alpha \subseteq \omega$, put $\bar{x}_\alpha \triangleq \langle x_\beta\rangle_{\beta\in\alpha}$ and $V_\alpha \triangleq (\mathrm{img}\,\bar{x}_\alpha)$, elements of which being viewed as *(propositional/sentential) variables of rank α*. Then, providing $\alpha \neq \varnothing$, whenever Σ is constant-free, we have the absolutely-free Σ-algebra $\mathfrak{Fm}^\alpha_\Sigma$ freely-generated by the set V_α, its endomorphisms/elements of its carrier $\mathrm{Fm}^\alpha_\Sigma$ being called *(propositional/sentential) Σ-substitutions/-formulas of rank α*. As usual, given any $n \in \omega$, by an *n-ary secondary connective of* Σ we mean any Σ-formula of rank $\max(1,n)$. Recall that

$$\forall h \in \hom(\mathfrak{A},\mathfrak{B}) : [(\mathrm{img}\,h) = B) \Rightarrow] \\ (\hom(\mathfrak{Fm}^\alpha_\Sigma,\mathfrak{B}) \supseteq [=]\{h \circ g \mid g \in \hom(\mathfrak{Fm}^\alpha_\Sigma,\mathfrak{A})\}), \quad (2.1)$$

where $\mathfrak{A}$ and $\mathfrak{B}$ are Σ-algebras. Any $\langle\phi,\psi\rangle \in \mathrm{Eq}^\alpha_\Sigma \triangleq (\mathrm{Fm}^\alpha_\Sigma)^2$ is referred to as a *Σ-equation/-indentity of rank α* and normally written in the standard equational form $\phi \approx \psi$. (In general, any mention of α is normally omitted, whenever $\alpha = \omega$.) In this way, given any $h \in \hom(\mathfrak{Fm}^\alpha_\Sigma,\mathfrak{A})$, $\ker h$ is the set of all Σ-identities of rank α *true/satisfied in* $\mathfrak{A}$ *under* h. Likewise, given a class K of Σ-algebras, $\theta^\alpha_\mathsf{K} \triangleq (\mathrm{Eq}^\alpha_\Sigma \cap \bigcap \ker[\hom(\mathfrak{Fm}^\alpha_\Sigma,\mathsf{K})]) \in \mathrm{Con}(\mathfrak{Fm}^\alpha_\Sigma)$ is the set of all all Σ-identities of rank α *true/satisfied in* K, in which case we set $\mathfrak{F}^\alpha_\mathsf{K} \triangleq (\mathfrak{Fm}^\alpha_\Sigma/\theta^\alpha_\mathsf{K})$. (In case both α as well as both K and all members of it are finite, the set $I \triangleq \{\langle h,\mathfrak{A}\rangle \mid h \in \hom(\mathfrak{Fm}^\alpha_\Sigma,\mathfrak{A}), \mathfrak{A} \in \mathsf{K}\}$ is finite — more precisely, $|I| = \sum_{\mathfrak{A}\in\mathsf{K}} |A|^\alpha$, in which case $g \triangleq (\prod_{i\in I} \pi_0(i)) \in \hom(\mathfrak{Fm}^\alpha_\Sigma, \prod_{i\in I}(\pi_1(i){\restriction}\,\mathrm{img}\,\pi_0(i)))$ with $(\ker g) = \theta \triangleq \theta^\alpha_\mathsf{K}$, and so, by the Homomorphism Theorem, $e \triangleq (g \circ \nu_\theta^{-1})$ is an isomorphism from $\mathfrak{F}^\alpha_\mathsf{K}$ onto the subdirect product $(\prod_{i\in I}(\pi_1(i){\restriction}\,\mathrm{img}\,\pi_0(i))){\restriction}(\mathrm{img}\,g)$ of $\langle\pi_1(i){\restriction}\,\mathrm{img}\,\pi_0(i)\rangle_{i\in I}$. In this way, the former is finite, for the latter is so — more precisely, $|F^\alpha_\mathsf{K}| \leqslant (\max_{\mathfrak{A}\in\mathsf{K}} |A|)^{|I|}$.)

A *"congruence-permutation term"/discriminator for* K is any $\tau \in \mathrm{Fm}^3_\Sigma$ such that, for each $\mathfrak{A}$ and all $\bar{a} \in A^{2/3}$, it holds that $[\tau^\mathfrak{A}(a_0,a_1,a_{1/2}) =]a_0 = \tau^\mathfrak{A}(a_1,a_1,a_0)$ [unless $a_0 = a_1$], in which case it is so for any homomorphic image of any subalgebra of $\mathfrak{A}$ /as well as a congruence-permutation term for $\mathfrak{A}$ (when taking $a_2 = a_1$), while, for any $\theta \in \mathrm{Con}(\mathfrak{A})$, any $\langle a,b\rangle \in (\theta \setminus \Delta_A)$ and any $c \in A$, we have $a = \tau^\mathfrak{A}(a,b,c)\ \theta\ \tau^\mathfrak{A}(a,a,c) = c$, in which case we get $\theta = A^2$, and so $\mathfrak{A}$ is simple, unless it is one-element. By [8] and Lemma 2.10 of [20], we have:

Lemma 2.1. *Let $n \in (\omega[\setminus 1])$, $\overline{\mathfrak{A}}$ an n-tuple of simple Σ-algebras and τ a congruence-permutation term for $\mathrm{img}\,\overline{\mathfrak{A}}$. Then, any subdirect product of $\overline{\mathfrak{A}}$ is isomorphic to the direct product of some [non-empty] subset of $\overline{\mathfrak{A}}$.*

The mapping $\mathrm{Var} : \mathrm{Fm}^\omega_\Sigma \to \wp_\omega(V_\omega)$ assigning the set of all *actually* occurring variables is defined in the standard recursive manner by induction on construction of Σ-formulas. The Σ-substitution extending $[x_i/x_{i+1}]_{i\in\omega}$ is denoted by σ_{+1}.

2.2.1. *Equational implicative systems.* According to [20], an *(equational) implicative system for* a class K of Σ-algebras is any $\mho \subseteq \mathrm{Eq}^4_\Sigma$ such that, for each $\mathfrak{A} \in \mathsf{K}$ and all $\bar{a} \in A^4$, it holds that:

$$((a_0 = a_1) \Rightarrow (a_2 = a_3)) \Leftrightarrow (\mathfrak{A} \models (\bigwedge \mho)[x_i/a_i]_{i\in 4}). \quad (2.2)$$

2.2.2. *Lattice-theoretic background.*

2.2.2.1. Semi-lattices. Let $\diamond$ be a (possibly, secondary) binary connective of Σ.

A Σ-algebra $\mathfrak{A}$ is called a *$\diamond$-semi-lattice*, provided it satisfies semilattice (viz., idempotencity, commutativity and associativity) identities for $\diamond$, in which case we have the partial ordering $\leq^{\mathfrak{A}}_{\diamond}$ on A, given by $(a \leq^{\mathfrak{A}}_{\diamond} b) \stackrel{\text{def}}{\Longleftrightarrow} (a = (a \diamond^{\mathfrak{A}} b))$, for all $a, b \in A$. Then, in case the poset $\langle A, \leq^{\mathfrak{A}}_{\diamond}\rangle$ has the least element (viz., *zero*) [in particular, when A is finite], this is denoted by $\flat^{\mathfrak{A}}_{\diamond}$, while $\mathfrak{A}$ is referred to as a *$\diamond$-semi-lattice with zero (a)* (whenever $a = \flat^{\mathfrak{A}}_{\diamond}$).

Lemma 2.2. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be $\diamond$-semi-lattices with zero and $h \in \hom(\mathfrak{A}, \mathfrak{B})$. Suppose $h[A] = B$. Then, $h(\flat^{\mathfrak{A}}_{\diamond}) = \flat^{\mathfrak{B}}_{\diamond}$.*

Proof. Then, there is some $a \in A$ such that $h(a) = \flat^{\mathfrak{B}}_{\diamond}$, in which case $(a \diamond^{\mathfrak{A}} \flat^{\mathfrak{A}}_{\diamond}) = \flat^{\mathfrak{A}}_{\diamond}$, and so $h(\flat^{\mathfrak{A}}_{\diamond}) = (h(a) \diamond^{\mathfrak{B}} h(\flat^{\mathfrak{A}}_{\diamond})) = (\flat^{\mathfrak{B}}_{\diamond} \diamond^{\mathfrak{B}} h(\flat^{\mathfrak{A}}_{\diamond})) = \flat^{\mathfrak{B}}_{\diamond}$, as required. □

2.2.2.2. Distributive lattices. Let $\barwedge$ and $\veebar$ be (possibly, secondary) binary connectives of Σ tacitly fixed throughout the paper.

A Σ-algebra $\mathfrak{A}$ is called a *[distributive] $(\barwedge, \veebar)$-lattice*, provided it satisfies [distributive] lattice identities for $\barwedge$ and $\veebar$ (viz., semilattice identities for both $\barwedge$ and $\veebar$ as well as mutual [both] absorption [and distributivity] identities for them), in which case $\leq^{\mathfrak{A}}_{\barwedge}$ and $\leq^{\mathfrak{A}}_{\veebar}$ are inverse to one another, and so, in case $\mathfrak{A}$ is a $\veebar$-semi-lattice with zero (in particular, when A is finite), $\flat^{\mathfrak{A}}_{\veebar}$ is the greatest element (viz., *unit*) of the poset $\langle A, \leq^{\mathfrak{A}}_{\barwedge}\rangle$. Then, in case $\mathfrak{A}$ is a {distributive} $(\barwedge, \veebar)$-lattice, it is said to be that *with zero/unit (a)*, whenever it is a $(\barwedge/\veebar)$-semilattice with zero (a).

Let $\Sigma_{+[,01]} \triangleq \{\wedge, \vee[, \bot, \top]\}$ be the [bounded] lattice signature with binary $\wedge$ (conjunction) and $\vee$ (disjunction) [as well as nullary $\bot$ and $\top$ (falsehood/zero and truth/unit constants, respectively)]. Then, a $\Sigma_{+[,01]}$-algebra $\mathfrak{A}$ is called a *[bounded] (distributive) lattice*, whenever it is a (distributive) $(\wedge, \vee)$-lattice [with zero $\bot^{\mathfrak{A}}$ and unit $\top^{\mathfrak{A}}$] {cf., e.g., [2]}.

Given any $n \in (\omega \setminus 2)$, by $\mathfrak{D}_{n[,01]}$ we denote the [bounded] distributive lattice given by the chain $n \div (n-1)$ ordered by $\leqslant$.

Let $\Sigma_{+,\sim[,01]} \triangleq (\Sigma_{+[,01]} \cup \{\sim\})$ with unary $\sim$ (negation) tacitly fixed throughout the paper.

2.3. **Propositional logics and matrices.** A *[finitary/unary/axiomatic] Σ-rule* is any couple $\langle \Gamma, \varphi\rangle$, where $\Gamma \in \wp_{[\omega/(2\setminus 1)/1]}(\mathrm{Fm}^{\omega}_{\Sigma})$ and $\varphi \in \mathrm{Fm}^{\omega}_{\Sigma}$, normally written in the standard sequent form $\Gamma \vdash \varphi$, $\varphi|(\psi \in \Gamma)$ being referred to as the|a *conclusion|premise of* it. A *(substitutional) Σ-instance* of it is then any Σ-rule of the form $\sigma(\Gamma \vdash \varphi) \triangleq (\sigma[\Gamma] \vdash \sigma(\varphi))$, where σ is a Σ-substitution, in this way determining the equally-denoted unary operation on $\wp_{[\omega/(2\setminus 1)/1]}(\mathrm{Fm}^{\omega}_{\Sigma}) \times \mathrm{Fm}^{1}_{\Sigma}$. As usual, axiomatic Σ-rules are called *Σ-axioms* and are identified with their conclusions. A[n] *[axiomatic/finitary/unary] Σ-calculus* is then any set of [axiomatic/finitary/unary] Σ-rules.

A *(propositional/sentential) Σ-logic* (cf., e.g., [6]) is any closure operator C over $\mathrm{Fm}^{\omega}_{\Sigma}$ that is *structural* in the sense that $\sigma[C(X)] \subseteq C(\sigma[X])$, for all $X \subseteq \mathrm{Fm}^{\omega}_{\Sigma}$ and all $\sigma \in \hom(\mathfrak{Fm}^{\omega}_{\Sigma}, \mathfrak{Fm}^{\omega}_{\Sigma})$, that is, $\mathrm{img}\, C$ is closed under inverse Σ-substitutions. In this way, given any set S of [finitary] Σ-logics, $\wp(\mathrm{Fm}^{\omega}_{\Sigma}) \cap \bigcap_{C' \in S}(\mathrm{img}\, C')$ is a[n inductive] closure system over $\mathrm{Fm}^{\omega}_{\Sigma}$, closed under inverse Σ-substitutions, in which case the dual closure operator is a [finitary] Σ-logic, and so this is the complete lattice join of S. Next, C is said to be *[inferentially] (in)consistent*, if $x_1 \notin (\in) C(\varnothing[\cup\{x_0\}])$, the only inconsistent Σ-logic being denoted by IC_{Σ}, the signature subscript being normally omitted, uinless any confusion is possible. Further, a Σ-rule $\Gamma \to \Phi$ is said to be *satisfied/derivable in C*, provided $\Phi \in C(\Gamma)$, Σ-axioms satisfied in C being referred to as *theorems of C*. Next, a

Σ-logic C' is said to be a *(proper) [K-]extension of* C [where $K \subseteq \infty$], whenever $(C[\upharpoonright\wp_{[K]}(\mathrm{Fm}^{\omega}_{\Sigma})]) \subseteq (\subsetneq)(C'[\upharpoonright\wp_{[K]}(\mathrm{Fm}^{\omega}_{\Sigma})])$, in which case C is said to be a *(proper) [K-]sublogic of* C'. In that case, C' and C are said to be *[K-]equivalent* ($C' \equiv_{[K]} C$, in symbols), provided they are [K-]extensions of one another. (In this connection, axiomatically/finitely stands for $1/\omega$, respectively.) Then, a[n axiomatic] Σ-calculus $\mathcal{C}$ is said to *axiomatize* C' *(relatively to* C*)*, if C' is the least Σ-logic (being an extension of C and) satisfying every rule in $\mathcal{C}$ [(in which case it is called an *axiomatic extension of* C)]. Further, a Σ-rule $\mathcal{R}$ is said to be *admissible in* C, provided the extension of C relatively axiomatized by $\mathcal{R}$ is axiomatically-equivalent to C. Clearly, $\mathcal{R}$ is admissible in C, whenever it is derivable in C. Then, C is said to be *structurally/deductively/inferentially complete|maximal*, whenever every Σ-rule, being admissible in C, is derivable in C. Clearly, C is structurally complete iff it has no proper axiomatically-equivalent extension. Then, as the join of the non-empty set of all Σ-logics axiomatically-equivalent to C is so, C has a unique structurally complete axiomatically-equivalent extension, called the *structural completion of* C. Furthermore, we have the finitary sublogic $C_{\exists}$ of C, defined by $C_{\exists}(X) \triangleq (\bigcup C[\wp_{\omega}(X)])$, for all $X \subseteq \mathrm{Fm}^{\omega}_{\Sigma}$, called the *finitarization of* C. Then, the extension of any finitary (in particular, diagonal) Σ-logic relatively axiomatized by a finitary Σ-calculus is a sublogic of its own finitarization, in which case it is equal to this, and so is finitary (in particular, the Σ-logic axiomatized by a finitary Σ-calculus is finitary; conversely, any [finitary] Σ-logic is axiomatized by the [finitary] Σ-calculus consisting of all those [finitary] Σ-rules, which are satisfied in C). Further, C is said to be *[weakly]* $\bar{\wedge}$*-conjunctive*, provided $C(\phi \mathbin{\bar{\wedge}} \psi)[\supseteq] = C(\{\phi, \psi\})$, for all $\phi, \psi \in \mathrm{Fm}^{\omega}_{\Sigma}$, in which case any extension of C is so. Likewise, C is said to be *[weakly]* $\veebar$*-disjunctive*, provided $C(X \cup \{\phi \veebar \psi\})[\subseteq] = (C(X \cup \{\phi\}) \cap C(X \cup \{\psi\}))$, where $(X \cup \{\phi, \psi\}) \subseteq \mathrm{Fm}^{\omega}_{\Sigma}$, in which case [resp. that is, the first two (viz., (2.3) with $i \in 2$) of] the following rules:

$$x_i \vdash (x_0 \veebar x_1), \tag{2.3}$$
$$(x_0 \veebar x_1) \vdash (x_1 \veebar x_0), \tag{2.4}$$
$$(x_0 \veebar x_0) \vdash x_0, \tag{2.5}$$

where $i \in 2$, are satisfied in C, and so in its extensions. Furthermore, C is said to *have Deduction Theorem (DT) with respect to* a (possibly, secondary) binary connective $\sqsupset$ of Σ (tacitly fixed throughout the paper), provided, for all $\phi \in X \subseteq \mathrm{Fm}^{\omega}_{\Sigma}$ and all $\psi \in C(X)$, it holds that $(\phi \sqsupset \psi) \in C(X \setminus \{\phi\})$, in which case the following axioms:

$$x_0 \sqsupset x_0, \tag{2.6}$$
$$x_0 \sqsupset (x_1 \sqsupset x_0) \tag{2.7}$$

are satisfied in C. Then, C is said to be *weakly* $\sqsupset$*-implicative*, if it has DT with respect to $\sqsupset$ and satisfies the *Modus Ponens* rule:

$$\{x_0, x_0 \sqsupset x_1\} \vdash x_1. \tag{2.8}$$

(In general, by C^{MP} we denote the extension of C relatively axiomatized by (2.8).) Likewise, C is said to be *(strongly)* $\sqsupset$*-implicative*, whenever it is weakly so as well as satisfies the *Peirce Law* axiom (cf. [10]):

$$(((x_0 \sqsupset x_1) \sqsupset x_0) \sqsupset x_0). \tag{2.9}$$

Then, C is said to be *[{axiomatically} (pre)maximally]* $\sim$*-paraconsistent*, provided it does not satisfy the *Ex Contradictione Quodlibet* rule:

$$\{x_0, \sim x_0\} \vdash x_1 \tag{2.10}$$

[and has no (more than one) proper $\sim$-paraconsistent {axiomatic} extension]. Likewise, C is said to be $\sqsupset$*-implicatively* $\sim$*-paraconsistent*, provided it does not satisfy the *Ex Contradictione Quodlibet* axiom:

$$\sim x_0 \sqsupset (x_0 \sqsupset x_1). \tag{2.11}$$

(Clearly, C is non-$\sim$-paraconsistent if[f] it is $\sqsupset$-implicatively so, whenever it satisfies (2.8) [and has DT with respect to $\sqsupset$].) In general, by $C^{[\mathrm{I}]\mathrm{NP}}$ we denote the least [$\sqsupset$-implicatively] non-$\sim$-paraconsistent extension of C, that is, the extension relatively axiomatized by (2.10) [resp. by (2.11)]. Further, C is said to be *(⟨pre⟩maximally {axiomatically}) [inferentially]* $(\veebar, \sim)$*-paracomplete*, whenever $(x_1 \veebar \sim x_1) \notin C(\varnothing[\cup\{x_0\}])$ (and C has no ⟨more than one⟩ proper {axiomatic} [inferentially] $(\veebar, \sim)$-paracomplete extension). In general, by C^{EM} we denote the extension of C relatively axiomatized by the *Excluded Middle Law* axiom:

$$x_0 \veebar \sim x_0. \tag{2.12}$$

Finally, C is said to be *theorem-less/purely-inferential*, whenever it has no theorem, that is, $\varnothing \in (\operatorname{img} C)$. Likewise, C is said to be *[non-]pseudo-axiomatic*, provided $\bigcap_{k\in\omega} C(x_k) \nsubseteq [\subseteq] C(\varnothing)$ [in which case it is $(\veebar, \sim)$-paracomplete/(in)consistent iff it is inferentially so]. In general, $(\operatorname{img} C) \cup \{\varnothing\}$ is closed under inverse Σ-substitutions, for $\operatorname{img} C$ is so, in which case the dual closure operator C_{+0} is the greatest purely-inferential sublogic of C, called the *purely-inferential/theorem-less version of* C, while:

$$(C_{+0} \upharpoonright \wp_{\infty\setminus 1}(\mathrm{Fm}^{\omega}_{\Sigma})) = (C \upharpoonright \wp_{\infty\setminus 1}(\mathrm{Fm}^{\omega}_{\Sigma})). \tag{2.13}$$

Likewise, $C_{-0} \triangleq ((C \upharpoonright \wp_{\infty\setminus 1}(\mathrm{Fm}^{\omega}_{\Sigma})) \cup \{\langle \varnothing, \bigcap_{k\in\omega} C(x_k)\rangle\}$ is the least non-pseudo-axiomatic extension of C called the *non-pseudo-axiomatic version of* C, in which case, by (2.13), we have:

$$(C_{+/-0})_{-/+0} = C, \tag{2.14}$$

whenever C is non-pseudo-axiomatic/purely-inferential, respectively, and so this provides an isomorphism between the posets of all non-pseudo-axiomatic and purely-inferential Σ-logics ordered by $\subseteq$.

Remark 2.3. By (2.14), the purely-inferential version of the axiomatic extension of a non-pseudo-axiomatic Σ-logic, relatively-axiomatized by an $\mathcal{A} \subseteq \mathrm{Fm}^{\omega}_{\Sigma}$, is relatively axiomatized by $\{x_0 \vdash \sigma_{+1}(\varphi) \mid \varphi \in \mathcal{A}\}$; □

Remark 2.4. Any purely-inferential inferentially consistent Σ-logic C is a proper sublogic of the unique purely-inferential inferentially inconsistent Σ-logic IC_{+0}, and so is not structurally complete, in which case IC_{+0} is the structural completion of C, for $(\operatorname{img} \mathrm{IC}_{+0}) = \{\mathrm{Fm}^{\omega}_{\Sigma}, \varnothing\}$, [relatively] axiomatized by $x_0 \vdash x_1$. □

A *(logical)* Σ*-matrix* (cf. [6]) is any couple of the form $\mathcal{A} = \langle \mathfrak{A}, D^{\mathcal{A}}\rangle$, where $\mathfrak{A}$ is a Σ-algebra, called the *underlying algebra of* $\mathcal{A}$, while $\lceil\mathcal{A}\rceil \triangleq A$ is called the *carrier/"underlying set" of* $\mathcal{A}$, whereas $D^{\mathcal{A}} \subseteq A$ is called the *truth predicate of* $\mathcal{A}$, elements of $A[\cap D^{\mathcal{A}}]$ being referred to as *[distinguished] values of* $\mathcal{A}$. (In general, matrices are denoted by Calligraphic letters [possibly, with indices], their underlying algebras being denoted by corresponding Fraktur letters [with same indices, if any].) This is said to be *n-valued/[in]consistent/truth(-non)-empty/truth-|false-{non-}singular*, where $n \in (\omega \setminus 1)$, provided $(|A| = n)/(D^{\mathcal{A}} \neq [=]A)/(D^{\mathcal{A}} = (\neq)\varnothing)/(|(D^{\mathcal{A}}|(A \setminus D^{\mathcal{A}}))| \in \{\notin\}2)$, respectively. Next, given any $\Sigma' \subseteq \Sigma$, $\mathcal{A}$ is said to be a *(*Σ*-)expansion of* its Σ'*-reduct* $(\mathcal{A}\upharpoonright\Sigma') \triangleq \langle \mathfrak{A}\upharpoonright\Sigma', D^{\mathcal{A}}\rangle$. (Any notation, being specified for single matrices, is supposed to be extended to classes of matrices member-wise.) Finally, $\mathcal{A}$ is said to be *finite[ly-generated]/"generated by* $B \subseteq A$*"*, whenever $\mathfrak{A}$ is so.

Given any $\alpha \in \wp_{\infty[\setminus 1]}(\omega)$ [whenever Σ is constant-free] and any class M of Σ-matrices, we have the closure operator $\mathrm{Cn}^{\alpha}_{\mathsf{M}}$ over $\mathrm{Fm}^{\alpha}_{\Sigma}$ dual to the closure system with basis $\{h^{-1}[D^{\mathcal{A}}] \mid \mathcal{A} \in \mathsf{M}, h \in \hom(\mathfrak{Fm}^{\alpha}_{\Sigma}, \mathfrak{A})\}$, in which case:

$$\mathrm{Cn}^{\alpha}_{\mathsf{M}}(X) = (\mathrm{Fm}^{\alpha}_{\Sigma} \cap \mathrm{Cn}^{\omega}_{\mathsf{M}}(X)), \tag{2.15}$$

for all $X \subseteq \mathrm{Fm}^{\alpha}_{\Sigma}$. Then, by (2.1), $\mathrm{Cn}^{\omega}_{\mathsf{M}}$ is a Σ-logic, called the *logic of* M, a Σ-logic C being said to be *[finitely-]defined by* M, provided it is [finitely-]equivalent to $\mathrm{Cn}^{\omega}_{\mathsf{M}}$. A Σ-logic is said to be *(unitary/uniform) n-valued*, where $n \in (\omega \setminus 1)$, whenever it is defined by an n-valued Σ-matrix, in which case it is finitary (cf. [6]), and so is the logic of any finite class of finite Σ-matrices.

As usual, Σ-matrices are treated as first-order model structures (viz., algebraic systems; cf. [9]) of the first-order signature $\Sigma \cup \{D\}$ with unary predicate D, any [in]finitary Σ-rule $\Gamma \vdash \phi$ being viewed as the [in]finitary equality-free basic strict Horn formula $(\bigwedge \Gamma) \to \phi$ under the standard identification of any propositional Σ-formula ψ with the first-order atomic formula $D(\psi)$, as well as being *true/satisfied in* a class M of Σ-matrices iff it being satisfied in the logic of M.

Remark 2.5. Since any Σ-formula contains just finitely many variables, and so there is a variable not occurring in it, the logic of any class of truth-non-empty Σ-matrices is non-pseudo-axiomatic. □

Remark 2.6. Since any rule with[out] premises is [not] true in any truth-empty matrix, taking Remark 2.5 into account, given any class M of Σ-matrices, the purely-inferential/non-pseudo-axiomatic version of the logic of M is defined by $\mathsf{M} \cup / \setminus \mathsf{S}$, where S is "any non-empty class of truth-empty Σ-matrices"/"the class of all truth-empty members of M", respectively. □

Let $\mathcal{A}$ and $\mathcal{B}$ be two Σ-matrices. A *(strict) [surjective] {matrix} homomorphism from $\mathcal{A}$ [on]to $\mathcal{B}$* is any $h \in \hom(\mathfrak{A}, \mathfrak{B})$ such that $[h[A] = B$ and$]$ $D^{\mathcal{A}} \subseteq (=) h^{-1}[D^{\mathcal{B}}]$, the set of all them being denoted by $\hom^{[\mathrm{S}]}_{(\mathrm{S})}(\mathcal{A}, \mathcal{B})$, in which case $\mathcal{B}/\mathcal{A}$ is said to be a *(strict) [surjective] {matrix} homomorphic image/counter-image of $\mathcal{A}/\mathcal{B}$*, respectively. Then, by (2.1), we have:

$$(\exists h \in \hom^{[\mathrm{S}]}_{\mathrm{S}}(\mathcal{A}, \mathcal{B})) \quad \Rightarrow \quad (\mathrm{Cn}^{\alpha}_{\mathcal{B}} \subseteq [=] \, \mathrm{Cn}^{\alpha}_{\mathcal{A}}), \tag{2.16}$$

$$(\exists h \in \hom^{\mathrm{S}}(\mathcal{A}, \mathcal{B})) \quad \Rightarrow \quad (\mathrm{Cn}^{\alpha}_{\mathcal{A}}(\varnothing) \subseteq \mathrm{Cn}^{\alpha}_{\mathcal{B}}(\varnothing)), \tag{2.17}$$

for all $\alpha \in \wp_{\infty[\setminus 1]}(\omega)$ [unless Σ has a nullary connective]. Further, $\mathcal{A}[\neq \mathcal{B}]$ is said to be a *[proper] submatrix of* $\mathcal{B}$, whenever $\Delta_A \in \hom_{\mathrm{S}}(\mathcal{A}, \mathcal{B})$, in which case we set $(\mathcal{B} \restriction A) \triangleq \mathcal{A}$. Injective/bijective strict homomorphisms from $\mathcal{A}$ to $\mathcal{B}$ are referred to as *embeddings/isomorphisms of/from $\mathcal{A}$ into/onto $\mathcal{B}$*, in case of existence of which $\mathcal{A}$ is said to be *embeddable/isomorphic into/to* $\mathcal{B}$, respectively.

Given a Σ-matrix $\mathcal{A}$, $\chi^{\mathcal{A}} \triangleq \chi^{D^{\mathcal{A}}}_{A}$ is referred to as the *characteristic function of* $\mathcal{A}$. Then, any $\theta \in \mathrm{Con}(\mathfrak{A})$ such that $\theta \subseteq \theta^{\mathcal{A}} \triangleq (\ker \chi^{\mathcal{A}})$, in which case ν_{θ} is a strict surjective homomorphism from $\mathcal{A}$ onto $(\mathcal{A}/\theta) \triangleq \langle \mathfrak{A}/\theta, D^{\mathcal{A}}/\theta \rangle$, is called a *congruence of* $\mathcal{A}$, the set of all them being denoted by $\mathrm{Con}(\mathcal{A})$. Given any $\theta, \vartheta \in \mathrm{Con}(\mathcal{A})$, the transitive closure $\theta \amalg \vartheta$ of $\theta \cup \vartheta$, being a congruence of $\mathfrak{A}$, is then that of $\mathcal{A}$, for $\theta^{\mathcal{A}}$, being an equivalence relation, is transitive. In particular, any maximal congruence of $\mathcal{A}$ (that exists, by Zorn Lemma, because $\mathrm{Con}(\mathcal{A}) \ni \Delta_A$ is both non-empty and inductive, for $\mathrm{Con}(\mathfrak{A})$ is so) is the greatest one to be denoted by $\eth(\mathcal{A})$. Finally, $\mathcal{A}$ is said to be *[hereditarily] simple*, provided it has no non-diagonal congruence [and no non-simple submatrix].

Remark 2.7. *Let $\mathcal{A}$ and $\mathcal{B}$ be two Σ-matrices and $h \in \hom_{\mathrm{S}}(\mathcal{A}, \mathcal{B})$. Then,*

(i) $\theta^{\mathcal{A}} = h^{-1}[\theta^{\mathcal{B}}]$.

Moreover, $f \triangleq \{\langle \theta, h^{-1}[\theta] \rangle \mid \theta \in \mathrm{Con}(\mathfrak{B})\} : \mathrm{Con}(\mathfrak{B}) \to \mathrm{Con}(\mathfrak{A})$. Therefore,

(ii) $f' \triangleq (f \upharpoonright \mathrm{Con}(\mathcal{B})) : \mathrm{Con}(\mathcal{B}) \to \mathrm{Con}(\mathcal{A})$.

In particular (when $\theta = \Delta_B \in \mathrm{Con}(\mathcal{B})$), $(\ker h) = h^{-1}[\Delta_B] \in \mathrm{Con}(\mathcal{A})$, in which case $(\ker h) \subseteq \eth(\mathcal{A})$, and so

(iii) *h is injective, whenever $\mathcal{A}$ is simple.* □

A Σ-matrix $\mathcal{A}$ is said to be a *[K-]model of* a Σ-logic C [where $K \subseteq \infty$], provided C is a [K-]sublogic of the logic of $\mathcal{A}$, the class of all them being denoted by $\mathrm{Mod}_{[K]}(C)$, respectively. Next, $\mathcal{A}$ is said to be *"($\sqsupset$-implicatively) $\sim$-paraconsistent"/"[inferentially] ($\veebar, \sim$)-paracomplete"*, whenever the logic of $\mathcal{A}$ is so. Further, $\mathcal{A}$ is said to be *[weakly] $\diamond$-conjunctive*, where $\diamond$ is a (possibly, secondary) binary connective of Σ, provided $(\{a, b\} \subseteq D^{\mathcal{A}})[\Leftarrow] \Leftrightarrow ((a \diamond^{\mathfrak{A}} b) \in D^{\mathcal{A}})$, for all $a, b \in A$, that is, the logic of $\mathcal{A}$ is [weakly] $\diamond$-conjunctive. Then, $\mathcal{A}$ is said to be *[weakly] $\diamond$-disjunctive*, whenever $\langle \mathfrak{A}, A \setminus D^{\mathcal{A}} \rangle$ is [weakly] $\diamond$-conjunctive, in which case [resp., that is] the logic of $\mathcal{A}$ is [weakly] $\diamond$-disjunctive, and so is the logic of any class of [weakly] $\diamond$-disjunctive Σ-matrices. Likewise, $\mathcal{A}$ is said to be *$\sqsupset$-implicative*, whenever $((a \in D^{\mathcal{A}}) \Rightarrow (b \in D^{\mathcal{A}})) \Leftrightarrow ((a \sqsupset^{\mathfrak{A}} b) \in D^{\mathcal{A}})$, for all $a, b \in A$, in which case it is $\uplus_{\sqsupset}$-disjunctive, where $(x_0 \uplus_{\sqsupset} x_1) \triangleq ((x_0 \sqsupset x_1) \sqsupset x_1)$, while the logic of $\mathcal{A}$ is $\sqsupset$-implicative, for both (2.8) and (2.9) $= ((x_0 \sqsupset x_1) \uplus_{\sqsupset} x_0)$ are true in any $\sqsupset$-implicative (and so $\uplus_{\sqsupset}$-disjunctive) Σ-matrix, while DT is immediate, and so is the logic of any class of $\sqsupset$-implicative Σ-matrices. Finally, given any (possibly secondary) unary connective $\wr$ of Σ, put $(x_0 \diamond^{\wr} x_1) \triangleq \wr(\wr x_0 \diamond \wr x_1)$ and $(x_0 \sqsupset^{\wr}_{\diamond} x_1) \triangleq (\wr x_0 \diamond x_1)$. Then, $\mathcal{A}$ is said to be *[weakly] (classically) $\wr$-negative*, provided, for all $a \in A$, $(a \in D^{\mathcal{A}})[\Leftarrow] \Leftrightarrow (\wr^{\mathfrak{A}} a \notin D^{\mathcal{A}})$, in which case it is [truth-non-empty], and so consistent.

Remark 2.8. Let $\diamond$ and $\wr$ be as above. Then, the following hold:

(i) any (weakly) $\wr$-negative Σ-matrix $\mathcal{A}$:
   a) is [weakly] $\diamond$-disjunctive/-conjunctive iff it is [weakly] $\diamond^{\wr}$-conjunctive/-disjunctive, respectively;
   b) defines a logic having PWC with respect to $\wr \in \Sigma$;
   c) is $\sqsupset^{\wr}_{\diamond}$-implicative, whenever it is $\diamond$-disjunctive;
   d) is not $\wr$-paraconsistent(/($\diamond, \wr$)-paracomplete), whenever $\wr \in \Sigma$(/ while $\mathcal{A}$ is weakly $\diamond$-disjunctive).

(ii) given any two Σ-matrices $\mathcal{A}$ and $\mathcal{B}$ and any $h \in \hom^{[\mathrm{S}]}_{\mathrm{S}}(\mathcal{A}, \mathcal{B})$, $\mathcal{A}$ is (weakly) $\wr$-negative|$\diamond$-conjunctive/-disjunctive/-implicative if[f] $\mathcal{B}$ is so;

(iii) the direct product of any tuple of Σ-matrices is not $\wr$-paraconsistent, where $\wr \in \Sigma$, whenever the tuple image contains a non-$\wr$-paraconsistent consistent Σ-matrix. □

Given a set I and an I-tuple $\overline{\mathcal{A}}$ of Σ-matrices, [any submatrix $\mathcal{B}$ of] the Σ-matrix $(\prod_{i \in I} \mathcal{A}_i) \triangleq \langle \prod_{i \in I} \mathfrak{A}_i, \prod_{i \in I} D^{\mathcal{A}_i} \rangle$ is called the [a] *[sub]direct product of $\overline{\mathcal{A}}$* [whenever, for each $i \in I$, $\pi_i[B] = A_i$]. As usual, if $(\mathrm{img}\, \overline{\mathcal{A}}) \subseteq \{\mathcal{A}\}$ (and $I = 2$), where $\mathcal{A}$ is a Σ-matrix, $\mathcal{A}^I \triangleq (\prod_{i \in I} \mathcal{A}_i)$ [resp., $\mathcal{B}$] is called the [a] *[sub]direct I-power (square) of $\mathcal{A}$*.

Given a class M of Σ-matrices, the class of all "strict surjective homomorphic [counter-]images"/"(consistent) submatrices" of members of M is denoted by $(\mathbf{H}^{[-1]}/\mathbf{S}_{(*)})(\mathsf{M})$, respectively. Likewise, the class of all [sub]direct products of tuples (of cardinality $\in K \subseteq \infty$) constituted by members of M is denoted by $\mathbf{P}^{[\mathrm{SD}]}_{(K)}(\mathsf{M})$. (Logic model classes, being actually infinitary equality-free universal Horn theory model classes, are well known to be closed under $\mathbf{P}$.)

Lemma 2.9. *Let M be a class of Σ-matrices. Then, $\mathbf{H}(\mathbf{H}^{-1}(\mathsf{M})) \subseteq \mathbf{H}^{-1}(\mathbf{H}(\mathsf{M}))$.*

Proof. Let $\mathcal{A}$ and $\mathcal{B}$ be Σ-matrices, $\mathcal{C} \in \mathsf{M}$ and $(h|g) \in \hom_{\mathrm{S}}^{\mathrm{S}}(\mathcal{B}, \mathcal{C}|\mathcal{A})$. Then, by Remark 2.7(ii), $(\ker(h|g)) \in \mathrm{Con}(\mathcal{B})$, in which case $(\ker(h|g)) \subseteq \theta \triangleq \eth(\mathcal{B}) \in \mathrm{Con}(\mathcal{B})$, and so, by the Homomorphism Theorem, $(\nu_\theta \circ (h|g)^{-1}) \in \hom_{\mathrm{S}}^{\mathrm{S}}(\mathcal{C}|\mathcal{A}, \mathcal{B}/\theta)$, as required. □

Lemma 2.10 (Finite Subdirect Product Lemma; cf. Lemma 2.7 of [21])**.** *Let* M *be a finite class of finite* Σ*-matrices and* $\mathcal{A}$ *a finitely-generated (in particular, finite) model of the logic of* M*. Then,* $\mathcal{A} \in \mathbf{H}^{-1}(\mathbf{H}(\mathbf{P}^{\mathrm{SD}}_{\omega}(\mathbf{S}_*(\mathsf{M}))))$.

Lemma 2.11. *Let* M *be a class of weakly* $\veebar$*-disjunctive* Σ*-matrices,* I *a finite set,* $\overline{\mathcal{C}} \in \mathsf{M}^I$*, and* $\mathcal{D}$ *a consistent* $\veebar$*-disjunctive submatrix of* $\prod \overline{\mathcal{C}}$*. Then, there is some* $i \in I$ *such that* $(\pi_i \restriction D) \in \hom_{\mathrm{S}}(\mathcal{D}, \mathcal{C}_i)$.

Proof. By contradiction. For suppose that, for every $i \in I$, $(\pi_i \restriction D) \notin \hom_{\mathrm{S}}(\mathcal{D}, \mathcal{C}_i)$, in which case $D^{\mathcal{D}} \subsetneq (\pi_i \restriction D)^{-1}[D^{\mathcal{C}_i}] = (D \cap \pi_i^{-1}[D^{\mathcal{C}_i}])$, for $(\pi_i \restriction D) \in \hom(\mathcal{D}, \mathcal{C}_i)$, and so there is some $a_i \in (D \setminus D^{\mathcal{D}})$ such that $\pi_i(a_i) \in D^{\mathcal{C}_i}$. By induction on the cardinality of any $J \subseteq I$, let us prove that there is some $b \in (D \setminus D^{\mathcal{D}})$ such that $\pi_j(b) \in D^{\mathcal{C}_j}$, for all $j \in J$, as follows. In case $J = \varnothing$, take any $b \in (D \setminus D^{\mathcal{D}}) \neq \varnothing$, for $\mathcal{D}$ is consistent. Otherwise, take any $j \in J$, in which case $K \triangleq (J \setminus \{j\}) \subseteq I$, while $|K| < |J|$, so, by the induction hypothesis, there is some $c \in (D \setminus D^{\mathcal{D}})$ such that $\pi_k(c) \in D^{\mathcal{C}_k}$, for all $k \in K$. Then, by the $\veebar$-disjunctivity of $\mathcal{D}$, $b \triangleq (c \veebar^{\mathfrak{D}} a_j) \in (D \setminus D^{\mathcal{D}})$, while $\pi_i(b) \in D^{\mathcal{C}_i}$, for all $i \in J = (K \cup \{j\})$, because $(\pi_i \restriction D) \in \hom(\mathfrak{D}, \mathfrak{C}_i)$, while $\mathcal{C}_i$ is weakly $\veebar$-disjunctive. In particular, when $J = I$, there is some $b \in (D \setminus D^{\mathcal{D}})$ such that $\pi_i(b) \in D^{\mathcal{C}_i}$, for all $i \in I$. This contradicts to the fact that $D^{\mathcal{D}} = (D \cap \bigcap_{i \in I} \pi_i^{-1}[D^{\mathcal{C}_i}])$, as required. □

By Lemmas 2.9, 2.10, 2.11 and Remark 2.8(ii), we immediately have:

Corollary 2.12. *Let* M *be a finite class of finite weakly* $\veebar$*-disjunctive* Σ*-matrices and* $\mathcal{A}$ *a finitely-generated (in particular, finite) consistent* $\veebar$*-disjunctive model of the logic of* M*. Then,* $\mathcal{A} \in \mathbf{H}^{-1}(\mathbf{H}(\mathbf{S}_*(\mathsf{M})))$.

Corollary 2.13. *Let* C *be a* Σ*-logic. (Suppose it is defined by a finite class* M *of finite [weakly* $\veebar$*-disjunctive]* Σ*-matrices.) Then, (i)⇔(ii)⇔(iii)(⇔(iv)), where:*

(i) C *is purely-inferential;*
(ii) C *has a truth-empty model;*
(iii) C *has a one-valued truth-empty model;*
(iv) $\mathbf{P}^{\mathrm{SD}}_{\omega[\cap 0]}(\mathbf{S}_*(\mathsf{M}))[\cup \mathbf{S}_*(\mathsf{M})]$ *has a truth-empty member.*

Proof. First, (ii)⇒(i) is immediate. The converse is by the fact that, by the structurality of C, $\langle \mathfrak{Fm}^{\omega}_{\Sigma}, C(\varnothing) \rangle$ is a model of C.

Next, (ii) is a particular case of (iii). Conversely, let $\mathcal{A} \in \mathrm{Mod}(C)$ be truth-empty. Then, $(\mathrm{img}\, \chi^{\mathcal{A}}) = \{0\}$, in which case $\theta^{\mathcal{A}} = A^2 \in \mathrm{Con}(\mathfrak{A})$, and so, by (2.16), $(\mathcal{A}/\theta^{\mathcal{A}}) \in \mathrm{Mod}(C)$ is both one-valued and truth-empty.

(Finally, (iv)⇒(ii) is by (2.16). Conversely, (iii)⇒(iv) is by Lemma 2.10 [resp., Corollary 2.12 and the $\veebar$-disjunctivity of truth-empty Σ-matrices].) □

Theorem 2.14 (cf. Theorem 2.8 of [21])**.** *Let* K *and* M *be classes of* Σ*-matrices,* C *the logic of* M *and* C' *an extension of* C*. Suppose [both* M *and all members of it are finite and]* $\mathbf{P}^{\mathrm{SD}}_{[\omega]}(\mathbf{S}_*(\mathsf{M})) \subseteq \mathsf{K}$ *(in particular,* $\mathbf{S}(\mathbf{P}_{[\omega]}(\mathsf{M})) \subseteq \mathsf{K}$ *{in particular,* $\mathsf{K} \supseteq \mathsf{M}$ *is closed under both* $\mathbf{S}$ *and* $\mathbf{P}_{[\omega]}\langle$ *in particular,* $\mathsf{K} = \mathrm{Mod}(C)\rangle\}$*). Then,* C' *is [finitely-]defined by* $\mathrm{Mod}(C') \cap \mathsf{K}$*, and so by* $\mathrm{Mod}(C')$.

Corollary 2.15 (cf. Corollary 2.9 of [21])**.** *Let* M *be a class of* Σ*-matrices and* $\mathcal{A}$ *an axiomatic* Σ*-calculus. Then, the axiomatic extension of the logic of* M *relatively axiomatized by* $\mathcal{A}$ *is defined by* $\mathbf{S}_*(\mathsf{M}) \cap \mathrm{Mod}(\mathcal{A})$.

Given any Σ-logic C and any $\Sigma' \subseteq \Sigma$, in which case $\mathrm{Fm}^{\alpha}_{\Sigma'} \subseteq \mathrm{Fm}^{\alpha}_{\Sigma}$ and $\hom(\mathfrak{Fm}^{\alpha}_{\Sigma'}, \mathfrak{Fm}^{\alpha}_{\Sigma'}) = \{h{\restriction}\mathrm{Fm}^{\alpha}_{\Sigma'} \mid h \in \hom(\mathfrak{Fm}^{\alpha}_{\Sigma}, \mathfrak{Fm}^{\alpha}_{\Sigma}), h[\mathrm{Fm}^{\alpha}_{\Sigma'}] \subseteq \mathrm{Fm}^{\alpha}_{\Sigma'}\}$, for all $\alpha \in \wp_{\infty\setminus 1}(\omega)$, we have the Σ'-logic C', defined by $C'(X) \triangleq (\mathrm{Fm}^{\omega}_{\Sigma'} \cap C(X))$, for all $X \subseteq \mathrm{Fm}^{\omega}_{\Sigma'}$, called the *$\Sigma'$-fragment of C*, in which case C is said to be a *(Σ-)expansion of C'*, while, given any class M of Σ-matrices, C' is defined by $\mathsf{M}{\restriction}\Sigma'$, whenever C is defined by M.

3. Preliminary key adnanced generic issues

3.1. False-singular consistent weakly conjunctive matrices.

Lemma 3.1. *Let $\mathcal{A}$ be a false-singular weakly $\overline{\wedge}$-conjunctive Σ-matrix, $f \in (A \setminus D^{\mathcal{A}})$, I a finite set, $\overline{\mathcal{C}}$ an I-tuple constituted by consistent submatrices of $\mathcal{A}$ and $\mathcal{B}$ a subdirect product of $\overline{\mathcal{C}}$. Then, $(I \times \{f\}) \in B$.*

Proof. By induction on the cardinality of any $J \subseteq I$, let us prove that there is some $a \in B$ including $(J \times \{f\})$. First, when $J = \varnothing$, take any $a \in B \neq \varnothing$, in which case $(J \times \{f\}) = \varnothing \subseteq a$. Now, assume $J \neq \varnothing$. Take any $j \in J \subseteq I$, in which case $K \triangleq (J \setminus \{j\}) \subseteq I$, while $|K| < |J|$, and so, as $\mathcal{C}_j$ is a consistent submatrix of the false-singular Σ-matrix $\mathcal{A}$, we have $f \in C_j = \pi_j[B]$. Hence, there is some $b \in B$ such that $\pi_j(b) = f$, while, by induction hypothesis, there is some $a \in B$ including $(K \times \{f\})$. Therefore, since $J = (K \cup \{j\})$, while $\mathcal{A}$ is both weakly $\overline{\wedge}$-conjunctive and false-singular, we have $B \ni c \triangleq (a \mathbin{\overline{\wedge}^{\mathfrak{B}}} b) \supseteq (J \times \{f\})$. Thus, when $J = I$, we eventually get $B \ni (I \times \{f\})$, as required. $\square$

3.2. Equality determinants versus matrix simplicity.

A *(binary) relational Σ-scheme* is any Σ-calculus of the form $\varepsilon \subseteq (\wp(\mathrm{Fm}^2_{\Sigma}) \times \mathrm{Fm}^2_{\Sigma})$, in which case, given any Σ-matrix $\mathcal{A}$, we set $\theta^{\mathcal{A}}_{\varepsilon} \triangleq \{\langle a, b\rangle \in A^2 \mid \mathcal{A} \models (\bigwedge \varepsilon)[x_0/a, x_1/b]\} \subseteq A^2$. Given a one more Σ-matrix $\mathcal{B}$ and any $h \in \hom_{(\mathrm{S})}(\mathcal{A}, \mathcal{B})$ [being strict, unless ε is axiomatic], we have:

$$h^{-1}[\theta^{\mathcal{B}}_{\varepsilon}](\subseteq)[\supseteq]\theta^{\mathcal{A}}_{\varepsilon}. \tag{3.1}$$

A *unitary relational Σ-scheme* is any $\Upsilon \subseteq \mathrm{Fm}^1_{\Sigma}$, in which case we have the unary relational Σ-scheme $\varepsilon_{\Upsilon} \triangleq \{(\upsilon[x_0/x_i]) \vdash (\upsilon[x_0/x_{1-i}]) \mid i \in 2, \upsilon \in \Upsilon\}$.

A *(binary) equality determinant for* a class of Σ-matrices M is any relational Σ-scheme ε such that, for each $\mathcal{A} \in \mathsf{M}$, $\theta^{\mathcal{A}}_{\varepsilon} = \Delta_A$, that includes a finite one, whenever both M and all members of it are finite.

Then, according to [18], a *unitary equality determinant for* a class of Σ-matrices M is any unitary relational Σ-scheme Υ such that ε_{Υ} is an equality determinant for M that includes a finite one, whenever both M and all members of it are finite. (It is unitary equality determinants that are equality determinants in the sense of [18].)

Lemma 3.2. *Let $\mathcal{A}$ be a Σ-matrix, $\theta \in \mathrm{Con}(\mathcal{A})$ and ε a relational Σ-scheme. Then, $\theta \subseteq \theta^{\mathcal{A}}_{\varepsilon}$, whenever $\Delta_A \subseteq \theta^{\mathcal{A}}_{\varepsilon}$. In particular, $\mathcal{A}$ is simple, whenever ε is an equality determinant for it.*

Proof. Let $\mathcal{B} \triangleq (\mathcal{A}/\theta)$, in which case $h \triangleq \nu_{\theta} \in \hom^{\mathrm{S}}_{\mathrm{S}}(\mathcal{A}, \mathcal{B})$. Consider any $\langle a, b\rangle \in \theta$, in which case $h(a) = h(b)$. Therefore, if $\Delta_A \subseteq \theta^{\mathcal{A}}_{\varepsilon}$, then we have $\langle a, a\rangle \in \theta^{\mathcal{A}}_{\varepsilon}$, in which case, by (3.1), we get $\langle h(a), h(b)\rangle = \langle h(a), h(a)\rangle \in \theta^{\mathcal{B}}_{\varepsilon}$, and so we eventually get $\langle a, b\rangle \in \theta^{\mathcal{A}}_{\varepsilon}$, as required. $\square$

Lemma 3.3. *Let $\mathcal{A}$ and $\mathcal{B}$ be Σ-matrices, ε an equality determinant for $\mathcal{B}$ and h an embedding of $\mathcal{A}$ into $\mathcal{B}$. Then, ε is an equality determinant for $\mathcal{A}$.*

Proof. In that case, by (3.1), we have $\theta^{\mathcal{A}}_{\varepsilon} = h^{-1}[\theta^{\mathcal{B}}_{\varepsilon}]$. In this way, the injectivity of h completes the argument. $\square$

Theorem 3.4. *Let $\mathcal{A}$ be a Σ-matrix. Then, the following are equivalent:*

(i) *$\mathcal{A}$ is hereditarily simple;*
(ii) *$\mathcal{A}$ has an equality determinant;*
(iii) *$\mathcal{A}$ has a unary equality determinant.*

Proof. First, (ii) is a particular case of (iii), (ii)$\Rightarrow$(i) being by Lemmas 3.2 and 3.3.

Finally, assume (i) holds. Let $\varepsilon \triangleq \{\phi_i \vdash \phi_{1-i} \mid i \in 2, \bar{\phi} \in (\mathrm{Fm}^2_\Sigma)^2, (\phi_0[x_1/x_0]) = (\phi_1[x_1/x_0])\}$. Then, $\Delta_A \subseteq \theta^{\mathcal{A}}_\varepsilon$. Conversely, consider any $\bar{a} \in (A^2 \setminus \Delta_A)$. Let $\mathcal{B}$ be the submatrix of $\mathcal{A}$ generated by $\operatorname{img}\bar{a}$. Then, it is simple, by (i). Therefore, $\theta \triangleq \mathrm{Cg}^{\mathfrak{B}}(\bar{a}) \nsubseteq \theta^{\mathcal{B}}$, for $\theta \ni \bar{a} \notin \Delta_B$ is a non-diagonal congruence of $\mathfrak{B}$. Let $\vartheta \triangleq \{\langle \varphi^{\mathfrak{B}}[x_0/a_j; x_{k+1}/c_k]_{k\in(n-1)}, \varphi^{\mathfrak{B}}[x_0/a_{1-j}; x_{k+1}/c_k]_{k\in(n-1)}\rangle \mid j \in 2, n \in (\omega \setminus 1), \varphi \in \mathrm{Fm}^n_\Sigma, \bar{c} \in B^{n-1}\}$. Then, by Mal'cev's Principal Congruence Lemma [8], θ is the transitive closure of ϑ. Hence, $\theta^{\mathcal{B}}$, being transitive, does not include ϑ, in which case there are some $j \in 2$, some $n \in (\omega \setminus 1)$, some $\varphi \in \mathrm{Fm}^n_\Sigma$ and some $\bar{c} \in B^{n-1}$ such that $\langle \varphi^{\mathfrak{B}}[x_0/a_j; x_{k+1}/c_k]_{k\in(n-1)}, \varphi^{\mathfrak{B}}[x_0/a_{1-j}; x_{k+1}/c_k]_{k\in(n-1)}\rangle \notin \theta^{\mathcal{B}}$, in which case there is some $i \in 2$ such that $\varphi^{\mathfrak{B}}[x_0/a_i; x_{k+1}/c_k]_{k\in(n-1)} \in D^{\mathcal{B}} \not\ni \varphi^{\mathfrak{B}}[x_0/a_{1-i}; x_{k+1}/c_k]_{k\in(n-1)}$, while, as $\mathfrak{B}$ is generated by $\operatorname{img}\bar{a}$, there is some $\bar{\psi} \in (\mathrm{Fm}^2_\Sigma)^{n-1}$ such that $c_k = \psi^{\mathfrak{B}}[x_l/a_l]_{l\in 2}$, for all $k \in (n-1)$, and so $\phi_i^{\mathfrak{B}}[x_l/a_l]_{l\in 2} \in D^{\mathcal{B}} \not\ni \phi^{\mathfrak{B}}_{1-i}[x_l/a_l]_{l\in 2}$, where, for each $m \in 2$, $\phi_m \triangleq (\varphi[x_0/x_m; x_{k+1}/\psi_k]_{k\in(n-1)} \in \mathrm{Fm}^2_\Sigma$. Moreover, $(\phi_0[x_1/x_0]) = (\varphi[x_{k+1}/(\psi_k[x_0/x_1])]_{k\in(n-1)} = (\phi_1[x_1/x_0])$, in which case $(\phi_i \vdash \phi_{1-i}) \in \varepsilon$, and so $\bar{a} \notin \theta^{\mathcal{B}}_\varepsilon = (\theta^{\mathcal{A}}_\varepsilon \cap B^2)$, in view of (3.1) with $h = \Delta_B$ as well as $\mathcal{A}$ and $\mathcal{B}$ instead of one another. Thus, $\bar{a} \notin \theta^{\mathcal{A}}_\varepsilon$, for $\bar{a} \in B^2$, in which case ε is a unary equality determinant for $\mathcal{A}$, and so (iii) holds. □

Lemma 3.5. *Any axiomatic equality determinant ε for a class M of Σ-matrices is so for $\mathbf{P}(\mathsf{M})$.*

Proof. In that case, members of M are models of the infinitary universal strict Horn theory $\varepsilon[x_1/x_0] \cup \{(\bigwedge \varepsilon) \to (x_0 \approx x_1)\}$ with equality, and so are well-known to be those of $\mathbf{P}(\mathsf{M})$, as required. □

3.3. **Classical matrices and logics.** A two-valued Σ-matrix $\mathcal{A}$ is said to be $\sim$-*classical*, whenever it is $\sim$-negative, in which case it is both consistent and truth-non-empty, and so is both false- and truth-singular, the unique element of $(A \setminus D^{\mathcal{A}})/D^{\mathcal{A}}$ being denoted by $(0/1)_{\mathcal{A}}$, respectively (the index $_{\mathcal{A}}$ is often omitted, unless any confusion is possible), in which case $A = \{0, 1\}$, while $\sim^{\mathfrak{A}} i = (1 - i)$, for each $i \in 2$, whereas $\theta^{\mathcal{A}}$ is diagonal, for $\chi^{\mathcal{A}}$ is so, and so $\mathcal{A}$ is simple (in particular, hereditarily so, for it has no proper submatrix) but is not $\sim$-paraconsistent, in view of Remark 2.8(i)**d)**.

A Σ-logic is said to be $\sim$-*[sub]classical*, whenever it is [a sublogic of] the logic of a $\sim$-classical Σ-matrix, in which case it is inferentially consistent. Then, $\sim$ is called a *subclassical negation for* a Σ-logic C, whenever the $\sim$-fragment of C is $\sim$-subclassical, in which case:

$$\sim^m x_0 \notin C(\sim^n x_0), \tag{3.2}$$

for all $m, n \in \omega$ such that the integer $m - n$ is odd.

Lemma 3.6. *Let $\mathcal{A}$ be a $\sim$-classical Σ-matrix, C the logic of $\mathcal{A}$ and $\mathcal{B}$ a truth-non-empty consistent model of C. Then, $\mathcal{A}$ is a strict surjective homomorphic image of a submatrix of $\mathcal{B}$, in which case $\mathcal{A}$ is isomorphic to any $\sim$-classical model of C, and so C has no proper $\sim$-classical extension.*

Proof. Take any $a \in D^{\mathcal{B}} \neq \varnothing$ and any $b \in (B \setminus D^{\mathcal{B}}) \neq \varnothing$. Then, by (2.16), the submatrix $\mathcal{D}$ of $\mathcal{B}$ generated by $\{a, b\}$ is a finitely-generated consistent truth-non-empty model of C. Therefore, by Corollary 2.12, there are some set I, some submatrix

$\mathcal{E}$ of $\mathcal{A}^I$, some Σ-matrix $\mathcal{F}$, some $g \in \hom_{\mathrm{S}}^{\mathrm{S}}(\mathcal{D},\mathcal{F})$ and some $h \in \hom_{\mathrm{S}}^{\mathrm{S}}(\mathcal{E},\mathcal{F})$, in which case $\mathcal{E}$ is both truth-non-empty and consistent (in particular, $I \neq \varnothing$), for $\mathcal{D}$ is so, and so there is some $d \in D^{\mathcal{E}} \neq \varnothing$, in which case $E \ni d \triangleq (I \times \{1\})$, and so $E \ni \sim^{\mathfrak{E}} d = (I \times \{0\})$. Hence, as $I \neq \varnothing$, $e \triangleq \{\langle x, (I \times \{x\})\rangle \mid x \in A\}$ is an embedding of $\mathcal{A}$ into $\mathcal{E}$, in which case $f \triangleq (h \circ e) \in \hom_{\mathrm{S}}(\mathcal{A},\mathcal{F})$ is injective, in view of Remark 2.7(iii). Then, $G \triangleq (\operatorname{img} f)$ forms a subalgebra of $\mathfrak{F}$, in which case $H \triangleq g^{-1}[G]$ forms a subalgebra of $\mathfrak{D}$, and so $f^{-1} \circ (g \upharpoonright G)$ is a strict surjective homomorphism from $(\mathcal{D} \upharpoonright H) \in \mathbf{S}(\mathcal{B})$ onto $\mathcal{A}$. In this way, (2.16), Remark 2.7(iii) and the fact that any $\sim$-classical Σ-matrix is simple and has no proper submatrix complete the argument. □

A $\sim$-classical Σ-matrix $\mathcal{A}$ is said to be *canonical*, whenever $A = 2$ and $a_{\mathcal{A}} = a$, for all $a \in A$, any isomorphism between canonical ones being clearly diagonal, so any isomorphic canonical ones being equal. In general, the bijection $e_{\mathcal{A}} \triangleq \{\langle i, i_{\mathcal{A}}\rangle \mid i \in 2\} : 2 \to A$ is an isomorphism from the canonical $\sim$-classical Σ-matrix $\langle e_{\mathcal{A}}^{-1}[\mathfrak{A}], \{1\}\rangle$ onto $\mathcal{A}$. In this way, in view of (2.16) and Lemma 3.6, any $\sim$-classical Σ-logic is defined by a unique canonical $\sim$-classical Σ-matrix, said to be *characteristic for/of* the logic.

Corollary 3.7. *Any $\sim$-classical Σ-logic has no proper inferentially consistent extension, and so is structurally complete iff it has a theorem.*

Proof. Let $\mathcal{A}$ be a $\sim$-classical Σ-matrix, C the logic of $\mathcal{A}$ and C' an inferentially consistent extension of C. Then, $x_1 \notin T \triangleq C'(x_0) \ni x_0$. On the other hand, by the structurality of C', $\langle \mathfrak{Fm}_{\Sigma}^{\omega}, T\rangle$ is a consistent truth-non-empty model of C' (in particular, of C). In this way, (2.16), Remark 2.4 and Lemma 3.6 complete the argument. □

3.4. **Structural completions versus free models.** Let M be a class of Σ-matrices, C the logic of M, $\mathsf{K} \triangleq \pi_0[\mathsf{M}]$ and $\alpha \in \wp_{\omega[\setminus 1]}(\omega)$ [whenever Σ is constant-free]. Then, for any $\mathfrak{A} \in \mathsf{M}$ and any $h \in \hom(\mathrm{Fm}_{\Sigma}^{\alpha}, \mathfrak{A})$, $h \in \hom_{\mathrm{S}}(\mathcal{B},\mathcal{A})$, where $\mathcal{B} \triangleq \langle \mathfrak{Fm}_{\Sigma}^{\alpha}, h^{-1}[D^{\mathcal{A}}]\rangle$, in which case, by Remark 2.7(i), we have $\theta_{\mathsf{K}}^{\alpha} \subseteq (\ker h) = h^{-1}[\Delta_A] \subseteq h^{-1}[\theta^{\mathcal{A}}] = \theta^{\mathcal{B}}$, and so $\theta_{\mathsf{K}}^{\alpha} \subseteq \theta^{\mathcal{D}}$, where $\mathcal{D} \triangleq \langle \mathfrak{Fm}_{\Sigma}^{\alpha}, \mathrm{Cn}_{\mathsf{M}}^{\alpha}(\varnothing)\rangle \in \mathrm{Mod}(C)$, in view of the structurality of C. Thus, $\theta_{\mathsf{K}}^{\alpha} \in \mathrm{Con}(\mathcal{D})$, in which case, by (2.16), $\mathcal{F}_{\mathsf{M}}^{\alpha} \triangleq (\mathcal{D}/\theta_{\mathsf{K}}^{\alpha}) \in \mathrm{Mod}(C)$, while $\mathfrak{F}_{\mathsf{M}}^{\alpha} = \mathfrak{F}_{\mathsf{K}}^{\alpha}$.

Theorem 3.8. *Let Σ be a signature [with(out) nullary symbols], M a [finite (non-empty)] class of [finite] Σ-matrices, C the logic of M, [$f \in \prod_{\mathcal{A}\in\mathsf{M}} \wp_{\omega(\setminus 1)}(A)$] $\alpha \triangleq (\omega[\cap \bigcup_{\mathcal{A}\in\mathsf{M}} |f(\mathcal{A})|])$ and $\mathcal{B}$ a submatrix of $\mathcal{F}_{\mathsf{M}}^{\alpha}$. Suppose every $\mathcal{A} \in \mathsf{M}$ is a surjective homomorphic image of $\mathcal{B}$, unless $\mathcal{B} = \mathcal{F}_{\mathsf{M}}^{\alpha}$, [and generated by $f(\mathcal{A})$]. Then, the structural completion of C is defined by $\mathcal{B}$.*

Proof. Then, by (2.16), the logic C' of $\mathcal{F}_{\mathsf{M}}^{\omega[/\alpha]}$ is defined by $\mathcal{D}_{\omega[/\alpha]} \triangleq \langle \mathfrak{Fm}_{\Sigma}^{\omega[/\alpha]}, \mathrm{Cn}_{\mathsf{M}}^{\omega[/\alpha]}(\varnothing)\rangle \in \mathrm{Mod}(C)$, in view of the structurality of C [/and (2.15)], in which case it is an extension of C, and so $C(\varnothing) \subseteq C'(\varnothing)$. For proving the converse inclusion, consider the following complementary cases:

- $\alpha = \omega$.
  Then, applying the diagonal Σ-substitution, we get $C'(\varnothing) \subseteq D^{\mathcal{D}_{\omega}} = C(\varnothing)$.
- $\alpha \neq \omega$.
  Consider any $\mathcal{A} \in \mathsf{M}$, in which case it is generated by $f(\mathcal{A})$ of cardinality $\leqslant \alpha$, and so there is some surjective $h \in \hom(\mathfrak{Fm}_{\Sigma}^{\alpha}, \mathfrak{A})$. Then, $D^{\mathcal{D}_{\alpha}} = \mathrm{Cn}_{\mathsf{M}}^{\alpha}(\varnothing) \subseteq h^{-1}[D^{\mathcal{A}}]$, in which case $h \in \hom^{\mathrm{S}}(\mathcal{D}_{\alpha},\mathcal{A})$, and so, by (2.17), $C'(\varnothing) \subseteq C(\varnothing)$.

Next, $\mathcal{D}_\omega$ is a model of any extension C'' of C' such that $C''(\varnothing) = C(\varnothing)$, in view of its structurality [and so is its submatrix $\mathcal{D}_\alpha$, in view of (2.15) and (2.16)], in which case C' is the structural completion of C. Finally, by (2.16), $\mathcal{B}$ is a model of C'. Conversely, if $\mathcal{B} = \{\neq\}\mathcal{F}^\alpha_{\mathsf{M}}$, then {each $\mathcal{A} \in \mathsf{M}$ is a surjective homomorphic image of $\mathcal{B}$, in which case, by (2.17)} $\mathrm{Cn}_{\mathcal{B}}(\varnothing) = C'(\varnothing)$, and so C', being structurally complete, is defined by $\mathcal{B}$, as required. □

The []-optional case of this theorem provides an effective procedure of finding finite matrix semantics of any finitely-valued logic, practical applications of which are demonstrated in Paragraphs 8.3.1.1 and 8.3.2.1 below.

4. Three-valued logics with subclassical negation versus super-classical matrices

A Σ-matrix $\mathcal{A}$ is said to be $\sim$*-super-classical*, if $\mathcal{A}{\restriction}\{\sim\}$ has a $\sim$-classical submatrix, in which case $\mathcal{A}$ is both consistent and truth-non-empty, while, by (2.16), $\sim$ is a subclassical negation for the logic of $\mathcal{A}$, and so we have the "if" part of the following preliminary marking the framework of the present subsection:

Theorem 4.1. *Let $\mathcal{A}$ be a Σ-matrix. [Suppose $|A| \leqslant 3$.] Then, $\sim$ is a subclassical negation for the logic of $\mathcal{A}$ if[f] $\mathcal{A}$ is $\sim$-super-classical.*

Proof. [Assume $\sim$ is a subclassical negation for the logic of $\mathcal{A}$. First, by (3.2) with $m = 1$ and $n = 0$, there is some $a \in D^{\mathcal{A}}$ such that $\sim^{\mathfrak{A}} a \notin D^{\mathcal{A}}$. Likewise, by (3.2) with $m = 0$ and $n = 1$, there is some $b \in (A \setminus D^{\mathcal{A}})$ such that $\sim^{\mathfrak{A}} b \in D^{\mathcal{A}}$, in which case $a \neq b$, and so $|A| \neq 1$. Then, if $|A| = 2$, we have $A = \{a, b\}$, in which case $\mathcal{A}$ is $\sim$-classical, and so $\sim$-super-classical. Now, assume $|A| = 3$.

Claim 4.2. *Let $\mathcal{A}$ be a three-valued Σ-matrix, $\bar{a} \in A^2$ and $i \in 2$. Suppose $\sim$ is a subclassical negation for the logic of $\mathcal{A}$ and, for each $j \in 2$, $(a_j \in D^{\mathcal{A}}) \Leftrightarrow (\sim^{\mathfrak{A}} a_j \notin D^{\mathcal{A}}) \Leftrightarrow (a_{1-j} \notin D^{\mathcal{A}})$. Then, either $\sim^{\mathfrak{A}} a_i = a_{1-i}$ or $\sim^{\mathfrak{A}}\sim^{\mathfrak{A}} a_i = a_i$.*

Proof. By contradiction. For suppose both $\sim^{\mathfrak{A}} a_i \neq a_{1-i}$ and $\sim^{\mathfrak{A}}\sim^{\mathfrak{A}} a_i \neq a_i$. Then, in case $a_i \in / \notin D^{\mathcal{A}}$, as $|A| = 3$, we have both $(D^{\mathcal{A}}/(A \setminus D^{\mathcal{A}})) = \{a_i\}$, in which case $\sim^{\mathfrak{A}} a_{1-i} = a_i$, and $((A \setminus D^{\mathcal{A}})/D^{\mathcal{A}}) = \{a_{1-i}, \sim^{\mathfrak{A}} a_i\}$, respectively. Consider the following exhaustive cases:

- $\sim^{\mathfrak{A}}\sim^{\mathfrak{A}} a_i = a_{1-i}$.
  Then, $\sim^{\mathfrak{A}}\sim^{\mathfrak{A}}\sim^{\mathfrak{A}} a_i = a_i$. This contradicts to (3.2) with $(n/m) = 0$ and $(m/n) = 3$, respectively.
- $\sim^{\mathfrak{A}}\sim^{\mathfrak{A}} a_i = \sim^{\mathfrak{A}} a_i$.
  Then, for each $c \in ((A \setminus D^{\mathcal{A}})/D^{\mathcal{A}})$, $\sim^{\mathfrak{A}}\sim^{\mathfrak{A}}\sim^{\mathfrak{A}} c = \sim^{\mathfrak{A}} a_i \notin / \in D^{\mathcal{A}}$. This contradicts to (3.2) with $(n/m) = 3$ and $(m/n) = 0$, respectively.

Thus, in any case, we come to a contradiction, as required. □

Finally, consider the following exhaustive cases:

- both $\sim^{\mathfrak{A}} a = b$ and $\sim^{\mathfrak{A}} b = a$.
  Then, $\{a, b\}$ forms a subalgebra of $\mathfrak{A}{\restriction}\{\sim\}$, $(\mathcal{A}{\restriction}\{\sim\}){\restriction}\{a, b\}$ being a $\sim$-classical submatrix of $\mathcal{A}{\restriction}\{\sim\}$, as required.
- $\sim^{\mathfrak{A}} a \neq b$.
  Then, by Claim 4.2, $\sim^{\mathfrak{A}}\sim^{\mathfrak{A}} a = a$, in which case $\{a, \sim^{\mathfrak{A}} a\}$ forms a subalgebra of $\mathfrak{A}{\restriction}\{\sim\}$, $(\mathcal{A}{\restriction}\{\sim\}){\restriction}\{a, \sim^{\mathfrak{A}} a\}$ being a $\sim$-classical submatrix of $\mathcal{A}{\restriction}\{\sim\}$, as required.
- $\sim^{\mathfrak{A}} b \neq a$.
  Then, by Claim 4.2, $\sim^{\mathfrak{A}}\sim^{\mathfrak{A}} b = b$, in which case $\{b, \sim^{\mathfrak{A}} b\}$ forms a subalgebra of $\mathfrak{A}{\restriction}\{\sim\}$, $(\mathcal{A}{\restriction}\{\sim\}){\restriction}\{b, \sim^{\mathfrak{A}} b\}$ being a $\sim$-classical submatrix of $\mathcal{A}{\restriction}\{\sim\}$, as required.] □

The following counterexample shows that the optional condition $|A| \leqslant 3$ is essential for the optional "only if" part of Theorem 4.1 to hold:

Example 4.3. Let $n \in \omega$ and $\mathcal{A}$ any Σ-matrix with $A \triangleq (n \cup (2 \times 2))$, $D^{\mathcal{A}} \triangleq \{\langle 1,0\rangle, \langle 1,1\rangle\}$, $\sim^{\mathfrak{A}}\langle i,j\rangle \triangleq \langle 1-i, (1-i+j) \mod 2\rangle$, for all $i,j \in 2$, and $\sim^{\mathfrak{A}}k \triangleq \langle 1,0\rangle$, for all $k \in n$. Then, for any subalgebra $\mathfrak{B}$ of $\mathfrak{A}{\restriction}\{\sim\}$, we have $(2 \times 2) \subseteq B$, in which case $4 \leqslant |B|$, and so $\mathcal{A}$ is not $\sim$-super-classical, for $4 \nleqslant 2$. On the other hand, 2×2 forms a subalgebra of $\mathfrak{A}{\restriction}\{\sim\}$, $\mathcal{B} \triangleq (\mathcal{A}{\restriction}\{\sim\}){\restriction}(2 \times 2)$ being $\sim$-negative, in which case $\chi^{\mathcal{A}}{\restriction}(2 \times 2)$ is a surjective strict homomorphism from $\mathcal{B}$ onto the canonical $\sim$-classical $\{\sim\}$-matrix $\mathcal{C}$, and so, by (2.16), $\sim$ is a subclassical negation for the logic of $\mathcal{A}$. □

Let $\mathcal{A}$ be a three-valued $\sim$-super-classical (in particular, both consistent and truth-non-empty) Σ-matrix and $\mathcal{B}$ a $\sim$-classical submatrix of $\mathcal{A}{\restriction}\{\sim\}$. Then, as $4 \nleqslant 3$, $\mathcal{A}$ is either false-singular, in which case the unique non-distinguished value $0_{\mathcal{A}}$ of $\mathcal{A}$ is that $0_{\mathcal{B}}$ of $\mathcal{B}$, so $1^{\sim}_{\mathcal{A}} \triangleq \sim^{\mathfrak{A}}0_{\mathcal{A}} = \sim^{\mathfrak{B}}0_{\mathcal{B}} = 1_{\mathcal{B}}$, or truth-singular, in which case the unique distinguished value $1_{\mathcal{A}}$ of $\mathcal{A}$ is that $1_{\mathcal{B}}$ of $\mathcal{B}$, so $0^{\sim}_{\mathcal{A}} \triangleq \sim^{\mathfrak{A}}1_{\mathcal{A}} = \sim^{\mathfrak{B}}1_{\mathcal{B}} = 0_{\mathcal{B}}$, but not both, for $|A| = 3 \neq 2$. Thus, in case $\mathcal{A}$ is false-/truth-singular, $B = 2^{\sim}_{\mathcal{A}} \triangleq \{0^{/\sim}_{\mathcal{A}}, 1^{\sim/}_{\mathcal{A}}\}$ is uniquely determined by $\mathcal{A}$ and $\sim$, the unique element of $A \setminus 2^{\sim}_{\mathcal{A}}$ being denoted by $(\frac{1}{2})^{\sim}_{\mathcal{A}}$. (The indexes $_{\mathcal{A}}$ and, especially, $^{\sim}$ are often omitted, unless any confusion is possible.) Strict homomorphisms from $\mathcal{A}$ to itself retain both 0 and 1, in which case surjective ones retain $\frac{1}{2}$, and so:

$$\hom^{[\mathrm{S}]}_{\mathrm{S}}(\mathcal{A}, \mathcal{A}) \supseteq [=]\{\Delta_A\}, \tag{4.1}$$

the inclusion [not] being allowed to be proper (cf. Example 4.9 below). Then, $\mathcal{A}$ is said to be *canonical*, provided $A = (3 \div 2)$ and $a_{\mathcal{A}} = a$, for all $a \in A$.

Lemma 4.4. *Let $\mathcal{A}$ and $\mathcal{B}$ be canonical three-valued $\sim$-super-classical Σ-matrices and e an isomorphism from $\mathcal{A}$ onto $\mathcal{B}$. Then, e is diagonal, in which case $\mathcal{A} = \mathcal{B}$.*

Proof. Then, $\mathcal{A}$ is "false-/-truth-singular"|$\sim$-negative iff $\mathcal{B}$ is so"|, in view of Remark 2.8(ii)", in which case $D^{\mathcal{A}} = D^{\mathcal{B}}$, while $\sim^{\mathfrak{A}}\frac{1}{2}$ is equal to 0/1 iff $\sim^{\mathfrak{B}}\frac{1}{2}$ is so. Moreover, since $\mathfrak{A}$ and $\mathfrak{B}$ are isomorphic, we have $(\sim^{\mathfrak{A}}\frac{1}{2} = \frac{1}{2}) \Leftrightarrow (\exists a \in A : \sim^{\mathfrak{A}}a = a) \Leftrightarrow (\exists b \in B : \sim^{\mathfrak{B}}b = b) \Leftrightarrow (\sim^{\mathfrak{B}}\frac{1}{2} = \frac{1}{2})$. Hence, $\sim^{\mathfrak{A}} = \sim^{\mathfrak{B}}$. In this way, e is an isomorphism from the three-valued $\sim$-super-classical $\mathcal{A}{\restriction}\{\sim\}$ onto $(\mathcal{B}{\restriction}\{\sim\}) = (\mathcal{A}{\restriction}\{\sim\})$, in which case, by (4.1), e is diagonal, and so $\mathcal{A} = \mathcal{B}$, as required. □

Lemma 4.5. *Any three-valued $\sim$-super-classical Σ-matrix $\mathcal{A}$ is isomorphic to a unique canonical one.*

Proof. Then, the mapping $e : (3 \div 2) \mapsto A, a \mapsto a_{\mathcal{A}}$ is a bijection, in which case it is an isomorphism from the canononical three-valued $\sim$-super-classical Σ-matrix $\langle e^{-1}[\mathfrak{A}], e^{-1}[D^{\mathcal{A}}]\rangle$ onto $\mathcal{A}$. In this way, Lemma 4.4 completes the argument. □

As an immediate consequence of (2.16), Theorem 4.1 and Lemma 4.5, we have:

Corollary 4.6. *Three-valued Σ-logics with subclassical negation $\sim$ are exactly logics of canonical three-valued $\sim$-super-classical Σ-matrices.*

From now on, unless otherwise specified, C is supposed to be the logic of an arbitrary but fixed canonincal three-valued $\sim$-super-classical Σ-matrix $\mathcal{A}$. In view of Corollary 4.6, this exhaust all three-valued Σ-logics with subclassical negation $\sim$. Then, C is " (weakly) $\bar{\wedge}$-conjunctive"/"weakly $\veebar$-disjunctive" iff $\mathcal{A}$ is so. It appears that such does hold for both disjunctivity and implicativity too, as it ensues from the following two lemmas:

Lemma 4.7. *Let $\mathcal{B}$ be a Σ-matrix and C' the logic of $\mathcal{B}$. Suppose [either] $\mathcal{B}$ is false-singular (in particular, $\sim$-classical) [or both $\mathcal{B}$ is $\sim$-super-classical and $|B| \leqslant 3$]. Then, the following are equivalent:*

(i) *C' is $\veebar$-disjunctive;*
(ii) *$\mathcal{B}$ is $\veebar$-disjunctive;*
(iii) (2.3) *with $i = 0$,* (2.4) *and* (2.5) *[as well as* (2.8) *for the* material implication $(x_0 \sqsupset x_1) \triangleq (\sim x_0 \veebar x_1)$*] are satisfied in C' (viz., true in $\mathcal{B}$).*

Proof. First, (ii)⇒(i) is immediate.

Next, assume (i) holds. Then, (2.3) with $i = 0$, (2.4) and (2.5) are immediate. [In addition, suppose $\mathcal{B}$ is not false-singular, in which case it is $\sim$-super-classical, while $|B| \leqslant 3$, and so it is both truth-singular and, therefore, not $\sim$-paraconsistent. Hence, $x_1 \in (C'(\{x_0, x_1\}) \cap C'(\{x_0, \sim x_0\})) = C'(\{x_0, \sim x_0 \veebar x_1\})$, so (2.8) is satisfied in C'.] Thus, (iii) holds.

Finally, assume (iii) holds. Consider any $a, b \in B$. In case $(a/b) \in D^{\mathcal{B}}$, by (2.3) with $i = 0$ /"and (2.4)", we have $(a \veebar^{\mathfrak{B}} b) \in D^{\mathcal{B}}$. Now, assume $(\{a, b\} \cap D^{\mathcal{B}}) = \varnothing$. Then, in case $a = b$ (in particular, $\mathcal{B}$ is false-singular), by (2.5), we get $D^{\mathcal{B}} \not\ni (a \veebar^{\mathfrak{B}} a) = (a \veebar^{\mathfrak{B}} b)$. [Otherwise, $\mathcal{B}$ is not false-singular, in which case it is $\sim$-super-classical, while $|B| \leqslant 3$, whereas (2.8) is true in $\mathcal{B}$, and so, for some $c \in (B \setminus D^{\mathcal{B}}) = \{a, b\}$, it holds that $\sim^{\mathfrak{B}} c \in D^{\mathcal{B}}$, while $\sim^{\mathfrak{B}} \sim^{\mathfrak{B}} c = c$. Let d be the unique element of $\{a, b\} \setminus \{c\}$, in which case $\{a, b\} = \{c, d\}$. Then, since $\sim^{\mathfrak{B}} c \in D^{\mathcal{B}}$, we conclude that $(c \veebar^{\mathfrak{B}} d) = (\sim^{\mathfrak{B}} \sim^{\mathfrak{B}} c \veebar^{\mathfrak{B}} d) \notin D^{\mathcal{B}}$, for, otherwise, by (2.8), we would get $d \in D^{\mathcal{B}}$. Hence, by (2.4), we eventually get $(a \veebar^{\mathfrak{B}} b) \notin D^{\mathcal{B}}$.] Thus, (ii) holds. □

Lemma 4.8. *Let $\mathcal{B}$ be a Σ-matrix and C' the logic of $\mathcal{B}$. Suppose [either] $\mathcal{B}$ is false-singular (in particular, $\sim$-classical) [or both $\mathcal{B}$ is $\sim$-super-classical and $|B| \leqslant 3$]. Then, the following [but (i)] are equivalent:*

(i) *C' is weakly $\sqsupset$-implicative;*
(ii) *C' is $\sqsupset$-implicative;*
(iii) *$\mathcal{B}$ is $\sqsupset$-implicative;*
(iv) (2.6)*,* (2.7) *and* (2.8) *[as well as both* (2.9) *and* (2.11)*] are satisfied in C' (viz., true in $\mathcal{B}$).*

In particular, any $\sim$-classical/"three-valued $\sim$-paraconsistent" Σ-logic /"with subclassical negation $\sim$" is $\sqsupset$-implicative iff it is weakly so.

Proof. First, (iii)⇒(ii) is immediate, while (i) is a particular case of (ii).

Next, assume (i[i]) holds. Then, (2.6), (2.7) and (2.8) [as well as (2.9)] are immediate. [In addition, suppose $\mathcal{B}$ is not false-singular, in which case it is $\sim$-super-classical, while $|B| \leqslant 3$, and so it is both truth-singular and, therefore, non-$\sim$-paraconsistent, and so is C'. Hence, by Deduction Theorem, (2.11) is satisfied in C'.] Thus, (iv) holds.

Finally, assume (iv) holds. Consider any $a, b \in B$. In case $b \in D^{\mathcal{B}}$, by (2.7) and (2.8), we have $(a \sqsupset^{\mathfrak{B}} b) \in D^{\mathcal{B}}$. Likewise, in case $\{a, a \sqsupset^{\mathfrak{B}} b\} \subseteq D^{\mathcal{B}}$, by (2.8), we have $b \in D^{\mathcal{B}}$. Now, assume $(\{a, b\} \cap D^{\mathcal{B}}) = \varnothing$. Then, in case $a = b$ (in particular, $\mathcal{B}$ is false-singular), by (2.6), we get $D^{\mathcal{B}} \ni (a \sqsupset^{\mathfrak{B}} a) = (a \sqsupset^{\mathfrak{B}} b)$. [Otherwise, $\mathcal{B}$ is not false-singular, in which case it is $\sim$-super-classical, while $|B| \leqslant 3$, whereas both (2.9) and (2.11) and true in $\mathcal{B}$, and so, for some $c \in (B \setminus D^{\mathcal{B}}) = \{a, b\}$, it holds that $\sim^{\mathfrak{B}} c \in D^{\mathcal{B}}$. Let d be the unique element of $\{a, b\} \setminus \{c\}$, in which case $\{a, b\} = \{c, d\}$. Then, since $\sim^{\mathfrak{B}} c \in D^{\mathcal{B}}$, by (2.8) and (2.11), we conclude that $(c \sqsupset^{\mathfrak{B}} d) \in D^{\mathcal{B}}$. Let us prove, by contradiction, that $(d \sqsupset^{\mathfrak{B}} c) \in D^{\mathcal{B}}$. For suppose $(d \sqsupset^{\mathfrak{B}} c) \notin D^{\mathcal{B}}$, in which case $(d \sqsupset^{\mathfrak{B}} c) = (c/d)$, and so we have $((d \sqsupset^{\mathfrak{B}} c) \sqsupset^{\mathfrak{B}} d) = ((c \sqsupset^{\mathfrak{B}} d)/(d \sqsupset^{\mathfrak{B}} d)) \in D^{\mathcal{B}}/$, by (2.6). Hence, by (2.8) and (2.9), we get $d \in D^{\mathcal{B}}$. This contradiction shows that $(d \sqsupset^{\mathfrak{B}} c) \in D^{\mathcal{B}} \ni (c \sqsupset^{\mathfrak{B}} d)$.

In particular, we eventually get $(a \sqsupset^{\mathfrak{B}} b) \in D^{\mathcal{B}}$.] Thus, (iii) holds, as required/", in view of Corollary 4.6". □

Three-valued logics with subclassical negation $\sim$ (even both implicative [and so disjunctive; cf. Lemma 4.8] and conjunctive ones) need not, generally speaking, be non-$\sim$-classical, as it ensues from the following elementary example:

Example 4.9. Let $\Sigma \triangleq \Sigma_{+,\sim}$ and $(\mathcal{B}/\mathcal{E})|\mathcal{F}$ the canonical "$\sim$-negative false-/truth-singular three-valued $\sim$-super-classical"|$\sim$-classical Σ-matrix with $(((\mathfrak{B}/\mathfrak{E})|\mathfrak{F})\restriction\Sigma_+) \triangleq \mathfrak{D}_{3|2}$. Then, $(\mathcal{B}/\mathcal{E})|\mathcal{F}$ is both $\wedge$-conjunctive and $\vee$-disjunctive, and so $\sqsupset^{\sim}_{\vee}$-implicative, in view of Remark 2.8(i)**c)**. And what is more, $\chi^{\mathcal{B}/\mathcal{E}} \in \hom^{\mathrm{S}}_{\mathrm{S}}(\mathcal{B}/\mathcal{E}, \mathcal{F})$. Therefore, by (2.16), $\mathcal{B}/\mathcal{E}$ define the same $\sim$-classical Σ-logic of $\mathcal{F}$. On the other hand, $\mathcal{B}$, being false-singular, is not isomorphic to $\mathcal{E}$, not being so. Moreover, $h \triangleq (\Delta_2 \circ \chi^{\mathcal{B}/\mathcal{E}})$ is a non-diagonal (for $h(\frac{1}{2}) = (1/0) \neq \frac{1}{2}$) strict homomorphism from $\mathcal{B}/\mathcal{E}$ to itself, so the non-[]-optional inclusion in (4.1) may be proper. □

On the other hand, $\sim$-classical three-valued Σ-logics with subclassical negation $\sim$ and with[out] theorems are [not] structurally complete, in view of Corollary 3.7. This makes the following subsection especially acute.

4.1. **Classical three-valued logics with subclassical negation.**

Lemma 4.10. *The following are equivalent:*

(i) *$\mathcal{A}$ is a strict surjective homomorphic counter-image of a $\sim$-classical Σ-matrix;*
(ii) *$\mathcal{A}$ is not simple;*
(iii) *$\mathcal{A}$ is not hereditarily simple;*
(iv) *$\theta^{\mathcal{A}} \in \mathrm{Con}(\mathfrak{A})$.*

Proof. First, (i)⇒(ii) is by Remark 2.7(iii) and the fact that $3 \not\leq 2$. Next, (iii) is a particular case of (ii). The converse is by the fact that any proper submatrix of $\mathcal{A}$, being either one-valued or $\sim$-classical, is simple. Further, (ii)⇒(iv) is by the following claim:

Claim 4.11. *Let $\mathcal{B}$ be a three-valued as well as both consistent and truth-non-empty Σ-matrix. Then, any non-diagonal congruence θ of it is equal to $\theta^{\mathcal{B}}$.*

Proof. First, we have $\theta \subseteq \theta^{\mathcal{B}}$. Conversely, consider any $\bar{a} \in \theta^{\mathcal{B}}$. Then, in case $a_0 = a_1$, we have $\bar{a} \in \Delta_B \subseteq \theta$. Otherwise, take any $\bar{b} \in (\theta \setminus \Delta_B) \neq \varnothing$, in which case $\bar{b} \in \theta^{\mathcal{B}}$, for $\theta \subseteq \theta^{\mathcal{B}}$. Then, as $|B| = 3 \not\geq 4$, there are some $i, j \in 2$ such that $a_i = b_j$. Hence, if a_{1-i} was not equal to b_{1-j}, then we would have both $|\{a_i, a_{1-i}, b_{1-j}\}| = 3 = |B|$, in which case we would get $\{a_i, a_{1-i}, b_{1-j}\} = B$, and $\chi^{\mathcal{B}}(b_{1-j}) = \chi^{\mathcal{B}}(b_j) = \chi^{\mathcal{B}}(a_i) = \chi^{\mathcal{B}}(a_{1-i})$, and so $\mathcal{B}$ would be either truth-empty or inconsistent. Therefore, both $a_{1-i} = b_{1-j}$ and $a_i = b_j$. Thus, since θ is symmetric, we eventually get $\bar{a} \in \theta$, for $\bar{b} \in \theta$, as required. □

Finally, assume (iv) holds. Then, $\theta \triangleq \theta^{\mathcal{A}}$, including itself, is a congruence of $\mathcal{A}$, in which case $\nu_\theta \in \hom^{\mathrm{S}}_{\mathrm{S}}(\mathcal{A}, \mathcal{A}/\theta)$, while $\mathcal{A}/\theta$ is $\sim$-classical, and so (i) holds. □

Set $h_{+/2} : 2^2 \to (3 \div 2), \langle i, j\rangle \mapsto \frac{i+j}{2}$.

Theorem 4.12. *The following are equivalent:*

(i) *C is $\sim$-classical;*
(ii) *$\mathcal{A}$ is either a strict surjective homomorphic counter-image of a $\sim$-classical Σ-matrix or a strict surjective homomorphic image of a submatrix of a direct power of a $\sim$-classical Σ-matrix;*
(iii) *either $\mathcal{A}$ is a strict surjective homomorphic counter-image of a $\sim$-classical Σ-matrix or $\mathcal{A}$ is a strict surjective homomorphic image of the direct square of a $\sim$-classical Σ-matrix;*

(iv) *either $\mathcal{A}$ is not simple or both 2 forms a subalgebra of $\mathfrak{A}$ and $\mathcal{A}$ is a strict surjective homomorphic image of $(\mathcal{A}\upharpoonright 2)^2$;*

(v) *either $\theta^{\mathcal{A}} \in \mathrm{Con}(\mathfrak{A})$ or both 2 forms a subalgebra of $\mathfrak{A}$, $\mathcal{A}$ is truth-singular and $h_{+/2} \in \hom((\mathfrak{A}\upharpoonright 2)^2, \mathfrak{A})$.*

Proof. We use Lemma 4.10 tacitly. First, (ii/iii/iv) is a particular case of (iii/iv/v), respectively. Next, (iv)⇒(i) is by (2.16). Further, (i)⇒(ii) is by Lemma 2.10 and Remark 2.7(iii).

Now, let $\mathcal{B}$ be a $\sim$-classical Σ-matrix, I a set, $\mathcal{D}$ a submatrix of $\mathcal{B}^I$ and $h \in \hom_{\mathrm{S}}^{\mathrm{S}}(\mathcal{D}, \mathcal{A})$, in which case $\mathcal{D}$ is both consistent and truth-non-empty, for $\mathcal{A}$ is so, and so $I \neq \varnothing$. Take any $a \in D^{\mathcal{B}} \neq \varnothing$. Then, as $\mathcal{B}$ is truth-singular, $D \ni a = (I \times \{1_{\mathcal{B}}\}) \in D^{\mathcal{D}}$, in which case $D \ni b \triangleq \sim^{\mathfrak{D}} a = (I \times \{0_{\mathcal{B}}\}) \notin D^{\mathcal{D}}$, for $I \neq \varnothing$, while $\sim^{\mathfrak{D}} b = a$, and so $E \triangleq \{a, b\}$ forms a subalgebra of $\mathfrak{D}\upharpoonright\{\sim\}$, $\mathcal{E} \triangleq ((\mathcal{D}\upharpoonright\{\sim\})\upharpoonright E)$ being $\sim$-classical with $1_{\mathcal{E}} = a$ and $0_{\mathcal{E}} = b$, and so being $(\mathcal{A}\upharpoonright\{\sim\})\upharpoonright h[E])$, in view of Remark 2.8(ii). Hence, $h(a/b) = (1/0)$. Therefore, there is some $c \in (D \setminus \{a, b\})$ such that $h(c) = \frac{1}{2}$. In this way, $I \neq J \triangleq \{i \in I \mid \pi_i(c) = 1_{\mathcal{B}}\} \neq \varnothing$. Given any $\bar{a} \in B^2$, set $(a_0 \| a_1) \triangleq ((J \times \{a_0\}) \cup ((I \setminus J) \times \{a_1\})) \in B^I$. Then, $D \ni a = (1_{\mathcal{B}} \| 1_{\mathcal{B}})$ and $D \ni b = (0_{\mathcal{B}} \| 0_{\mathcal{B}})$ as well as $D \ni c = (1_{\mathcal{B}} \| 0_{\mathcal{B}})$, in which case $D \ni \sim^{\mathfrak{D}} c = (0_{\mathcal{B}} \| 1_{\mathcal{B}})$, and so $e \triangleq \{\langle\langle x, y\rangle, (x\|y)\rangle \mid x, y \in B\}$ is an embedding of $\mathcal{B}^2$ into $\mathcal{D}$ such that $\{a, b, c\} \subseteq (\mathrm{img}\, e)$. Hence, since $h[\{a, b, c\}] = A$, we conclude that $(h \circ e) \in \hom_{\mathrm{S}}^{\mathrm{S}}(\mathcal{B}^2, \mathcal{A})$. Thus, (ii)⇒(iii) holds.

Likewise, let $\mathcal{B}$ be a $\sim$-classical Σ-matrix and $g \in \hom_{\mathrm{S}}^{\mathrm{S}}(\mathcal{B}^2, \mathcal{A})$. Then, $e' \triangleq (\Delta_B \times \Delta_B)$ is an embedding of $\mathcal{B}$ into $\mathcal{B}^2$, in which case, by Remark 2.7(iii), $g' \triangleq (g \circ e')$ is an embedding of $\mathcal{B}$ into $\mathcal{A}$, and so $E \triangleq (\mathrm{img}\, g')$ forms a two-element subalgebra of $\mathfrak{A}$, g' being an isomorphism from $\mathcal{B}$ onto $\mathcal{E} \triangleq (\mathcal{A}\upharpoonright E)$, in which case $h \triangleq ((g'^{-1} \circ (\pi_0\upharpoonright E^2)) \times (g'^{-1} \circ (\pi_1\upharpoonright E^2)))$ is an isomorphism from $\mathcal{E}^2$ onto $\mathcal{B}^2$. Therefore, as $\mathfrak{A}\upharpoonright\{\sim\}$ has no two-element subalgebra other than that with carrier 2, $E = 2$. And what is more, $(g \circ h) \in \hom_{\mathrm{S}}^{\mathrm{S}}(\mathcal{E}^2, \mathcal{A})$. Thus, (iii)⇒(iv) holds.

Finally, assume (iv) holds, while $\mathcal{A}$ is simple. Then, $\mathcal{A}$ is truth-singular, for $\mathcal{F} \triangleq (\mathcal{A}\upharpoonright 2)$ is so. Let $f \in \hom_{\mathrm{S}}^{\mathrm{S}}(\mathcal{F}^2, \mathcal{A})$. Then, $\langle 1, 1\rangle \in D^{\mathcal{F}^2}$, in which case $f(\langle 1, 1\rangle) \in D^{\mathcal{A}}$, and so $f(\langle 1, 1\rangle) = 1$. Hence, $f(\langle 0, 0\rangle) = f(\sim^{\mathfrak{A}^2}\langle 1, 1\rangle) = \sim^{\mathfrak{A}} f(\langle 1, 1\rangle) = \sim^{\mathfrak{A}} 1 = 0$. Moreover, $\sim^{\mathfrak{A}^2}\langle 0/1, 1/0\rangle = \langle 1/0, 0/1\rangle \notin D^{\mathcal{F}^2}$. Hence, $f(\langle 0/1, 1/0\rangle) \notin D^{\mathcal{A}} \not\ni \sim^{\mathfrak{A}} f(\langle 0/1, 1/0\rangle)$. Therefore, $f(\langle 0/1, 1/0\rangle) = \frac{1}{2}$. Thus, $f = h_{+/2}$, so (v) holds. □

Corollary 4.13. *[Providing $\mathcal{A}$ is either false-singular or $\bar{\wedge}$-conjunctive or $\veebar$-disjunctive] C is $\sim$-classical if[f] $\mathcal{A}$ is not (hereditarily) simple.*

Proof. The "if" part is by Theorem 4.12(iv)⇒(i) (and Lemma 4.10(iii)⇒(ii)). [The converse is proved by contradiction. For suppose C is $\sim$-classical, while $\mathcal{A}$ is simple. Then, by Lemma 4.10(iv)⇒(ii) and Theorem 4.12(i)⇒(v), 2 forms a subalgebra of $\mathfrak{A}$, while $h \triangleq h_{+/2} \in \hom((\mathfrak{A}\upharpoonright 2)^2, \mathfrak{A})$, whereas $\mathcal{A}$ is truth-singular, in which case it is not false-singular, and so $\bar{\wedge}$-conjunctive|$\veebar$-disjunctive, and so is $\mathcal{A}\upharpoonright 2$, in view of Remark 2.8(ii). Hence, $(i(\bar{\wedge}|\veebar)^{\mathfrak{A}} j) = (\min|\max)(i, j)$, for all $i, j \in 2$. Therefore, $\frac{1}{2} = h(01) = h((01)(\bar{\wedge}|\veebar)^{\mathfrak{A}^2}(01)) = (h(01)(\bar{\wedge}|\veebar)^{\mathfrak{A}^2} h(01)) = (\frac{1}{2}(\bar{\wedge}|\veebar)^{\mathfrak{A}^2}\frac{1}{2}) = (h(01)(\bar{\wedge}|\veebar)^{\mathfrak{A}^2} h(10)) = h((01)(\bar{\wedge}|\veebar)^{\mathfrak{A}^2}(10)) = h((00)|(11)) = (0|1)$. This contradiction completes the argument.] □

Generally speaking, the optional stipulation cannot be omitted in the formulation of Corollary 4.13, even if C is *weakly* conjunctive/disjunctive, as it follows from:

Example 4.14. Let $\Sigma \triangleq \{\diamond, \sim\}$ with binary $\diamond$ and $\mathcal{A}$ truth-singular with $(a \diamond^{\mathfrak{A}} b) \triangleq (0/1)$ and $\sim^{\mathfrak{A}} a \triangleq (1 - a)$, for all $a, b \in A$. Then, $\mathcal{A}$ is weakly $\diamond$-conjunctive/-disjunctive, respectively, while $\langle 0, \frac{1}{2}\rangle \in \theta^{\mathcal{A}} \not\ni \langle 1, \frac{1}{2}\rangle = \langle \sim^{\mathfrak{A}} 0, \sim^{\mathfrak{A}} \frac{1}{2}\rangle$, in which case

$\theta^{\mathcal{A}} \notin \mathrm{Con}(\mathfrak{A})$, and so, by Lemma 4.10(ii)⇒(iv), $\mathcal{A}$ is simple. On the other hand, 2 forms a subalgebra of $\mathfrak{A}$, while $h_{+/2} \in \hom((\mathfrak{A}{\upharpoonright}2)^2, \mathfrak{A})$. Hence, by Theorem 4.12(v)⇒(i), C is $\sim$-classical. □

4.1.1. *Uniqueness of three-valued super-classical matrices defining non-classical logics.* A $(2[+1])$-*ary* $[\frac{1}{2}$-*relative*$]$ $\{$*classical*$\}$ *semi-conjunction for* $\mathcal{A}$ is an arbitrary $\varphi \in \mathrm{Fm}_\Sigma^{2[+1]}$ such that both $\varphi^{\mathfrak{A}}(0,1[,\frac{1}{2}]) = 0$ and $\varphi^{\mathfrak{A}}(1,0[,\frac{1}{2}]) \in \{0[,\frac{1}{2}]\}$. (Clearly, any binary semi-conjunction for $\mathcal{A}$ is a ternary $\frac{1}{2}$-relative one.)

Lemma 4.15. *Let $\mathcal{B}$ be a $\sim$-paraconsistent model of C. Suppose either $\mathcal{A}$ has a ternary $\frac{1}{2}$-relative semi-conjunction or $\{\frac{1}{2}\}$ does not form a subalgebra of $\mathfrak{A}$ or $\mathcal{B}$ is weakly $\sim$-negative or*

$$x_0 \vdash {\sim}x_0 \tag{4.2}$$

is not true in $\mathcal{B}$. Then, $\mathcal{A}$ is embeddable into a strict surjective homomorphic image of a $\sim$-paraconsistent submatrix of $\mathcal{B}$.

Proof. Then, C (viz., $\mathcal{A}$) is $\sim$-paraconsistent, and so, by Remark 2.8(i)**d)**, is not $\sim$-classical, in which case, by Theorem 4.12(iv)⇒(i), $\mathcal{A}$ is simple. Moreover, [in case (4.2) is not true in $\mathcal{B}$] there are some $a, b[, c] \in B$ such that $D^{\mathcal{B}} \supseteq \{{\sim}^{\mathfrak{B}}a[, c]\}$ is disjoint with $\{b[, {\sim}^{\mathfrak{B}}c]\}$. Therefore, by (2.16), the submatrix $\mathcal{D}$ of $\mathcal{B}$ generated by $\{a, b[, c]\}$ is a finitely-generated $\sim$-paraconsistent model of C [in which (4.2) is not true]. Hence, by Lemma 2.10, there are some finite set I, some $\overline{\mathcal{C}} \in \mathbf{S}_*(\mathcal{A})^I$, some subdirect product $\mathcal{E}$ of it, some strict surjective homomorphic image $\mathcal{F}$ of $\mathcal{D}$ and some $h \in \mathrm{hom}_\mathrm{S}(\mathcal{E}, \mathcal{F})$, in which case, by (2.16), $\mathcal{E}$ is $\sim$-paraconsistent, and so consistent (in particular, $I \neq \varnothing$) [while (4.2) is not true in $\mathcal{E}$]. Given any $a' \in A$ and any $J \subseteq I$, set $(J : a') \triangleq (J \times \{a'\}) \in A^J$. Likewise, given any $\bar{a} \in A^2$ and any $J \subseteq I$, set $(a_0 \|_J a_1) \triangleq ((J : a_0) \cup ((I \setminus J) : a_1)) \in A^I$. Then, there are some $d \in (E \setminus D^{\mathcal{E}})$ and some $e[, f] \in D^{\mathcal{E}}$ such that ${\sim}^{\mathfrak{E}}e \in D^{\mathcal{E}}[\not\ni {\sim}^{\mathfrak{E}}f]$, in which case $e = (I : \frac{1}{2})$ and $J \triangleq \{i \in I \mid \pi_i(d) = 0\} \neq \varnothing[\neq K \triangleq \{i \in I \mid \pi_i(f) = 1\}]$, for $\mathcal{A}$ is $\sim$-paraconsistent, and so false-singular. Consider the following complementary cases:

- $\{\frac{1}{2}\}$ forms a subalgebra of $\mathfrak{A}$,
  in which case ${\sim}^{\mathfrak{A}}\frac{1}{2} = \frac{1}{2}$. We are going to prove that there is some non-empty $L \subseteq I$ such that $(0 \|_L \frac{1}{2}) \in E$. For consider the following exhaustive subcases:
  - $\mathcal{A}$ has a ternary $\frac{1}{2}$-relative semi-conjunction φ.
    Let $g \triangleq \varphi^{\mathfrak{E}}(d, {\sim}^{\mathfrak{E}}d, e)$. Consider the following exhaustive subsubcases:
    * $\varphi^{\mathfrak{A}}(1, 0, \frac{1}{2}) = 0$.
      Let $L \triangleq \{i \in I \mid \pi_i(d) \neq \frac{1}{2}\} \supseteq J$. Then, $E \ni g = (0 \|_L \frac{1}{2})$.
    * $\varphi^{\mathfrak{A}}(1, 0, \frac{1}{2}) = \frac{1}{2}$.
      Let $L \triangleq J$. Then, $E \ni g = (0 \|_L \frac{1}{2})$.
  - $\mathcal{B}$ is weakly $\sim$-negative.
    Then, by Remark 2.8(ii), $\mathcal{E}$ is weakly $\sim$-negative, in which case ${\sim}^{\mathfrak{E}}d \in D^{\mathcal{E}}$, and so $d \in \{0, \frac{1}{2}\}^I$. Let $L \triangleq J$. Then, $E \ni d = (0 \|_L \frac{1}{2})$.
  - (4.2) is not true in $\mathcal{B}$.
    Let $L \triangleq K$. Then, $f \in D^{\mathcal{E}} \subseteq \{\frac{1}{2}, 1\}^I$, in which case $E \ni f = (1 \|_L \frac{1}{2})$, and so $E \ni {\sim}^{\mathfrak{E}}f = (0 \|_L \frac{1}{2})$.

  In this way, $(0 \|_L \frac{1}{2}) \in E \ni e = (\frac{1}{2} \|_L \frac{1}{2})$, in which case $E \ni {\sim}^{\mathfrak{E}}(0 \|_L \frac{1}{2}) = (1 \|_L \frac{1}{2})$, and so, as $L \neq \varnothing$, while $\{\frac{1}{2}\}$ forms a subalgebra of $\mathfrak{A}$, $h' \triangleq \{\langle x, (x \|_L \frac{1}{2})\rangle \mid x \in A\}$ is an embedding of $\mathcal{A}$ into $\mathcal{E}$.
- $\{\frac{1}{2}\}$ does not form a subalgebra of $\mathfrak{A}$,
  in which case there is some $\varphi \in \mathrm{Fm}_\Sigma^1$ such that $\varphi^{\mathfrak{A}}(\frac{1}{2}) \in 2$, and so $A =$

$\{\frac{1}{2}, \varphi^{\mathfrak{A}}(\frac{1}{2}), \sim^{\mathfrak{A}}\varphi^{\mathfrak{A}}(\frac{1}{2})\}$. Hence, $\{I : x \mid x \in A\} = \{e, \varphi^{\mathfrak{E}}(e), \sim^{\mathfrak{E}}\varphi^{\mathfrak{E}}(e)\} \subseteq E$. Therefore, as $I \neq \varnothing$, $h' \triangleq \{\langle x, I : x\rangle \mid x \in A\}$ is an embedding of $\mathcal{A}$ into $\mathcal{E}$.

Thus, $(h \circ h') \in \hom_{\mathrm{S}}(\mathcal{A}, \mathcal{F})$ is injective, in view of Remark 2.7(iii), as required. □

Theorem 4.16. *Let $\mathcal{B}$ be a [canonical] three-valued $\sim$-super-classical Σ-matrix. Suppose C is defined by $\mathcal{B}$ as well as non-$\sim$-classical. Then, $\mathcal{B}$ is isomorphic [and so equal] to $\mathcal{A}$.*

Proof. In that case, both $\mathcal{A}$ and $\mathcal{B}$ are simple, in view of Theorem 4.12(iv)⇒(i). Consider the following complementary cases:

- $\mathcal{B}$ is $\sim$-paraconsistent,
  in which it is false-singular, and so weakly $\sim$-negative. Then, any proper submatrix of $\mathcal{B}$ is either $\sim$-classical or one-valued (in which case it is either truth-empty or inconsistent, and so its logic is inferentially inconsistent), and so is not $\sim$-paraconsistent (cf. Remark 2.8(i)**d**)). Therefore, by Remark 2.7(iii) and Lemma 4.15, there is an embedding of $\mathcal{A}$ into $\mathcal{B}$, being then an isomorphism from $\mathcal{A}$ onto $\mathcal{B}$, because $|A| = 3 \leqslant n$, for no $n \in 3 = |B|$.
- $\mathcal{B}$ (and so $\mathcal{A}$) is not $\sim$-paraconsistent.
  Then, as $\mathcal{B}$ is simple and finite, by Lemma 2.10 and Remark 2.7(iii), there are some finite set I, some $\overline{\mathcal{C}} \in \mathbf{S}_*(\mathcal{A})^I$, some subdirect product $\mathcal{D}$ of it and some $g \in \hom^{\mathrm{S}}_{\mathrm{S}}(\mathcal{D}, \mathcal{B})$, in which case $\mathcal{D}$ is both truth-non-empty and consistent (in particular, $I \neq \varnothing$), for $\mathcal{B}$ is so. Given any $x \in A$, set $(I : x) \triangleq (I \times \{x\}) \in A^I$. Then, by the following claim, $a \triangleq (I : 1) \in D \ni b \triangleq (I : 0)$:

  Claim 4.17. *Let I be a finite set, $\overline{\mathcal{C}} \in \mathbf{S}_*(\mathcal{A})^I$ and $\mathcal{D}$ a subdirect product of it. Suppose $\mathcal{A}$ is weakly conjunctive, whenever it is $\sim$-paraconsistent, and $\mathcal{D}$ is truth-non-empty, otherwise. Then, $\{I \times \{j\} \mid j \in 2\} \subseteq D$.*

  Proof. Consider the following complementary cases:
  - $\mathcal{A}$ is $\sim$-paraconsistent,
    in which case it is false-singular and weakly conjunctive, and so, by Lemma 3.1, $b \triangleq (I \times \{0\}) \in D$.
  - $\mathcal{A}$ is not $\sim$-paraconsistent,
    in which case $\mathcal{D}$ is truth-non-empty. Take any $a \in D^{\mathcal{D}} \neq \varnothing$. Let $b \triangleq \sim^{\mathfrak{D}} a \in D$. Consider any $i \in I$. Then, $\pi_i(a) \in D^{\mathcal{A}}$. Consider the following complementary subcases:
    * $\frac{1}{2} \in D^{\mathcal{A}}$,
      in which case, since $\mathcal{A}$ is not $\sim$-paraconsistent but is consistent, $\pi_i(b) = \sim^{\mathfrak{A}}\pi_i(a) \notin D^{\mathcal{A}}$, and so, as $1 \in D^{\mathcal{A}}$, $\pi_i(b) = 0$.
    * $\frac{1}{2} \notin D^{\mathcal{A}}$,
      in which case, as $0 \notin D^{\mathcal{A}}$, $\pi_i(a) = 1$, and so $\pi_i(b) = \sim^{\mathfrak{A}}\pi_i(a) = 0$.

    In this way, $D \ni b = (I \times \{0\})$.

  Then, $D \ni \sim^{\mathfrak{D}} b = (I \times \{1\})$. □

  Consider the following complementary subcases:
  - 2 does not form a subalgebra of $\mathfrak{A}$,
    in which case there is some $\varphi \in \mathrm{Fm}^2_\Sigma$ such that $\varphi^{\mathfrak{A}}(1, 0) = \frac{1}{2}$, and so $D \in \varphi^{\mathfrak{D}}(a, b) = (I : \frac{1}{2})$. In this way, as $I \neq \varnothing$, $e \triangleq \{\langle x, I : x\rangle \mid x \in A\}$ is an embedding of $\mathcal{A}$ into $\mathcal{D}$, in which case, by Remark 2.7(iii), $(g \circ e) \in \hom_{\mathrm{S}}(\mathcal{A}, \mathcal{B})$ is injective, and so bijective, because $|A| = 3 \leqslant n$, for no $n \in 3 = |B|$.
  - 2 forms a subalgebra of $\mathfrak{A}$,
    in which case $\mathcal{E} \triangleq (\mathcal{A}{\restriction}2)$ is $\sim$-classical, while $a, b \in E^I$. Moreover, $a \in D^{\mathcal{D}} \not\ni b$, for $I \neq \varnothing$, while $\sim^{\mathfrak{D}}(a/b) = (b/a)$, in which case $\mathcal{F} \triangleq$

$((\mathcal{D}\restriction\{\sim\})\restriction\{a,b\})$ is $\sim$-classical (in particular, simple) with $0_{\mathcal{F}} = b$ and $1_{\mathcal{F}} = a$, whereas $(g\restriction F) \in \hom_{\mathrm{S}}(\mathcal{F}, \mathcal{B}\restriction\{\sim\})$, and so, by Remarks 2.7(iii) (implying the injectivity of $g\restriction F$) and 2.8(ii), $(\mathcal{B}\restriction\{\sim\})\restriction g[F]$ is $\sim$-classical, while $g(a) \in D^{\mathcal{B}} \not\ni g(b)$. Hence, $g(a) = 1_{\mathcal{B}}$ and $g(b) = 0_{\mathcal{B}}$. Then, $(\frac{1}{2})_{\mathcal{B}} \in B = g[D]$, in which case there is some $c \in D$ such that $g(c) = (\frac{1}{2})_{\mathcal{B}}$. Let $\mathcal{G}$ be the submatrix of $\mathcal{D}$ generated by $\{a, b, c\}$, in which case $f \triangleq (g\restriction G) \in \hom^{\mathrm{S}}_{\mathrm{S}}(\mathcal{G}, \mathcal{B})$, for $g[\{a, b, c\}] = B$. Let $J \triangleq \{i \in I \mid \pi_i(c) = \frac{1}{2}\}$, in which case $\pi_i(c) \in E$, for all $i \in (I \setminus J)$, and so, if J was empty, then c would be in E^I, in which case $\mathcal{G}$ would be a submatrix of $\mathcal{E}^I$, and so, by (2.16), C, being defined by $\mathcal{B}$, would be $\sim$-classical. Therefore, $J \neq \varnothing$. Take any $j \in J$. Let us prove, by contradiction, that $(\pi_j\restriction G) \in \hom^{\mathrm{S}}_{\mathrm{S}}(\mathcal{G}, \mathcal{A})$. For suppose $(\pi_j\restriction G) \notin \hom^{\mathrm{S}}_{\mathrm{S}}(\mathcal{G}, \mathcal{A})$. Then, as $(\pi_j\restriction G) \in \hom^{\mathrm{S}}(\mathcal{G}, \mathcal{A})$, for $\pi_j[\{a, b, c\}] = A$, there is some $d \in (G \setminus D^{\mathcal{G}})$ such that $\pi_j(d) \in D^{\mathcal{A}}$. Consider the following complementary subsubcases:

* $\mathcal{A}$ is not truth-singular.
  Then, by Lemma 2.10 and Remark 2.7(iii), $\mathcal{A}$, being simple and finite, is a strict surjective homomorphic image of a subdirect product of a tuple constituted by submatrices of $\mathcal{B}$, in which case this is not truth-singular, and so is false-singular. Therefore, as $d \notin D^{\mathcal{G}}$, we have $f(d) \notin D^{\mathcal{B}}$, in which case $f(d) = 0_{\mathcal{B}}$, for $\mathcal{B}$ is false-singular, and so $\sim^{\mathfrak{B}} f(d) = 1_{\mathcal{B}} \in D^{\mathcal{B}}$. On the other hand, as $\mathcal{A}$ is not $\sim$-paraconsistent but is consistent, $\pi_j(\sim^{\mathfrak{G}} d) = \sim^{\mathfrak{A}} \pi_j(d) \notin D^{\mathcal{A}}$, in which case $\sim^{\mathfrak{G}} d \notin D^{\mathcal{G}}$, and so $\sim^{\mathfrak{B}} f(d) = f(\sim^{\mathfrak{G}} d) \notin D^{\mathcal{B}}$.
* $\mathcal{A}$ is truth-singular.
  Then, $\pi_j(d) = 1 = \pi_i(d)$, for all $i \in J$, because $\pi_j(e) = \pi_i(e)$, for all $e \in \{a, b, c\}$, and so for all $e \in G \ni d$, in which case $d \in E^I \supseteq \{a, b\}$, and so the submatrix $\mathcal{H}$ of $\mathcal{G}$ generated by $\{a, b, d\}$ is a submatrix of $\mathcal{E}^I$. Moreover, $\pi_j(\sim^{\mathfrak{G}} d) = \sim^{\mathfrak{A}} \pi_j(d) = 0 \notin D^{\mathcal{A}}$, in which case $(\{d, \sim^{\mathfrak{G}} d\} \cap D^{\mathcal{G}}) = \varnothing$, and so $(\{f(d), \sim^{\mathfrak{B}} f(d)\} \cap D^{\mathcal{B}}) = \varnothing$. Hence, $f(d) = (\frac{1}{2})_{\mathcal{B}}$, in which case $f[\{a, b, d\}] = B$, and so $(f\restriction H) \in \hom^{\mathrm{S}}_{\mathrm{S}}(\mathcal{H}, \mathcal{B})$. In this way, by (2.16), C, being defined by $\mathcal{B}$, is $\sim$-classical.

Thus, anyway, we come to a contradiction. Therefore, $(\pi_j\restriction G) \in \hom^{\mathrm{S}}_{\mathrm{S}}(\mathcal{G}, \mathcal{A})$. Hence, since $f \in \hom^{\mathrm{S}}_{\mathrm{S}}(\mathcal{G}, \mathcal{B})$, by Remark 2.7(iii) and Lemma 2.9, $\mathcal{A}$ and $\mathcal{B}$, being both simple, are isomorphic.

[Then, Lemma 4.4 completes the argument.] □

In view of Corollary 4.6 [and Theorem 4.16], any [non-$\sim$-classical] three-valued Σ-logic with subclassical negation $\sim$ is defined by a [unique] canonical three-valued $\sim$-super-classical Σ-matrix [said to be *characteristic for/of* the logic], $\mathcal{A}$ being characteristic for C, unless this is $\sim$-classical. On the other hand, the uniqueness is not, generally speaking, the case for $\sim$-classical (even both implicative {and so disjunctive; cf. Lemma 4.8} and conjunctive) ones, in view of Corollary 4.6 and Example 4.9.

Corollary 4.18. *Let $\Sigma' \supseteq \Sigma$ be a signature and C' a three-valued Σ'-expansion of C. Suppose C is not $\sim$-classical. Then, C' is defined by a unique Σ'-expansion of $\mathcal{A}$.*

Proof. In that case, $\sim$ is a subclassical negation for C', being, in its turn, non-$\sim$-classical. Hence, by Corollary 4.6, C' is defined by a canonical three-valued

$\sim$-super-classical Σ'-matrix $\mathcal{A}'$, in which case C is defined by the canonical three-valued $\sim$-super-classical Σ-matrix $\mathcal{A}'{\upharpoonright}\Sigma$, and so, by Theorem 4.16, this is equal to $\mathcal{A}$. Finally, as any Σ'-expansion of $\mathcal{A}$ is canonical, Theorem 4.16 completes the argument. □

5. Paraconsistent extensions

Set $M_2 \triangleq (2^2 \setminus \Delta_2)$.

Theorem 5.1. *Suppose $\mathcal{A}$ is false-singular (in particular, $\sim$-paraconsistent) [and C is $\sim$-subclassical]. Then, the following are equivalent:*

(i) *C has no proper $\sim$-paraconsistent [$\sim$-subclassical] extension;*
(ii) *C has no proper $\sim$-paraconsistent non-$\sim$-subclassical extension;*
(iii) *either $\mathcal{A}$ has a ternary $\frac{1}{2}$-relative semi-conjunction or $\{\frac{1}{2}\}$ does not form a subalgebra of $\mathfrak{A}$ (in particular, $\sim^{\mathfrak{A}}\frac{1}{2} \neq \frac{1}{2}$);*
(iv) *$L_3 \triangleq (M_2 \cup \{\langle\frac{1}{2},\frac{1}{2}\rangle\})$ does not form a subalgebra of $\mathfrak{A}^2$;*
(v) *$\mathcal{A}$ has no truth-singular $\sim$-paraconsistent subdirect square;*
(vi) *$\mathcal{A}^2$ has no truth-singular $\sim$-paraconsistent submatrix;*
(vii) *C has no truth-singular $\sim$-paraconsistent model;*
(viii) *$\mathcal{A}_{\frac{1}{2}} \triangleq \langle\mathfrak{A}, \{\frac{1}{2}\}\rangle$ is not a $\sim$-paraconsistent model of C;*
(ix) *C has no truth-singular $\sim$-paraconsistent model with underlying algebra $\mathfrak{A}$.*

In particular, C has a $\sim$-paraconsistent proper extension iff it has a [non-]non-$\sim$-subclassical one, and if any three-valued expansion of C does so.

Proof. First, assume (iii) holds. Consider any $\sim$-paraconsistent extension C' of C, in which case $x_1 \notin T \triangleq C'(\{x_0, \sim x_0\}) \supseteq \{x_0, \sim x_0\}$, and so, by the structurality of C', $\langle \mathfrak{Fm}^{\omega}_{\Sigma}, T\rangle$ is a $\sim$-paraconsistent model of C' (in particular, of C). Hence, by Lemma 4.15 and (2.16), $\mathcal{A}$ is a model of C', in which case $C' = C$, and so both (i) and (ii) hold.

Next, assume L_3 forms a subalgebra of $\mathfrak{A}^2$. Then, by (2.16), $\mathcal{B} \triangleq (\mathcal{A}^2{\upharpoonright}L_3) \in \mathrm{Mod}(C)$ is a subdirect square of $\mathcal{A}$, because $\pi_i[L_3] = A$, for each $i \in 2$. Moreover, M_2 is disjoint with $D^{\mathcal{B}} \ni \langle\frac{1}{2},\frac{1}{2}\rangle$, for $0 \notin D^{\mathcal{A}} \ni \frac{1}{2}$, because $\mathcal{A}$ is false-singular, in which case we have $D^{\mathcal{B}} = \{\langle\frac{1}{2},\frac{1}{2}\rangle\} = (L_3 \cap \Delta_A)$, and so $\mathcal{B}$ is both truth-singular and, being consistent, for $L_3 \supseteq M_2 \neq \varnothing$, $\sim$-paraconsistent, for $L_3 \ni \sim^{\mathfrak{A}^2}\langle\frac{1}{2},\frac{1}{2}\rangle = \langle\sim^{\mathfrak{A}}\frac{1}{2}, \sim^{\mathfrak{A}}\frac{1}{2}\rangle \in \Delta_A$. Moreover, $(\pi_0{\upharpoonright}L_3) \in \hom^{\mathrm{S}}_{\mathrm{S}}(\mathcal{B}, \mathcal{A}_{\frac{1}{2}})$. Hence, by (2.16), $\mathcal{A}_{\frac{1}{2}} \in \mathrm{Mod}(C)$ is $\sim$-paraconsistent. Thus, (v/viii)⇒(iv) holds, while (v/viii/ix) is a particular case of (vi/ix/vii), respectively, whereas (vii)⇒(vi) is by (2.16).

Now, let $\mathcal{B} \in \mathrm{Mod}(C)$ be both $\sim$-paraconsistent and truth-singular, in which case (4.2) is true in $\mathcal{B}$, and so is its logical consequence

$$\{x_0, x_1, \sim x_1\} \vdash \sim x_0, \tag{5.1}$$

not being true in $\mathcal{A}$ under $[x_0/1, x_1/\frac{1}{2}]$ [but, being a logical consequence of (2.10)$[x_0/x_1, x_1/\sim x_0]$, true in any $\sim$-classical model $\mathcal{C}'$ of C, in view of Remark 2.8(i)**d)**]. Thus, the logic of $\{\mathcal{B}[,\mathcal{C}']\}$ is a proper $\sim$-paraconsistent [$\sim$-subclassical] extension of C, so (i)⇒(vii) holds. And what is more, (4.2), being true in $\mathcal{B}$, is not true in any $\sim$-[super-]classical Σ-matrix [in particular, in $\mathcal{A}$], in view of [Theorem 4.1 and] (3.2) with $n = 0$ and $m = 1$. Thus, the logic of $\mathcal{B}$ is a proper $\sim$-paraconsistent non-$\sim$-subclassical extension of C, so (ii)⇒(vii) holds.

Finally, assume $\mathcal{A}$ has no ternary $\frac{1}{2}$-relative semi-conjunction and $\{\frac{1}{2}\}$ forms a subalgebra of $\mathfrak{A}$, in which case $\sim^{\mathfrak{A}}\frac{1}{2} = \frac{1}{2}$. Let $\mathfrak{B}$ be the subalgebra of $\mathfrak{A}^2$ generated by L_3. If $\langle 0,0\rangle$ was in B, then there would be some $\varphi \in \mathrm{Fm}^3_{\Sigma}$ such that $\varphi^{\mathfrak{A}}(0,1,\frac{1}{2}) = 0 = \varphi^{\mathfrak{A}}(1,0,\frac{1}{2})$, in which case it would be a ternary $\frac{1}{2}$-relative semi-conjunction for

$\mathcal{A}$. Likewise, if either $\langle\frac{1}{2},0\rangle$ or $\langle 0,\frac{1}{2}\rangle$ was in B, then there would be some $\varphi\in\mathrm{Fm}^3_\Sigma$ such that $\varphi^{\mathfrak{A}}(0,1,\frac{1}{2})=0$ and $\varphi^{\mathfrak{A}}(1,0,\frac{1}{2})=\frac{1}{2}$, in which case it would be a ternary $\frac{1}{2}$-relative semi-conjunction for $\mathcal{A}$. Therefore, as $\sim^{\mathfrak{A}}1=0$ and $\sim^{\mathfrak{A}}\frac{1}{2}=\frac{1}{2}$, we conclude that $(\{\langle 0,\frac{1}{2}\rangle,\langle 1,\frac{1}{2}\rangle,\langle\frac{1}{2},1\rangle,\langle\frac{1}{2},0\rangle,\langle 0,0\rangle,\langle 1,1\rangle\}\cap B)=\varnothing$. Thus, $B=L_3$ forms a subalgebra of $\mathfrak{A}^2$. In this way, (iv)⇒(iii) holds.

After all, Corollary 4.18 completes the argument, for any expansion of $\mathcal{A}$ inherits ternary $\frac{1}{2}$-relative semi-conjunctions (if any). □

Theorem 5.1(i)⇔(iii[iv]) is especially useful for [effective dis]proving the maximal $\sim$-paraconsistency of C, as we show below [cf. Example 7.6]. And what is more, since, by Remark 2.8(i)**d)**, $\mathcal{A}$ has no proper $\sim$-paraconsistent submatrix, by Corollaries 2.15 and 4.6, we immediately have the following "axiomatic" version of Theorem 5.1:

Corollary 5.2. *Any [non-]non-$\sim$-paraconsistent three-valued Σ-logic with subclassical negation $\sim$ has no $\sim$-paraconsistent [proper axiomatic] extension [and so is axiomatically maximally $\sim$-paraconsistent].*

Remark 5.3. Suppose either $\mathcal{A}$ is both false-singular and weakly $\bar{\wedge}$-conjunctive or both 2 forms a subalgebra of $\mathfrak{A}$ and $\mathcal{A}{\restriction}2$ is weakly $\bar{\wedge}$-conjunctive. Then, $(x_0\,\bar{\wedge}\,x_1)$ is a binary semi-conjunction for $\mathcal{A}$. □

By Corollary 4.6, Theorem 5.1(iii)⇒(i) and Remark 5.3, we first have:

Corollary 5.4 (cf. the reference [Pyn 95b] of [14])**.** *Any weakly conjunctive three-valued Σ-logic with subclassical negation $\sim$ has no proper $\sim$-paraconsistent extension.*

The principal advance of this universal *maximal paraconsistency* result with regard to its particular case obtained in the reference [Pyn 95b] of [14] but for merely $\sim$-subclassical logics, subsuming particular results first obtained *ad hoc* for LP (being $\wedge$-conjunctive) in [14], HZ (being $\vee^{\sim}$-conjunctive; cf. the last paragraph of Subsubsection 8.1.1 below) in [17] and LA (being $\wedge$-conjunctive) in [20], and so providing these with a first *generic* insight, as well as yielding a *first* proof of the maximal paraconsistency of P^1 [22] (being conjunctive too; cf. either Remark 8.10 below or [13]), in its turn, subsuming its *axiomatic* maximal paraconsistency discovered in [22] and equally subsumed by either Corollary 5.2 or both Corollary 6.6 below (in particular, Theorem 6.3 of [13]) and Remark 2.8(i)**d)**, consists in extending the latter beyond subclassical logics towards those with merely subclassical negation, in which case, contrary to the latter, the former is equally applicable to arbitrary three-valued expansions (cf. Corollary 4.18 in this connection) of logics under consideration, because expansions retain (weak) conjunction, subclassical negation and paraconsistency, but do not, generally speaking, inherit the property of being subclassical, and so the former, as opposed to the latter, covers *arbitrary three-valued expansions* of LP (including those of its three-valued expansion LA), HZ and P^1. In view of Example 7.15 below, the stipulation of the weak conjunctivity cannot be omitted in the formulation of Corollary 5.4.

5.1. **Premaximal paraconsistency.** Let $C_{\frac{1}{2}}$ be the logic of $\mathcal{A}_{\frac{1}{2}}$.

Lemma 5.5. *Let $\mathcal{B}\in\mathrm{Mod}(C)$. Suppose C is a non-purely-inferential $\sim$-paraconsistent sublogic of $C_{\frac{1}{2}}$. Then, $\mathcal{B}$ is consistent iff it is $\sim$-paraconsistent. In particular, $\mathcal{A}_{\frac{1}{2}}$ is $\sim$-paraconsistent.*

Proof. The "if" part is immediate. Conversely, assume $\mathcal{B}$ is consistent. Then, by the structurality of C, applying the Σ-substitution extending $[x_i/x_0]_{i\in\omega}$ to any theorem of C, we conclude that there is some $\phi\in(\mathrm{Fm}^1_\Sigma\cap C(\varnothing))$, and so, as $\mathcal{A}_{\frac{1}{2}}\in\mathrm{Mod}(C)$,

$\phi^{\mathfrak{A}}(a) = \frac{1}{2}$, for all $a \in A$. Take any $b \in (B \setminus D^{\mathcal{B}}) \neq \varnothing$, for $\mathcal{B}$ is consistent. Then, by (2.16), the submatrix $\mathcal{D}$ of $\mathcal{B}$ generated by $\{b\}$ is a finitely-generated consistent model of C. Hence, by Lemma 2.10, there are some set I and some submatrix $\mathcal{E} \in \mathbf{H}^{-1}(\mathbf{H}(\mathcal{D}))$ of $\mathcal{A}^I$. Take any $e \in E \neq \varnothing$. Then, $\phi^{\mathfrak{E}}(e) = (I \times \{\frac{1}{2}\}) \in D^{\mathcal{E}}$, in which case $\sim^{\mathfrak{E}}\phi^{\mathfrak{E}}(e) \in D^{\mathcal{E}}$, for $\mathcal{A}$ is $\sim$-paraconsistent, and so $\mathcal{E}$, being consistent, for $\mathcal{D}$ is so, is $\sim$-paraconsistent. Thus, $\mathcal{B}$ is so, in view of (2.16), as required. $\square$

Theorem 5.6. *Suppose C has a proper $\sim$-paraconsistent extension. Then, the following hold:*

(i) *$C_{\frac{1}{2}}$ is the proper ($\sim$-para)consistent extension of C relatively axiomatized by (4.2);*

(ii) *$C_{\frac{1}{2}}$ has no proper inferentially consistent (in particular, $\sim$-paraconsistent) extension;*

(iii) *the following are equivalent:*

a) *C has a theorem;*

b) *2 does not form a subalgebra of $\mathfrak{A}$;*

c) *C is not $\sim$-subclassical;*

d) *$C_{\frac{1}{2}}$ is the only proper ($\sim$-para)consistent extension of C;*

e) *$C_{\frac{1}{2}}$ has no proper sublogic being a proper extension of C.*

In particular, any three-valued $\sim$-paraconsistent Σ-logic with subclassical negation $\sim$ is premaximally $\sim$-paraconsistent extension iff it is either maximally $\sim$-paraconsistent or not $\sim$-subclassical/purely-inferential (in particular, weakly disjunctive [in particular, implicative]).

Proof. Then, C (viz., $\mathcal{A}$) is $\sim$-paraconsistent (in which case it is false-singular, and so weakly $\sim$-negative). Hence, by Theorem 5.1(iii/iv/viii)$\Rightarrow$(i), $\mathcal{A}_{\frac{1}{2}} \in \mathrm{Mod}(C)$ is $\sim$-paraconsistent, while $\mathcal{A}$ has no ternary $\frac{1}{2}$-relative semi-conjunction, whereas $\{\frac{1}{2}\}|L_3$ forms a subalgebra of $\mathfrak{A}|\mathfrak{A}^2$, respectively (in particular, $\sim^{\mathfrak{A}}\frac{1}{2} = \frac{1}{2}$).

(i) Then, (4.2), not being true in $\mathcal{A}$ under $[x_0/1]$, is true in $\mathcal{A}_{\frac{1}{2}}$. In this way, the logic of $\mathcal{A}_{\frac{1}{2}}$ is a proper ($\sim$-para)consistent extension of C satisfying (4.2). Conversely, consider any Σ-rule $\Gamma \vdash \phi$ not satisfied in the extension C' of C relatively axiomatized by (4.2), in which case, as $\sim[\Gamma] \subseteq C'(\Gamma)$, the Σ-rule $(\Gamma \cup \sim[\Gamma]) \vdash \phi$ is not satisfied in C', and so in its sublogic C. Then, there is some $h \in \hom(\mathfrak{Fm}^{\omega}_{\Sigma}, \mathfrak{A})$ such that $h[\Gamma \cup \sim[\Gamma]] \subseteq D^{\mathcal{A}} = \{\frac{1}{2}, 1\} \not\ni h(\phi)$. In particular, $h(\phi) \neq \frac{1}{2}$. And what is more, for each $\psi \in \Gamma$, both $h(\psi) \in D^{\mathcal{A}}$ and $\sim^{\mathfrak{A}}h(\psi) = h(\sim\psi) \in D^{\mathcal{A}}$, in which case $h(\psi) = \frac{1}{2}$, for $\sim^{\mathfrak{A}}1 = 0 \notin D^{\mathcal{A}}$, and so $h[\Gamma] \subseteq \{\frac{1}{2}\} = D^{\mathcal{A}_{\frac{1}{2}}} \not\ni h(\phi)$. Thus, $C' = C_{\frac{1}{2}}$.

(ii) Consider any inferentially consistent extension C' of $C_{\frac{1}{2}}$, in which case $x_1 \notin T \triangleq C'(x_0) \ni x_0$. Then, by the structurality of C', $\langle \mathfrak{Fm}^{\omega}_{\Sigma}, T\rangle$ is a model of C' (in particular, of $C_{\frac{1}{2}}$), and so is its finitely-generated consistent truth-non-empty submatrix $\mathcal{B} \triangleq \langle \mathfrak{Fm}^2_{\Sigma}, T \cap \mathrm{Fm}^2_{\Sigma}\rangle$, in view of (2.16). Hence, by Lemma 2.10, there are some set I and some submatrix $\mathcal{D} \in \mathbf{H}^{-1}(\mathbf{H}(\mathcal{B}))$ of $\mathcal{A}^I_{\frac{1}{2}}$, in which case, by (2.16), $\mathcal{D}$ is a consistent truth-non-empty model of C', for $\mathcal{B}$ is so, and so $I \neq \varnothing$, while there are some $a \in D^{\mathcal{D}}$ and some $b \in (D \setminus D^{\mathcal{D}})$. Then, $D \ni a = (I \times \{\frac{1}{2}\}) \neq b$, in which case either $J \triangleq \{i \in I \mid \pi_i(b) = 1\}$ or $K \triangleq \{i \in I \mid \pi_i(b) = 0\}$ is non-empty. Given any $\bar{c} \in A^3$, set $(c_0\|c_1\|c_2) \triangleq ((J\times\{c_0\})\cup(K\times\{c_1\})\cup((I\setminus(J\cup K))\times\{c_2\})) \in A^I$. In this way, $D \ni a = (\frac{1}{2}\|\frac{1}{2}\|\frac{1}{2})$ and $D \ni b = (1\|0\|\frac{1}{2})$, in which case $D \ni \sim^{\mathfrak{D}}b = (0\|1\|\frac{1}{2})$. Consider the following exhaustive cases:

- $J \neq \varnothing \neq K$.

  Then, as $\{\frac{1}{2}\}|L_3$ forms a subalgebra of $\mathfrak{A}|\mathfrak{A}^2$, $\{\langle\langle x, y\rangle, (x\|y\|\frac{1}{2})\rangle \mid \langle x, y\rangle \in$

$L_3\}$ is an embedding of $\mathcal{E} \triangleq (\mathcal{A}^2{\upharpoonright}L_3)$ into $\mathcal{D}$, in which case, by (2.16), $\mathcal{E}$ is a model of C', for $\mathcal{D}$ is so, and so is $\mathcal{A}_{\frac{1}{2}}$, for $(\pi_0{\upharpoonright}L_3) \in \mathrm{hom}^{\mathrm{S}}_{\mathrm{S}}(\mathcal{E}, \mathcal{A}_{\frac{1}{2}})$.

- $K = \varnothing$,
  in which case $J \neq \varnothing$, while $D \ni a = (\frac{1}{2}\|\frac{1}{2}\|\frac{1}{2})$, whereas $D \ni b = (0\|\frac{1}{2}\|\frac{1}{2})$, and so $D \ni {\sim}^{\mathfrak{D}} b = (1\|\frac{1}{2}\|\frac{1}{2})$. Then, as $\{\frac{1}{2}\}$ forms a subalgebra of $\mathfrak{A}$, $\{\langle x, (x\|\frac{1}{2}\|\frac{1}{2})\rangle \mid x \in A\}$ is an embedding of $\mathcal{A}_{\frac{1}{2}}$ into $\mathcal{D}$, in which case, by (2.16), $\mathcal{A}_{\frac{1}{2}}$ is a model of C', for $\mathcal{D}$ is so.
- $J = \varnothing$,
  in which case $K \neq \varnothing$, while $D \ni a = (\frac{1}{2}\|\frac{1}{2}\|\frac{1}{2})$, whereas $D \ni b = (\frac{1}{2}\|0\|\frac{1}{2})$, and so $D \ni {\sim}^{\mathfrak{D}} b = (\frac{1}{2}\|1\|\frac{1}{2})$. Then, as $\{\frac{1}{2}\}$ forms a subalgebra of $\mathfrak{A}$, $\{\langle x, (\frac{1}{2}\|x\|\frac{1}{2})\rangle \mid x \in A\}$ is an embedding of $\mathcal{A}_{\frac{1}{2}}$ into $\mathcal{D}$, in which case, by (2.16), $\mathcal{A}_{\frac{1}{2}}$ is a model of C', for $\mathcal{D}$ is so.

Thus, in any case, $\mathcal{A}_{\frac{1}{2}} \in \mathrm{Mod}(C')$, and so $C' = C_{\frac{1}{2}}$.

(iii) First, assume **a)** holds. Consider any consistent extension C' of C, in which case $C'(\varnothing) \supseteq C(\varnothing) \neq \varnothing$, and so, if C' was inferentially inconsistent, then it, being structural, would be inconsistent, and the following complementary cases:

- (4.2) is satisfied in C',
  in which case, by (i), C' is an inferentially consistent extension of $C_{\frac{1}{2}}$, and so, by (ii), $C' = C_{\frac{1}{2}}$.
- (4.2) is not satisfied in C',
  in which case ${\sim}x_0 \notin T \triangleq C'(x_0) \ni x_0$. Then, by the structurality of C', $\mathcal{B} \triangleq \langle \mathfrak{Fm}^{\omega}_{\Sigma}, T\rangle$ is a model of C' (in particular, of C), in which (4.2) is not true under the diagonal Σ-substitution, in which case, by Lemma 5.5, $\mathcal{B}$, being consistent, is $\sim$-paraconsistent, for C is so, and so, by (2.16) and Lemma 4.15, $\mathcal{A}$ is a model of C', for $\mathcal{B}$ is so, in which case $C' = C$.

Thus, by (i), **d)** holds.

Next, **d)**$\Rightarrow$**e)** is by the ($\sim$-para)consistency of $\mathcal{A}_{\frac{1}{2}}$, and so of any sublogic of $C_{\frac{1}{2}}$.

Now, let $\mathcal{B}$ be a $\sim$-classical model of C. Then, (5.1), being a logical consequence of $((2.10)[x_0/x_1, x_1/{\sim}x_0])/(4.2)$, is true in $\mathcal{B}/\mathcal{A}_{\frac{1}{2}}$, for (2.10)/(4.2) is so, in view of "Remark 2.8(i)**d)**"/(i), respectively. However, it is not true in $\mathcal{A}$ under $[x_0/1, x_1/\frac{1}{2}]$. Moreover, by (3.2) with $n = 0$ and $m = 1$, (4.2) is not true in $\mathcal{B}$. In this way, by (i), the logic of $\{\mathcal{A}_{\frac{1}{2}}, \mathcal{B}\}$ is a proper extension/sublogic of $C_{/\frac{1}{2}}$. Thus, **e)**$\Rightarrow$**c)** holds.

Further, if 2 forms a subalgebra of $\mathfrak{A}$, then, by (2.16), $\mathcal{A}{\upharpoonright}2$ is a $\sim$-classical model of C. Therefore, **c)**$\Rightarrow$**b)** holds.

Finally, assume **b)** holds. Then, there is some $\varphi \in \mathrm{Fm}^2_{\Sigma}$ such that $\varphi^{\mathfrak{A}}(1,0) = \frac{1}{2} = \varphi^{\mathfrak{A}}(\frac{1}{2},\frac{1}{2})$, for $\{\frac{1}{2}\}$ forms a subalgebra of $\mathfrak{A}$, in which case, if $\varphi^{\mathfrak{A}}(0,1)$ was equal to 0, then φ would be a ternary $\frac{1}{2}$-relative semi-conjunction for $\mathcal{A}$, and so $\varphi^{\mathfrak{A}}(0,1) \in D^{\mathcal{A}} \supseteq \{\varphi^{\mathfrak{A}}(1,0), \varphi^{\mathfrak{A}}(\frac{1}{2},\frac{1}{2})\}$. In this way, $(\varphi[x_1/{\sim}x_0]) \in C(\varnothing)$, and so **a)** holds.

After all, Corollary 4.6, Lemma 4.8 and Remark 2.8(i)**d)** complete the proof. $\square$

In this way, Corollary 4.6 as well as Theorem[s] 5.1(i)$\Leftrightarrow$(iv) [and 5.6(iii)**b)**$\Leftrightarrow$**d)**] provide an effective algebraic criterion of the [pre]maximal $\sim$-paraconsistency of three-valued $\sim$-paraconsistent Σ-logics with subclassical negation $\sim$.

6. Classical extensions

Next, $\mathcal{A}$ is said to *satisfy [Diagonal] Generation Condition ([D]GC)*, provided either $\langle 0,0\rangle$ or $\langle\frac{1}{2},0[+\frac{1}{2}]\rangle$ or $\langle 0[+1],\frac{1}{2}[+\frac{1}{2}]\rangle$ belongs to [i.e., Δ_A is not disjoint with] the carrier of the subalgebra of $\mathfrak{A}^2$ generated by $[M_2\cup]\{\langle 1,\frac{1}{2}\rangle\}$.

Lemma 6.1. *Let I be a finite set, $\overline{\mathcal{C}}\in\mathbf{S}_*(\mathcal{A})^I$ and $\mathcal{D}$ a consistent truth-non-empty non-$\sim$-paraconsistent subdirect product of it. Suppose $\mathcal{A}$ is not a model of the logic of $\mathcal{D}$, while either $\mathcal{A}$ is either non-$\sim$-paraconsistent or weakly conjunctive, or $\mathcal{D}$ is $\sim$-negative or both $\mathcal{A}$ either has a binary semi-conjunction or satisfies GC, and either 2 forms a subalgebra of $\mathfrak{A}$ or $L_4\triangleq(A^2\setminus(2^2\cup\{\frac{1}{2}\}^2))$ forms a subalgebra of $\mathfrak{A}^2$ or $\mathcal{A}$ satisfies DGC. Then, the following hold:*

(i) *if 2 forms a subalgebra of $\mathfrak{A}$, then $\mathcal{A}\restriction 2$ is embeddable into $\mathcal{D}$;*
(ii) *if 2 does not form a subalgebra of $\mathfrak{A}$, then $\mathcal{A}$ is both $\sim$-paraconsistent (in particular, false-singular) and not weakly conjunctive, while L_4 forms a subalgebra of $\mathfrak{A}^2$, whereas $\mathcal{A}^2\restriction L_4$ is embeddable into $\mathcal{D}$.*

Proof. In that case, $I\neq\varnothing$, for $\mathcal{D}$ is consistent. Consider the following complementary cases:

(1) $(I\times\{i\})\in D$, for some $i\in 2$,
in which case $D\ni{\sim}^{\mathfrak{D}}(I\times\{i\})=(I\times\{1-i\})$, and so, if 2 did not form a subalgebra of $\mathfrak{A}$, then there would be some $\varphi\in\mathrm{Fm}^2_\Sigma$ such that $\varphi^{\mathfrak{A}}(0,1)=\frac{1}{2}$, in which case D would contain $\varphi^{\mathfrak{D}}(I\times\{0\},I\times\{1\})=(I\times\{\frac{1}{2}\})$, and so, as $I\neq\varnothing$, $\{\langle a,I\times\{a\}\rangle\mid a\in A\}$ would be an embedding of $\mathcal{A}$ into $\mathcal{D}$ (in particular, by (2.16), $\mathcal{A}$ would be a model of the logic of $\mathcal{D}$). Therefore, 2 forms a subalgebra of $\mathfrak{A}$, in which case $\{\langle j,I\times\{j\}\rangle\mid j\in 2\}$ is an embedding of $\mathcal{A}\restriction 2$ into $\mathcal{D}$, and so (i,ii) hold, in that case.

(2) $(I\times\{i\})\in D$, for no $i\in 2$,
in which case, by Claim 4.17, $\mathcal{A}$ is both not weakly conjunctive and $\sim$-paraconsistent, and so false-singular. In particular,
$$e\triangleq(I\times\{\tfrac{1}{2}\})\notin D,\tag{6.1}$$
for, otherwise, we would have $\{e,{\sim}^{\mathfrak{D}}e\}\subseteq D^{\mathcal{D}}$, contrary to the fact that $\mathcal{D}$ is not $\sim$-paraconsistent but is consistent. Take any $a\in D^{\mathcal{D}}\neq\varnothing$, for $\mathcal{D}$ is truth-non-empty, Then, $a\in\{\frac{1}{2},1\}^I$, in which case, by (2) with $i=1$ and (6.1), $I\neq J\triangleq\{i\in I\mid\pi_i(a)=1\}\neq\varnothing$, and so $b\triangleq{\sim}^{\mathfrak{D}}a\in(D\setminus D^{\mathcal{D}})$. Given any $\bar{a}\in A^2$, set $(a_0\|a_1)\triangleq((J\times\{a_0\})\cup((I\setminus J)\times\{a_1\}))\in A^I$. Then, $a=(1\|\frac{1}{2})$.

Let us prove, by contradiction, that ${\sim}^{\mathfrak{A}}\frac{1}{2}=\frac{1}{2}$. For suppose ${\sim}^{\mathfrak{A}}\frac{1}{2}\neq\frac{1}{2}$. Then, as $\mathcal{A}$ is $\sim$-paraconsistent, we have ${\sim}^{\mathfrak{A}}\frac{1}{2}\in D^{\mathcal{A}}=\{\frac{1}{2},1\}$, in which case we get ${\sim}^{\mathfrak{A}}\frac{1}{2}=1$, and so both $b=(0\|1)\in D$ and ${\sim}^{\mathfrak{B}}b=(1\|0)\in D$ do not belong to $D^{\mathcal{D}}$, for $I\neq J\neq\varnothing$. Hence, $\mathcal{D}$ is not $\sim$-negative. Moreover, if $\mathcal{A}$ had a binary semi-conjunction φ, then D would contain $\varphi^{\mathfrak{A}}(b,{\sim}^{\mathfrak{B}}b)=(0\|0)=(I\times\{0\})$, contrary to (2) with $i=0$. Likewise, if $\mathcal{A}$ satisfied GC, then there would be some $\psi\in\mathrm{Fm}^1_\Sigma$ such that $\psi^{\mathfrak{A}}(\langle 1,\frac{1}{2}\rangle)$ would be in $\{\langle 0,\frac{1}{2}\rangle,\langle\frac{1}{2},0\rangle,\langle 0,0\rangle\}$, in which case ${\sim}^{\mathfrak{A}}\psi^{\mathfrak{A}}(\langle 1,\frac{1}{2}\rangle)$ would be equal to $\langle 1,1\rangle$, and so D would contain ${\sim}^{\mathfrak{D}}\psi^{\mathfrak{D}}(a)=(1\|1)=(I\times\{1\})$, contrary to (2) with $i=1$. This contradicts to the fact that $\mathcal{A}$ is neither weakly conjunctive nor non-$\sim$-paraconsistent. Thus, ${\sim}^{\mathfrak{A}}\frac{1}{2}=\frac{1}{2}$, in which case $b=(0\|\frac{1}{2})$. Consider the following complementary subcases:

(i) 2 forms a subalgebra of $\mathfrak{A}$.
Let us prove, by contradiction, that so does $\{\frac{1}{2}\}$. For suppose $\{\frac{1}{2}\}$ does not form a subalgebra of $\mathfrak{A}$. Then, there is some $\psi\in\mathrm{Fm}^1_\omega$ such

that $\psi^{\mathfrak{A}}(\frac{1}{2}) \in 2$, in which case $\psi^{\mathfrak{A}}[A] \subseteq 2$, for 2 forms a subalgebra of $\mathfrak{A}$, and so $\psi^{\mathfrak{A}} : A \to 2$ is not injective, for $|A| = 3 \nleq 2 = |2|$. Therefore, we have the following exhaustive subsubcases:

- $\psi^{\mathfrak{A}}(\frac{1}{2}) = \psi^{\mathfrak{A}}(0)$.
  Then, $(I \times \{1\}) \in \{\psi^{\mathfrak{D}}(b), \sim^{\mathfrak{D}} \psi^{\mathfrak{D}}(b)\} \subseteq D$.
- $\psi^{\mathfrak{A}}(\frac{1}{2}) = \psi^{\mathfrak{A}}(1)$.
  Then, $(I \times \{1\}) \in \{\psi^{\mathfrak{D}}(a), \sim^{\mathfrak{D}} \psi^{\mathfrak{D}}(a)\} \subseteq D$.
- $\psi^{\mathfrak{A}}(1) = \psi^{\mathfrak{A}}(0)$.
  Then, $(I \times \{1\}) \in \{\psi^{\mathfrak{D}}(\psi^{\mathfrak{D}}(a)), \sim^{\mathfrak{D}} \psi^{\mathfrak{D}}(\psi^{\mathfrak{D}}(a))\} \subseteq D$.

Thus, anyway, $(I \times \{1\}) \in D$. This contradicts to (2) with $i = 1$. In this way, $\{\frac{1}{2}\}$ forms a subalgebra of $\mathfrak{A}$. Then, as $J \neq \varnothing$, while $(1\|\frac{1}{2}) = a \in D \ni b = (0\|\frac{1}{2})$, $\{\langle i, (i\|\frac{1}{2})\rangle \mid i \in 2\}$ is an embedding of $\mathcal{A}{\restriction}2$ into $\mathcal{D}$.

(ii) 2 does not form a subalgebra of $\mathfrak{A}$.
Then, there is some $\varphi \in \mathrm{Fm}^2_\Sigma$ such that $\varphi^{\mathfrak{A}}(0,1) = \frac{1}{2}$, in which case $\psi \triangleq \varphi[x_1/{\sim}x_0] \in \mathrm{Fm}^1_\Sigma$, while $\psi^{\mathfrak{A}}(0) = \varphi^{\mathfrak{A}}(0,1) = \frac{1}{2}$, and so, as $D \ni \psi^{\mathfrak{D}}(b)$, by (6.1), we have $\psi^{\mathfrak{D}}(\frac{1}{2}) \in 2$. Hence, we get $c \triangleq (\frac{1}{2}\|1) \in \{\psi^{\mathfrak{D}}(b), \sim^{\mathfrak{D}}\psi^{\mathfrak{D}}(b)\} \subseteq D$, in which case $D \ni d \triangleq \sim^{\mathfrak{D}} c = (\frac{1}{2}\|0)$, and so $\{(u\|v) \mid \langle u,v\rangle \in L_4\} = \{a,b,c,d\} \subseteq D$. Let us prove, by contradiction, that L_4 forms a subalgebra of $\mathfrak{A}^2$. For suppose L_4 does not form a subalgebra of $\mathfrak{A}^2$, in which case there is some $\phi \in \mathrm{Fm}^4_\Sigma$ such that $\phi^{\mathfrak{A}^2}(\langle 1,\frac{1}{2}\rangle, \langle 0,\frac{1}{2}\rangle, \langle \frac{1}{2},1\rangle, \langle \frac{1}{2},0\rangle) \in (A^2 \setminus L_4) = (2^2 \cup \{\frac{1}{2}\}^2)$, and so $D \ni f \triangleq \phi^{\mathfrak{D}}(a,b,c,d) = (x\|y)$, where $\langle x,y\rangle \in (2^2 \cup \{\frac{1}{2}\}^2)$. Then, by (2) and (6.1), $\langle x,y\rangle \in (2^2 \setminus \Delta_2)$, in which case $0 \in \{x,y\}$, and so $f \in (D \setminus D^{\mathcal{D}}) \ni (y\|x) = \sim^{\mathfrak{D}} f$, for $I \neq J \neq \varnothing$. Hence, $\mathcal{D}$ is not $\sim$-negative. Therefore, $\mathcal{A}$ satisfies DGC, for it is is neither weakly conjunctive nor non-$\sim$-paraconsistent, in which case there are some $\xi \in \mathrm{Fm}^3_\Sigma$ and some $z \in A$ such that $\xi^{\mathfrak{A}^2}(\langle 1,\frac{1}{2}\rangle, \langle 1,0\rangle, \langle 0,1\rangle) = \langle z,z\rangle$, and so $D \ni \xi^{\mathfrak{D}}(a, (1\|0), (0\|1)) = (z\|z)$, for $\{(1\|0),(0\|1)\} = \{f, \sim^{\mathfrak{D}} f\} \subseteq D \ni a$. This contradicts to (2) and (6.1). Therefore, L_4 forms a subalgebra of $\mathfrak{A}^2$. Hence, as $J \neq \varnothing \neq (I \setminus J)$, $\{\langle\langle u,v\rangle, (u\|v)\rangle \mid \langle u,v\rangle \in L_4\}$ is an embedding of $\mathcal{A}^2{\restriction}L_4$ into $\mathcal{D}$, as required. □

Corollary 6.2. *Let $\mathcal{B}$ be a $\sim$-classical model of C. Suppose C is not $\sim$-classical. Then, the following hold:*

(i) *if 2 forms a subalgebra of $\mathcal{A}$, then $\mathcal{A}{\restriction}2$ is isomorphic to $\mathcal{B}$;*
(ii) *if 2 does not form a subalgebra of $\mathcal{A}$, then both $\mathcal{B}$ is not disjunctive and C is both not weakly conjunctive and maximally $\sim$-paraconsistent, in which case $\mathcal{A}$ is $\sim$-paraconsistent, and so is false-singular, while L_4 forms a subalgebra of $\mathfrak{A}^2$, whereas $\theta^{\mathcal{A}^2{\restriction}L_4} \in \mathrm{Con}(\mathfrak{A}^2{\restriction}L_4)$, $\langle \chi^{\mathcal{A}^2{\restriction}L_4}[\mathfrak{A}^2{\restriction}L_4], \{1\}\rangle$ being isomorphic to $\mathcal{B}$.*

Proof. Then, $\mathcal{B}$ is finite and simple. Therefore, by Lemma 2.10 and Remark 2.7(iii), there are some finite set I, some $\overline{\mathcal{C}} \in \mathbf{S}_*(\mathcal{A})^I$, some subdirect product $\mathcal{D}$ of it and some $h \in \hom^{\mathrm{S}}_{\mathrm{S}}(\mathcal{D},\mathcal{B})$, in which case, by Remark 2.8(ii), $\mathcal{D}$ is $\sim$-negative, for $\mathcal{B}$ is so, and so both consistent and truth-non-empty, while, by (2.16), the logic C' of $\mathcal{D}$ is the $\sim$-classical (in particular, non-$\sim$-paraconsistent; cf. Remark 2.8(i)**d)**) one of $\mathcal{B}$, and so, by Corollary 3.7, $\mathcal{A}$, being both consistent and truth-non-empty, in which case C is inferentially-consistent, is not a model of C'. Consider the following complementary cases:

(i) 2 forms a subalgebra of $\mathfrak{A}$.
Then, by Lemma 6.1(i), there is some embedding e of $\mathcal{A}{\restriction}2$ into $\mathcal{D}$, in which

case, by Remark 2.7(iii), $h \circ e$ is that into $\mathcal{B}$, and so is an isomorphism from $\mathcal{A}\upharpoonright 2$ onto $\mathcal{B}$, for this has no proper submatrix.

(ii) 2 does not form a subalgebra of $\mathfrak{A}$.
Then, by Theorem 5.6(iii)**b)**$\Rightarrow$**c)** and Lemma 6.1(ii), C is both not weakly conjunctive and maximally $\sim$-paraconsistent, in which case $\mathcal{A}$ is $\sim$-paraconsistent, and so false-singular, while L_4 forms a subalgebra of $\mathfrak{A}^2$, whereas there is some embedding e of of $\mathcal{F} \triangleq (\mathcal{A}^2\upharpoonright L_4)$ into $\mathcal{D}$, in which case $g \triangleq (h \circ e) \in \hom^{\mathrm{S}}_{\mathrm{S}}(\mathcal{F}, \mathcal{B})$, for $\mathcal{B}$, being $\sim$-classical, has no proper submatrix, and so, by Remark 2.7(i), $(\ker \chi^{\mathcal{F}}) = \theta^{\mathcal{F}} = g^{-1}[\theta^{\mathcal{B}}] = g^{-1}[\Delta_B] = (\ker g) \in \mathrm{Con}(\mathfrak{F})$, in which case $\chi^{\mathcal{F}}$ is a strict surjective homomorphism from $\mathcal{F}$ onto $\mathcal{G} \triangleq \langle \chi^{\mathcal{F}}[\mathfrak{F}], \{1\}\rangle$, and so, by the Homomorphism Theorem, $\chi^{\mathcal{F}} \circ g^{-1}$ is an isomorphism from $\mathcal{B}$ onto $\mathcal{G}$. Finally, let us prove, by contradiction, that $\mathcal{B}$ is not disjunctive. For suppose $\mathcal{B}$ is $\veebar$-disjunctive, and so is $\mathcal{F}$, in view of Remark 2.8(ii). Then, as $\langle \frac{1}{2}, 1\rangle \in D^{\mathcal{F}}$, for $\mathcal{A}$ is false-singular, we have $\{\langle 0, \frac{1}{2}\rangle \veebar^{\mathfrak{F}} \langle \frac{1}{2}, 1\rangle, \langle \frac{1}{2}, 1\rangle \veebar^{\mathfrak{F}} \langle 0, \frac{1}{2}\rangle\} \subseteq D^{\mathcal{F}}$, in which case we get $\{0 \veebar^{\mathfrak{A}} \frac{1}{2}, \frac{1}{2} \veebar^{\mathfrak{A}} 0\} \subseteq D^{\mathcal{A}}$, and so we eventually get $(\langle 0, \frac{1}{2}\rangle \veebar^{\mathfrak{F}} \langle \frac{1}{2}, 0\rangle) \in D^{\mathcal{F}}$. This contradicts to the fact that $(\{\langle 0, \frac{1}{2}\rangle, \langle \frac{1}{2}, 0\rangle\} \cap D^{\mathcal{F}}) = \varnothing$. Thus, $\mathcal{B}$ is not disjunctive. □

Combining [Lemmas 3.6, 4.7 and] Corollary 6.2 with (2.16) [and Remark 2.8(ii)], we immediately get:

Theorem 6.3. *C has a [$\veebar$-disjunctive] $\sim$-classical extension iff either of the following [but (iii)] holds:*

(i) *C is $\sim$-classical [and $\veebar$-disjunctive];*

(ii) *2 forms a subalgebra of $\mathfrak{A}$ [with $\veebar$-disjunctive $\mathcal{A}\upharpoonright 2$], in which case $\mathcal{A}\upharpoonright 2$ is a canonical $\sim$-classical model of C isomorphic to any $\sim$-classical model of C, and so is a unique canonical one and defines a unique $\sim$-classical extension of C;*

(iii) *C is both not weakly conjunctive and maximally $\sim$-paraconsistent, in which case $\mathcal{A}$ is $\sim$-paraconsistent, and so false-singular, while L_4 forms a subalgebra of $\mathfrak{A}^2$, whereas $\theta^{\mathcal{A}^2\upharpoonright L_4} \in \mathrm{Con}(\mathfrak{A}^2\upharpoonright L_4)$, in which case $\langle \chi^{\mathfrak{A}^2\upharpoonright L_4}[\mathfrak{A}^2\upharpoonright L_4], \{1\}\rangle$ is a canonical $\sim$-classical model of C isomorphic to any $\sim$-classical model of C, and so is a unique canonical one and defines a unique $\sim$-classical extension of C.*

In view of Lemma 3.6 and Theorem 6.3, C, being $\sim$-subclassical, has a unique $\sim$-classical extension/"canonical model" to be denoted by $C^{\mathrm{PC}}/\mathcal{A}_{\mathrm{PC}}$, respectively, and referred to as *characteristic of|for C*, in which case $C^{\mathrm{PC}} = [\neq]C$, whenever C is [not] $\sim$-classical. It is remarkable that the $\veebar$-disjunctivity of C is not required in the []-optional version of Theorem 6.3, making this the right characterization of C's being *genuinely* $\sim$-subclassical in the sense of having a functionally complete $\sim$-classical extension. And what is more, by Lemma 4.7 and Theorem 6.3, we have:

Corollary 6.4. *[Suppose $\mathcal{A}$ is either truth-singular or weakly conjunctive or disjunctive (in particular, implicative).] Then, C is $\sim$-subclassical if[f] either of the following holds:*

(i) *C is $\sim$-classical;*

(ii) *2 forms a subalgebra of $\mathfrak{A}$, in which case $\mathcal{A}\upharpoonright 2$ is a canonical $\sim$-classical model of C isomorphic to any $\sim$-classical model of C, and so is a unique canonical one and defines a unique $\sim$-classical extension of C.*

The []-optional stipulation(s) in the formulation of Corollary 6.4 (resp., Theorem 6.3) cannot be omitted {or, even, "weakened"}, because of existence of three-valued {even, weakly disjunctive} non-$\sim$-classical ⟨even, $\sim$-paraconsistent⟩ $\sim$-subclassical

Σ-logics, the underlying algebras of the characteristic matrices of which do not have subalgebras with carrier 2, as it ensues from:

Example 6.5. Let $i \in 2$, $\Sigma \triangleq \{\amalg, \sim\}$ with binary $\amalg$, $\mathcal{B}$ the canonical $\sim$-classical Σ-matrix with $(j \amalg^{\mathfrak{B}} k) \triangleq i$, for all $j, k \in 2$, and $\mathcal{A}$ false-singular with $\sim^{\mathfrak{A}} \frac{1}{2} \triangleq \frac{1}{2}$ and

$$(a \amalg^{\mathfrak{A}} b) \triangleq \begin{cases} i & \text{if } a = \frac{1}{2}, \\ \frac{1}{2} & \text{otherwise}, \end{cases}$$

for all $a, b \in A$, in which case $\mathcal{A}$ is both $\sim$-paraconsistent and, providing $i = 1$, weakly $\amalg$-disjunctive, and so is C. Then, we have:

$$\begin{array}{rcl} (\langle \frac{1}{2}, a\rangle \amalg^{\mathfrak{A}^2} \langle b, \frac{1}{2}\rangle) & = & \langle i, \frac{1}{2}\rangle, \\ (\langle b, \frac{1}{2}\rangle \amalg^{\mathfrak{A}^2} \langle \frac{1}{2}, a\rangle) & = & \langle \frac{1}{2}, i\rangle, \\ (\langle \frac{1}{2}, a\rangle \amalg^{\mathfrak{A}^2} \langle \frac{1}{2}, b\rangle) & = & \langle i, \frac{1}{2}\rangle, \\ (\langle a, \frac{1}{2}\rangle \amalg^{\mathfrak{A}^2} \langle b, \frac{1}{2}\rangle) & = & \langle \frac{1}{2}, i\rangle, \end{array}$$

for all $a, b \in 2$. Hence, L_4 forms a subalgebra of $\mathfrak{A}^2$, while $\chi^{\mathcal{A}^2 \restriction L_4} \in \hom_{\mathrm{S}}^{\mathrm{S}}(\mathcal{A}^2 \restriction L_4, \mathcal{B})$, in which case, by (2.16), $\mathcal{B} \in \mathrm{Mod}(C)$, and so C is $\sim$-subclassical. However, $(0 \amalg^{\mathfrak{A}} 1) = \frac{1}{2}$, in which case 2 does not form a subalgebra of $\mathfrak{A}$, and so, by Corollary 6.4, C is neither disjunctive nor weakly conjunctive. □

Corollary 6.6. *Suppose $\mathcal{A}$ is $\sqsupset$-implicative (viz., C is so; cf. Lemma 4.8). Then, C has a proper consistent axiomatic extension iff it is non-$\sim$-classical (in particular, $\sim$-paraconsistent) and $\sim$-subclassical, in which case C^{PC} is a unique proper consistent axiomatic extension of C and is relatively axiomatized by $\bar{\phi} \sqsupset \psi$, where $\bar{\phi} \in (\mathrm{Fm}_{\Sigma}^{1})^{*}$ and $\psi \in (C^{\mathrm{PC}}(\mathrm{img}\,\bar{\phi}) \setminus C(\mathrm{img}\,\bar{\phi}))$ (in particular, by* (2.11)*).*

Proof. According to Corollary 2.15, any [proper] {consistent} axiomatic extension of C is defined by some {non-empty} $\mathsf{S} \subseteq \mathbf{S}_{*}(\mathcal{A})$ [not containing $\mathcal{A}$, in which case $\mathsf{S} \subseteq \{=\}\{\mathcal{A}\restriction 2\}$, if 2 forms a subalgebra of $\mathfrak{A}$, and $\mathsf{S} = \varnothing$, otherwise {and so (2.6), Corollaries 2.15, 3.7, 6.4 and Remark 2.8(ii)(,(i)**d**)) complete the argument}. □

This subsumes Theorem 6.3 of [13] proved *ad hoc* therein.

7. Theorems versus consistent and proper paraconsistent versus inferentially consistent non-subclassical extensions

Lemma 7.1. *Let S be a set of Σ-matrices and C' the logic of S. Then, the following are equivalent:*

(i) *C' has a theorem;*
(ii) *for any set I, any $e \in \mathsf{S}^{I}$, any function f with domain I, and any $S \subseteq \prod_{i \in I} \lceil e(i) \rceil^{f(i)}$, the submatrix of $\prod_{i \in I} e(i)^{f(i)}$ generated by S is truth-non-empty;*
(iii) *for any set I, any $e \in \mathsf{S}^{I}$, any function f with domain I, and any $\vec{\bar{g}} \in \prod_{i \in I} \lceil e(i) \rceil^{f(i)}$, the submatrix of $\prod_{i \in I} e(i)^{f(i)}$ generated by $\{\vec{\bar{g}}\}$ is truth-non-empty;*
(iv) *for any enumeration e of S and any $|\mathsf{S}|$-tuple $\vec{\bar{g}}$ such that, for every $i \in |\mathsf{S}|$, $\bar{g}^{i}$ is an enumeration of $\lceil e(i) \rceil$, the submatrix of $\prod_{i \in I} e(i)^{|\lceil e(i) \rceil|}$ generated by $\{\vec{\bar{g}}\}$ is truth-non-empty.*

Proof. First, (i)⇒(ii) is by (2.16) and Corollary 2.13(ii)⇒(i). Next, (iii/iv) is a particular case of (ii/iii), respectively. Finally, assume (iv) holds. Take any enumeration e of S and, for each $i \in |\mathsf{S}|$, any enumeration $\bar{g}^{i}$ of $\lceil e(i) \rceil$. Let $\mathcal{D}$ be the submatrix of $\prod_{i \in I} e(i)^{|\lceil e(i) \rceil|}$ generated by $\{\vec{\bar{g}}\}$. Then, $D^{\mathcal{D}} \neq \varnothing$, in which case there is some $\varphi \in \mathrm{Fm}_{\Sigma}^{1}$ such that $\varphi^{\mathfrak{D}}(\vec{\bar{g}}) \in D^{\mathcal{D}}$, and so for each $i \in |\mathsf{S}|$ and every

$j \in |\lceil e(i) \rceil|$, $\varphi^{e(i)}(g^i_j) = \pi_j(\pi_i(\varphi^{\mathfrak{D}}(\vec{g}))) \in D^{e(i)}$. In this way, $\varphi \in C'(\varnothing)$. Thus, (i) holds. □

In case both S and all members of it are finite, Lemma 7.1(i)⇔(iv) provides an effective algebraic criterion of C''s having a theorem.

A *semi-conjunction for/of* a canonical $\sim$-classical Σ-matrix $\mathcal{B}$ is any $\varphi \in \mathrm{Fm}^2_\Sigma$ such that $\varphi^{\mathfrak{A}}(i, 1-i) = 0$, for all $i \in 2$.

Corollary 7.2. *Let $\mathcal{B}$ be a canonical $\sim$-classical Σ-matrix and C' the logic of $\mathcal{B}$. Then, the following are equivalent:*

(i) *C' has a theorem;*
(ii) *M_2 does not form a subalgebra of $\mathfrak{B}^2$;*
(iii) *$\mathcal{B}$ has a semi-conjunction.*

Proof. First, given any semi-conjunction φ of $\mathcal{B}$, $\sim\varphi[x_1/\sim x_0]$ is a theorem of C'. Hence, (iii)⇒(i) holds.

Next, assume (ii) holds. Then, there are some $\phi \in \mathrm{Fm}^2_\Sigma$ and some $j \in 2$ such that $\phi^{\mathfrak{B}}(i, 1-i) = j$, for all $i \in 2$, in which case $\sim^j \phi$ is a semi-conjunction of $\mathcal{B}$, and so (iii) holds.

Finally, assume (i) holds. Then, by Lemma 7.1(i)⇒(ii), the submatrix $\mathcal{D}$ of $\mathcal{B}^2$ generated by M_2 is truth-non-empty, in which case the unique distinguished value $\langle 1, 1\rangle \notin M_2$ of $\mathcal{B}^2$ belongs to D, and so $D \neq M_2$. Thus, (ii) holds. □

Lemma 7.3. *Suppose C is $\sim$-subclassical. Then, the following are equivalent:*

(i) *C^{PC} has a theorem;*
(ii) *$\mathcal{A}$ has a binary semi-conjunction;*
(iii) *$M^{0/1}_{2[+2(+4)]}$ does not form a subalgebra of $(\mathfrak{A}^{([2])}(\restriction L_{2[+2]}))^2$, whenever $L_2 \triangleq 2$ does [not] form a subalgebra of $\mathfrak{A}$, while $\theta^{\mathcal{A}} \in (\notin) \mathrm{Con}(\mathcal{A})$, whereas $\mathcal{A}$ is false-/truth-singular, where, for all $i \in 2$, $M^i_2 \triangleq M_2$, $M^i_4 \triangleq (M_2 \cup \{\langle i, \frac{1}{2}\rangle, \langle \frac{1}{2}, i\rangle\})$ and $M^{\{i\}}_8 \triangleq \{\langle\{\langle j, \frac{1}{2}\rangle, \langle 1-j, l\rangle\}, \{\langle k, \frac{1}{2}\rangle, \langle 1-k, 1-l\rangle\}\rangle \mid j, k, l \in 2\}$.*

Proof. Let $\mathcal{B} \triangleq \mathcal{A}_{\mathrm{PC}}$. Consider the following complementary cases:

- C is $\sim$-classical,
  in which case, by Lemma 3.6, it is defined by $\mathcal{B}$, and so there are some submatrix $\mathcal{D}$ of $\mathcal{A}$ and some $g \in \mathrm{hom_S}(\mathcal{D}, \mathcal{B})$. Then, $\mathcal{D}$ is both consistent and truth-non-empty, for $\mathcal{B}$ is so, and so is not one-valued. Hence, $2 \subseteq D$. Assume $\mathcal{A}$ is false-/truth-singular. Then, both $\mathcal{B}$ and $\mathcal{D}$ are so with the unique non-distinguihed/distinguished value 0/1, in which case $g(0/1) = (0/1)$, and so $(1/0) = \sim^{\mathfrak{B}}(0/1) = \sim^{\mathfrak{B}} g(0/1) = g(\sim^{\mathfrak{D}}(0/1)) = g(\sim^{\mathfrak{A}}(0/1)) = g(1/0)$. Thus, $g(i) = i$, for all $i \in 2$. Consider the following complementary subcases:
  - 2 forms a subalgebra of $\mathfrak{A}$,
    and so of $\mathfrak{D}$, for $2 \subseteq D$, in which case $g\restriction 2$ is a diagonal strict homomorphism from $(\mathcal{D}\restriction 2) = (\mathcal{A}\restriction 2)$ onto $\mathcal{B}$. Hence, $\mathcal{B} = (\mathcal{A}\restriction 2)$. In particular, semi-conjunctions of $\mathcal{B}$ are exactly binary semi-conjunctions for $\mathcal{A}$. Moreover, $M_2 \subseteq 2^2$ forms a subalgebra of $\mathfrak{B}^2$, being a subalgebra of $\mathfrak{A}^2$, iff it forms a subalgebra of $\mathfrak{A}^2$.
  - 2 does not form a subalgebra of $\mathfrak{A}$.
    Then, $\mathcal{D} = \mathcal{A}$, for $2 \subseteq D$, while $(A \setminus 2) = \{\frac{1}{2}\}$ is a singleton. Therefore, as $\mathcal{B}$ is truth-/false-singular, $g(\frac{1}{2}) = (1/0) = g(1/0)$, in which case g is not injective, and so, by Remark 2.7(iii) and Lemma 4.10(iv)⇒(ii), $\theta^{\mathcal{A}} \in \mathrm{Con}(\mathfrak{A})$. Moreover, $f \triangleq ((g \circ (\pi_0 \restriction A^2)) \times (g \circ (\pi_1 \restriction A^2))) \in \mathrm{hom}(\mathfrak{A}^2, \mathfrak{B}^2)$ is surjective. Hence, M_2 forms a subalgebra of $\mathfrak{B}^2$ iff

$M_4^{0/1} = f^{-1}[M_2]$ forms a subalgebra of $\mathfrak{A}^2$. Next, given any binary semi-conjunction φ for $\mathcal{A}$ and any $i \in 2$, we have $\varphi^{\mathfrak{A}}(i, 1-i) = 0$, in which case we get $\varphi^{\mathfrak{B}}(i, 1-i) = \varphi^{\mathfrak{B}}(g(i), g(1-i)) = g(\varphi^{\mathfrak{A}}(i, 1-i)) = g(0) = 0$, and so φ is a semi-conjunction of $\mathcal{B}$. Conversely, consider any semi-conjunction φ of $\mathcal{B}$, in which case, for all $i \in 2$, $g((\sim^{\mathfrak{A}})^{0/1}\varphi^{\mathfrak{A}}(i, 1-i)) = (\sim^{\mathfrak{B}})^{0/1}\varphi^{\mathfrak{B}}(g(i), g(1-i))) = (\sim^{\mathfrak{B}})^{0/1}\varphi^{\mathfrak{B}}(i, 1-i) = (\sim^{\mathfrak{B}})^{0/1}0 = (0/1) \notin / \in D^{\mathcal{B}}$, and so $(\sim^{\mathfrak{A}})^{0/1}\varphi^{\mathfrak{A}}(i, 1-i) \notin / \in D^{\mathcal{A}}$, in which case $(\sim^{\mathfrak{A}})^{0/1}\varphi^{\mathfrak{A}}(i, 1-i) = (0/1)$, and so

$$(\sim^{\mathfrak{A}})^{0/2}\varphi^{\mathfrak{A}}(i, 1-i) = (\sim^{\mathfrak{A}})^{0/1}(\sim^{\mathfrak{A}})^{0/1}\varphi^{\mathfrak{A}}(i, 1-i) = (\sim^{\mathfrak{A}})^{0/1}(0/1) = 0.$$

In this way, $\sim^{0/2}\varphi$ is a binary semi-conjunction for $\mathcal{A}$.

- C is not $\sim$-classical,
  in which case, by Theorem 4.12(v)⇒(i), $\theta^{\mathcal{A}} \notin \mathrm{Con}(\mathcal{A})$. Consider the following complementary subcases:
  - 2 forms a subalgebra of $\mathfrak{A}$,
    in which case $\mathcal{B} = (\mathcal{A} \upharpoonright 2)$, in view of (2.16) and Theorem 6.3, and so binary semi-conjunctions for $\mathcal{A}$ are exactly semi-conjunctions of $\mathcal{B}$.
  - 2 does not form a subalgebra of $\mathfrak{A}$.
    Then, by Theorem 6.3, $\mathcal{A}$ is false-singular, while L_4 forms a subalgebra of $\mathfrak{A}^2$, whereas $\theta^{\mathcal{A}^2 \upharpoonright L_4} \in \mathrm{Con}(\mathfrak{A}^2 \upharpoonright L_4)$, in which case $\mathcal{B} = \langle h[\mathfrak{A}^2 \upharpoonright L_4], \{1\}\rangle$, where $h \triangleq \chi^{\mathfrak{A}^2 \upharpoonright L_4}$ is a strict surjective homomorphism from $\mathcal{D} \triangleq (\mathcal{A}^2 \upharpoonright L_4)$ onto $\mathcal{B}$, and so $g' \triangleq ((h \circ (\pi_0 \upharpoonright D^2)) \times (h \circ (\pi_1 \upharpoonright D^2))) \in \hom(\mathfrak{D}^2, \mathfrak{B}^2)$ is surjective. In particular, M_2 forms a subalgebra of $\mathfrak{B}^2$ iff $M_8 = g'^{-1}[M_2]$ forms a subalgebra of $\mathfrak{D}^2$. Moreover, as $0 \notin D^{\mathcal{A}} \ni \frac{1}{2}$, for $\mathcal{A}$ is false-singular, $a \triangleq \langle 1, \frac{1}{2}\rangle \in D^{\mathcal{D}} \not\ni b \triangleq \langle 0, \frac{1}{2}\rangle \in D$, in which case we have $h(a|b) \in | \notin D^{\mathcal{B}}$, and so $h(a|b) = (1|0)$. Consider any binary semi-conjunction φ for $\mathcal{A}$. Then, $D \ni \varphi^{\mathfrak{D}}(a|b, b|a) = \varphi^{\mathfrak{A}^2}(a|b, b|a)$, in which case, as $(\pi_0 \upharpoonright A^2) \in \hom(\mathfrak{A}^2, \mathfrak{A})$, we have $\pi_0(\varphi^{\mathfrak{D}}(a|b, b|a)) = \varphi^{\mathfrak{A}}(\pi_0(a|b), \pi_0(b|a)) = \varphi^{\mathfrak{A}}(1|0, 0|1) = 0$, and so $\varphi^{\mathfrak{D}}(a|b, b|a) \notin D^{\mathcal{D}}$. Hence, $\varphi^{\mathfrak{B}}(1|0, 0|1) = \varphi^{\mathfrak{B}}(h(a|b), h(b|a)) = h(\varphi^{\mathfrak{D}}(a|b, b|a)) \notin D^{\mathcal{B}}$, in which case $\varphi^{\mathfrak{B}}(1|0, 0|1) = 0$, and so φ is a semi-conjunction of $\mathcal{B}$. Conversely, consider any semi-conjunction φ of $\mathcal{B}$. Then, $h(\varphi^{\mathfrak{D}}(a|b, b|a)) = \varphi^{\mathfrak{B}}(h(a|b), h(b|a)) = \varphi^{\mathfrak{B}}(1|0, 0|1) = 0 \notin D^{\mathcal{B}}$, in which case $\langle \varphi^{\mathfrak{A}}(1|0, 0|1), \varphi^{\mathfrak{A}}(\frac{1}{2}, \frac{1}{2})\rangle = \varphi^{\mathfrak{D}}(a|b, b|a) \notin D^{\mathcal{D}}$. Consider the following complementary subsubcases:
    * $\varphi^{\mathfrak{A}}(\frac{1}{2}, \frac{1}{2}) = \frac{1}{2}$.
      Then, as $\frac{1}{2} \in D^{\mathcal{A}}$, for $\mathcal{A}$ is false-singular, $\varphi^{\mathfrak{A}}(1|0, 0|1) = 0$, and so φ is a binary semi-conjunction for $\mathcal{A}$.
    * $\varphi^{\mathfrak{D}}(\frac{1}{2}, \frac{1}{2}) \neq \frac{1}{2}$.
      Then, as 2^2 is disjoint with $L_4 = D \ni \varphi^{\mathfrak{D}}(a|b, b|a)$, $\varphi^{\mathfrak{A}}(1|0, 0|1) = \frac{1}{2}$, in which case, as $\frac{1}{2} \in D^{\mathcal{A}}$, for $\mathcal{A}$ is false-singular, $\varphi^{\mathfrak{A}}(\frac{1}{2}, \frac{1}{2}) = 0$, and so $\varphi[x_i/\varphi]_{i \in 2}$ is a binary semi-conjunction for $\mathcal{A}$.

In this way, Corollary 7.2 completes the argument. □

Corollary 7.4. *Suppose C is $\sim$-subclassical and and weakly $\veebar$-disjunctive. Then, $\mathcal{A}$ has a binary semi-conjunction.*

Proof. In that case, $C^{\mathrm{PC}} \supseteq C$ is weakly $\veebar$-disjunctive, and so, by Remark 2.8(i)**d)**, satisfies (2.12). In this way, Lemma 7.3(i)⇒(ii) completes the argument. □

By Corollaries 4.6, 7.4, Lemmas 4.7, 4.8 and Theorem{s} 5.1(iii)⇒(i) [including the last assertion] {and 5.6}, we get the following “disjunctive” analogue of Corollary 5.4, being essentially beyond the scopes of the reference [Pyn 95b] of [14], and so becoming a one more substantial advance of the present study with regard to that one:

Corollary 7.5. *Any [three-valued expansion of any] disjunctive (in particular, implicative) {non-}$\sim$-subclassical three-valued Σ-logic {with subclassical negation $\sim$} has no {more than one} proper $\sim$-paraconsistent extension. In particular, any disjunctive (in particular, implicative) $\sim$-paraconsistent three-valued Σ-logic with subclassical negation $\sim$ is premaximally $\sim$-paraconsistent.*

This (more precisely, the {}-non-optional part) is *immediately* applicable to arbitrary (not necessarily $\sim$-subclassical) three-valued expansions of the implicative $\sim$-subclassical P^1 and HZ. On the other hand, as opposed to Corollary 5.4, the condition of being $\sim$-subclassical in the formulation of the {}-non-optional part of Corollary 7.5 is essential, as it follows from:

Example 7.6. Let $\mathcal{A}$ be false-singular, $\Sigma \triangleq \{\sim[,\vee]\}$, $\sim^{\mathfrak{A}}\frac{1}{2} \triangleq \frac{1}{2}$ [and:

$$(a \vee^{\mathfrak{A}} b) \triangleq \begin{cases} a & \text{if } a = b, \\ \frac{1}{2} & \text{otherwise,} \end{cases}$$

for all $a, b \in A$, in which case (2.3), (2.4) and (2.5) are true in $\mathcal{A}$, and so, by Lemma 4.7, C is $\vee$-disjunctive]. Then, L_3 forms a subalgebra of $\mathfrak{A}^2$, in which case, by Theorem 5.1(i)⇒(iv), C is non-maximally $\sim$-paraconsistent [and so is not $\sim$-subclassical, in view of Corollary 7.5]. □

Theorem 7.7. *Suppose $\mathcal{A}$ is [not] false-singular, while C is $\sim$-subclassical. Then, the following are equivalent:*

(i) *C has a theorem;*
(ii) *C^{PC} has a theorem [and $\{\frac{1}{2}\}$ does not form a subalgebra of $\mathfrak{A}$];*
(iii) *$\mathcal{A}$ has a binary semi-conjunction [and $\{\frac{1}{2}\}$ does not form a subalgebra of $\mathfrak{A}$];*
(iv) *[$\{\frac{1}{2}\}$ does not form a subalgebra of $\mathfrak{A}$, and] providing L_2 does (not) form a subalgebra of $\mathfrak{A}$, while $\theta^{\mathcal{A}} \in \{\notin\} \mathrm{Con}(\mathcal{A})$, whereas $\mathcal{A}$ is false-/truth-singular, $M^{0/1}_{2(+2\{+4\})}$ does not form a subalgebra of $(\mathfrak{A}^{\{(2)\}}\{\restriction L_{2(+2)}\})^2$;*
(v) *Any consistent extension of C is a sublogic of C^{PC}.*

Proof. First, the equivalence of (ii–iv) is by Lemma 7.3. Next, (i)⇒(ii) is by the fact that $C(\varnothing) \subseteq C^{\mathrm{PC}}(\varnothing)$ [as well as both (2.16) and Corollary 2.13(ii)⇒(i), for $\frac{1}{2} \notin D^{\mathcal{A}}$]. Conversely, assume (ii,iii) hold. Then, in case C is $\sim$-classical, and so, by Lemma 3.6, $C = C^{\mathrm{PC}}$, (i) is a particular case of (ii). Otherwise, (i) is by (iii) and the following claim:

Claim 7.8. *Let φ be a binary semi-conjunction for $\mathcal{A}$. Suppose either $\mathcal{A}$ is false-singular or both C is $\sim$-subclassical but not $\sim$-classical, and $\{\frac{1}{2}\}$ does not form a subalgebra of $\mathfrak{A}$. Then, C has a theorem.*

Proof. Let $\mathcal{D}$ the submatrix of $\mathcal{A}^3$ generated by the enumeration $a \triangleq (10\frac{1}{2})$ of A. Consider the following complementary cases:

- $\mathcal{A}$ is false-singular.
  Consider the following exhaustive subcases:
  - $\sim^{\mathfrak{A}}\frac{1}{2} = \frac{1}{2}$.
    Then, $D \ni b \triangleq \sim^{\mathfrak{D}} a = (01\frac{1}{2})$. Let $x \triangleq \varphi^{\mathfrak{A}}(\frac{1}{2}, \frac{1}{2}) \in A$. Consider the following exhaustive subsubcases:
    * $x = \frac{1}{2}$.
      Then, $D \ni c \triangleq \varphi^{\mathfrak{D}}(a, b) = (00\frac{1}{2})$. In this way, $D \ni d \triangleq \sim^{\mathfrak{D}} c = (11\frac{1}{2}) \in (D^{\mathcal{A}})^3$.

  * $x = 0$.
    Then, $D \ni c \triangleq \varphi^{\mathfrak{D}}(a,b) = (000)$. In this way, $D \ni d \triangleq \sim^{\mathfrak{D}} c = (111) \in (D^{\mathcal{A}})^3$.
  * $x = 1$.
    Then, $D \ni c \triangleq \varphi^{\mathfrak{D}}(a,b) = (001)$, in which case $D \ni \sim^{\mathfrak{D}} c = (110)$, and so $D \ni d \triangleq \sim^{\mathfrak{D}} \varphi^{\mathfrak{D}}(c, \sim^{\mathfrak{D}} c) = (111) \in (D^{\mathcal{A}})^3$.
- $\sim^{\mathfrak{A}} \frac{1}{2} = 1$.
  Then, $D \ni b \triangleq \sim^{\mathfrak{D}} a = (011)$, in which case $D \ni \sim^{\mathfrak{D}} b = (100)$, and so $D \ni d \triangleq \sim^{\mathfrak{D}} \varphi^{\mathfrak{D}}(b, \sim^{\mathfrak{D}} b) = (111) \in (D^{\mathcal{A}})^3$.
- $\sim^{\mathfrak{A}} \frac{1}{2} = 0$.
  Then, $D \ni b \triangleq \sim^{\mathfrak{D}} a = (010)$, in which case $D \ni \sim^{\mathfrak{D}} b = (101)$, and so $D \ni d \triangleq \sim^{\mathfrak{D}} \varphi^{\mathfrak{D}}(b, \sim^{\mathfrak{D}} b) = (111) \in (D^{\mathcal{A}})^3$.

- $\mathcal{A}$ is not false-singular,
  in which case $\{\frac{1}{2}\}$ does not form a subalgebra of $\mathfrak{A}$, while, by Theorem 6.3, 2 forms a subalgebra of $\mathfrak{A}$, and so there is some $\psi \in \mathrm{Fm}^1_\Sigma$ such that $\psi^{\mathfrak{A}}[A] \subseteq 2$. Then, $D \ni b \triangleq \psi^{\mathfrak{D}}(a) \in 2^3$, in which case $D \ni c \triangleq \varphi^{\mathfrak{D}}(b, \sim^{\mathfrak{D}} b) = (3 \times \{0\})$, and so $D \ni d \triangleq \sim^{\mathfrak{D}} c = (3 \times \{1\}) \in (D^{\mathcal{A}})^3$.

Thus, anyway, $d \in ((D^{\mathcal{A}})^3 \cap D) = D^{\mathcal{D}}$, in which case $\mathcal{D}$ is truth-non-empty, and so Lemma 7.1(iv)⇒(i) completes the argument. □

Finally, if C has no theorem, then the purely inferential (and so consistent) IC_{+0} is an extension of C, for $C \subseteq \mathrm{IC}$, in which case $C = C_{+0} \subseteq \mathrm{IC}_{+0}$. And what is more, IC_{+0}, being inferentially inconsistent, for IC, being an inconsistent $(\infty \setminus 1)$-sublogic of IC_{+0}, is inferentially inconsistent, is not $\sim$-subclassical. Thus, (v)⇒(i) holds. Conversely, assume (i,iii) hold. Consider any consistent extension C' of C. In case $C' = C$, we have $C' = C \subseteq C^{\mathrm{PC}}$. Likewise, in case C is $\sim$-classical, by Lemma 3.6, we have $C = C^{\mathrm{PC}}$, and so, by (i) and Corollary 3.7, we get $C' = C^{\mathrm{PC}} \subseteq C^{\mathrm{PC}}$. Now, assume $C \neq C'$ is not $\sim$-classical. If C' was $\sim$-paraconsistent, then so would be its sublogic C, in which case $\mathcal{A}$, being $\sim$-paraconsistent, would be false-singular, and so, by (iii) and Theorem 5.1(iii)⇒(i), C' would be equal to C. Therefore, C' is not $\sim$-paraconsistent. Then, $x_0 \notin T \triangleq C'(\varnothing)$. Moreover, by the structurality of C', $\langle \mathfrak{Fm}^{\omega}_\Sigma, T \rangle$ is a model of C' (in particular, of C), and so is its consistent finitely-generated submatrix $\mathcal{B} \triangleq \langle \mathfrak{Fm}^1_\Sigma, T \cap \mathrm{Fm}^1_\Sigma \rangle$, in view of (2.16). Then, by Lemma 2.10, there are some finite set I, some $\overline{\mathcal{C}} \in \mathbf{S}_*(\mathcal{A})^I$ and some subdirect product $\mathcal{D} \in \mathbf{H}^{-1}(\mathbf{H}(\mathcal{B}))$ of it, in which case, by (2.16), $\mathcal{D}$ is a consistent model of C', for $\mathcal{B}$ is so, and so $\mathcal{D}$ is non-$\sim$-paraconsistent, for C' is so, while $\mathcal{A}$ is not a model of the logic of $\mathcal{D}$, for $C \subsetneq C'$. And what is more, by (i) and Corollary 2.13(iv)⇒(i), $\mathcal{D}$ is truth-non-empty. Hence, by (2.16), (iii), Lemma 6.1 and Theorem 6.3, a Σ-matrix defining C^{PC} is embeddable into $\mathcal{D}$, in which case $C' \subseteq C^{\mathrm{PC}}$, and so (v) holds. □

Corollary 7.2(i)⇔(ii) [resp., Theorem 7.7(i)⇔(iv)] provides an effective algebraic criterion of a [three-valued] $\sim$-[sub]classical Σ-logic's having a theorem. In this connection, in view of Corollary 7.4, the instance of the disjunctive K_3/LP without/with theorems and the same underlying algebra of their characteristic matrices, shows that the []-optional reservations in the formulation of Theorem 7.7 are indeed necessary/irrelevant in the "truth-/false-singular" case. This equally concerns the following immediate consequence of Remark 5.3, Corollary 7.4 and Theorem 7.7(i)⇔(iii):

Corollary 7.9. *Suppose C is both $\sim$-subclassical and weakly either conjunctive or disjunctive, while $\mathcal{A}$ is [not] false-singular. Then, C has a theorem [iff $\{\frac{1}{2}\}$ does not form a subalgebra of $\mathfrak{A}$].*

The following simple example shows that the stipulation of the weak conjunctivity/disjunctivity cannot be omitted in Corollary 7.9 and "Remark 5.3"/"Corollary 7.4", respectively:

Example 7.10. Let $\Sigma \triangleq \{\sim\}$ and $\mathcal{A}$ false-|truth-singular with $\sim^{\mathfrak{A}}\frac{1}{2} = (1|0)$, in which case $[A\backslash]2$ does [not] form a subalgebra of $\mathfrak{A}$, and so, by Theorem 6.3, C is $\sim$-subclassical, while $\langle\sim^{\mathfrak{A}}\frac{1}{2}, \sim^{\mathfrak{A}}(1|0)\rangle = \langle 1|0, 0|1\rangle \notin \theta^{\mathcal{A}} \ni \langle\frac{1}{2}, 1|0\rangle$, in which case $\theta^{\mathcal{A}} \notin \mathrm{Con}(\mathfrak{A})$, whereas M_2 forms a subalgebra of $(\mathfrak{A}\upharpoonright 2)^2$, in which case, by Lemma 7.3(ii)⇒(iii), $\mathcal{A}$ has no binary semi-conjunction, and so, by Theorem 7.7(i)⇒(iii), C has no theorem. In particular, by Corollary 7.9, C is weakly neither conjunctive nor disjunctive. And what is more, if $h \triangleq h_{+/2}$ would be a homomorphism from $(\mathfrak{A}\upharpoonright 2)^2$ to $\mathfrak{A}$, then we would have $(1|0) = \sim^{\mathfrak{A}}\frac{1}{2} = \sim^{\mathfrak{A}}h(\langle 1,0\rangle) = h(\sim^{\mathfrak{A}^2}\langle 1,0\rangle) = h(\langle 0,1\rangle) = \frac{1}{2}$. Therefore, $h \notin \hom((\mathfrak{A}\upharpoonright 2)^2, \mathfrak{A})$. Hence, by Theorem 4.12(i)⇒(v), C is not $\sim$-classical. □

Theorem 7.11. *[Suppose $\mathcal{A}$ is both $\sim$-paraconsistent and weakly conjunctive.] Then, C^{NP} is consistent if[f] C is $\sim$-subclassical.*

Proof. The "if" part is by Remark 2.8(i)**d)** and the consistency of any $\sim$-classical Σ-matrix/-logic. [Conversely, assume C^{NP} is consistent. Then, by Remark 5.3 and Claim 7.8, C has a theorem, in which case, by its structurality, applying the Σ-substitution extending $[x_i/x_0]_{i\in\omega}$ to any theorem of C, we get some $\varphi \in (C(\varnothing) \cap \mathrm{Fm}^1_\Sigma) \subseteq T \triangleq C^{\mathrm{NP}}(\varnothing) \not\ni x_0$. Moreover, by the structurality of C^{NP}, $\langle\mathfrak{Fm}^\omega_\Sigma, T\rangle$ is a model of $C^{[\mathrm{NP}]}$, and so is its consistent truth-non-empty finitely-generated submatrix $\mathcal{B} \triangleq \langle\mathfrak{Fm}^1_\Sigma, T \cap \mathrm{Fm}^1_\Sigma\rangle$, in view of (2.16). Hence, by Lemma 2.10, there are some finite set I, some $\overline{\mathcal{C}} \in \mathbf{S}_*(\mathcal{A})^I$ and some subdirect product $\mathcal{D} \in \mathbf{H}^{-1}(\mathbf{H}(\mathcal{B}))$ of it, in which case, this is both consistent, truth-non-empty and, by (2.16), non-$\sim$-paraconsistent, for $\mathcal{B}$ is so, and so $\mathcal{A}$, being $\sim$-paraconsistent, is not a model of the logic of it. In this way, Lemma 6.1(ii) and Theorem 6.3 complete the argument.] □

The logic IC_{+0} invoked in the proof of Theorem 7.7(v)⇒(i) (held in general) is, though being consistent, is inferentially inconsistent. A proper "inferential" version of this result is then as follows:

Theorem 7.12. *Suppose $\mathcal{A}$ is [not] truth-singular, while C is $\sim$-subclassical. Then, any inferentially consistent extension of C is a sublogic of C^{PC} [iff $\mathcal{A}$ has GC and C has no proper $\sim$-paraconsistent extension iff $\mathcal{A}$ satisfies GC and L_3 does not form a subalgebra of $\mathfrak{A}^2$].*

Proof. [First, the second "iff" part is by Theorem 5.1(i)⇔(iv). Likewise, by Theorem 5.1(i)⇒(ii), C has a $\sim$-paraconsistent (and so inferentially consistent) non-$\sim$-subclassical extension, whenever it has a proper $\sim$-paraconsistent one. Now, assume $\mathcal{A}$ does not satisfy GC. Let $\mathcal{B}$ be the submatrix of $\mathcal{A}^2$ generated by $\varnothing \neq \{\langle 1, \frac{1}{2}\rangle\} \subseteq D^{\mathcal{B}}$, for $\mathcal{A}$ is false-singular. Then, by (2.16) and the following claim, the logic of $\mathcal{B}$ is an inferentially consistent (for $\mathcal{B}$ is both consistent and truth-non-empty) extension of C, not being a sublogic of C^{PC}:

Claim 7.13. *Let $\mathcal{B}$ be the submatrix of $\mathcal{A}^2$ generated by $\{\langle 1, \frac{1}{2}\rangle\}$ and C' the logic of $\mathcal{B}$. Suppose $\mathcal{A}$ is false-singular and does not satisfy GC. Then, $(B \setminus D^{\mathcal{B}}) = M_2 \neq \varnothing$, in which case $\sim x_0 \vdash x_0$ is true in $\mathcal{B}$, and so, by (3.2) with $n = 1$ and $m = 0$, $\sim$ is not a subclassical negation for C' (in particular, $C' \neq C$ is not $\sim$-subclassical; cf. Corollary 4.6).*

Proof. Then, $(B \cap \{\langle 0,0\rangle, \langle 0, \frac{1}{2}\rangle, \langle\frac{1}{2}, 0\rangle\}) = \varnothing$, in which case $\sim^{\mathfrak{A}}\frac{1}{2} = 1$, and so $(B \setminus D^{\mathcal{B}}) = M_2 \neq \varnothing$. On the other hand, for every $a \in M_2$, $\sim^{\mathfrak{B}}a \in M_2$, so the rule $\sim x_0 \vdash x_0$ is true in $\mathcal{B}$, as required. □

Thus, the first "only if" part holds. Conversely, assume $\mathcal{A}$ has GC, while C has no proper $\sim$-paraconsistent extension.] Consider any inferentially consistent extension C' of C. In case $C' = C$, we have $C' = C \subseteq C^{\mathrm{PC}}$. Likewise, in case C is $\sim$-classical, by Lemma 3.6, we have $C = C^{\mathrm{PC}}$, and so, by Corollary 3.7, we get $C' = C^{\mathrm{PC}} \subseteq C^{\mathrm{PC}}$. Now, assume $C \neq C'$ is not $\sim$-classical. If C' was $\sim$-paraconsistent, then so would be its sublogic C, in which case $\mathcal{A}$, being $\sim$-paraconsistent, would be false-singular, and so, by the []-optional assumption, C' would be equal to C. Therefore, C' is not $\sim$-paraconsistent. Then, $x_1 \notin T \triangleq C'(x_0) \ni x_0$. Moreover, by the structurality of C', $\langle \mathfrak{Fm}^{\omega}_{\Sigma}, T\rangle$ is a model of C' (in particular, of C), and so is its consistent truth-non-empty finitely-generated submatrix $\mathcal{B} \triangleq \langle \mathfrak{Fm}^{2}_{\Sigma}, T \cap \mathrm{Fm}^{2}_{\Sigma}\rangle$, in view of (2.16). Then, by Lemma 2.10, there are some finite set I, some $\overline{\mathcal{C}} \in \mathbf{S}_*(\mathcal{A})^I$ and some subdirect product $\mathcal{D} \in \mathbf{H}^{-1}(\mathbf{H}(\mathcal{B}))$ of it, in which case, by (2.16), $\mathcal{D}$ is a consistent truth-non-empty model of C', for $\mathcal{B}$ is so, and so $\mathcal{D}$ is non-$\sim$-paraconsistent, for C' is so, while $\mathcal{A}$ is not a model of the logic of $\mathcal{D}$, for $C \subsetneq C'$. Hence, by (2.16), Lemma 6.1 and Theorem 6.3, a Σ-matrix defining C^{PC} is embeddable into $\mathcal{D}$, in which case $C' \subseteq C^{\mathrm{PC}}$, as required. □

Theorem 7.14. *Suppose $\mathcal{A}$ is either non-$\sim$-paraconsistent (in particular, truth-singular) or weakly conjunctive (viz., C is so). Then, C has a proper inferentially consistent extension iff it is $\sim$-subclassical but not $\sim$-classical, in which case C^{PC} is an extension of any inferentially consistent extension of C.*

Proof. The "if" part is by the inferential consistency of $\sim$-classical Σ-logics. Conversely, consider any proper inferentially consistent extension C' of C, in which case, by Corollary 3.7, C is not $\sim$-classical. Moreover, if C' was $\sim$-paraconsistent, then so would be its sublogic C, in which case this would be weakly conjunctive, and so, by Corollaries 4.6 and 5.4, C' would be equal to C. Therefore, C' is not $\sim$-paraconsistent. Then, $x_1 \notin T \triangleq C'(x_0) \ni x_0$. Moreover, by the structurality of C', $\langle \mathfrak{Fm}^{\omega}_{\Sigma}, T\rangle$ is a model of C' (in particular, of C), and so is its consistent truth-non-empty finitely-generated submatrix $\mathcal{B} \triangleq \langle \mathfrak{Fm}^{2}_{\Sigma}, T \cap \mathrm{Fm}^{2}_{\Sigma}\rangle$, in view of (2.16). Then, by Lemma 2.10, there are some finite set I, some $\overline{\mathcal{C}} \in \mathbf{S}_*(\mathcal{A})^I$ and some subdirect product $\mathcal{D} \in \mathbf{H}^{-1}(\mathbf{H}(\mathcal{B}))$ of it, in which case, by (2.16), $\mathcal{D}$ is a consistent truth-non-empty model of C', for $\mathcal{B}$ is so, and so $\mathcal{D}$ is non-$\sim$-paraconsistent, for C' is so, while $\mathcal{A}$ is not a model of the logic of $\mathcal{D}$, for $C \subsetneq C'$. Hence, by Lemma 6.1, 2 forms a subalgebra of $\mathfrak{A}$, while $\mathcal{A}{\restriction}2$ is embeddable into $\mathcal{D}$, whereas, by Theorem 6.3, C is $\sim$-subclassical, in which case C^{PC} is defined by $\mathcal{A}{\restriction}2$, and so, by (2.16), $C' \subseteq C^{\mathrm{PC}}$, as required. □

The initial stipulation in the formulation of Theorem 7.14 cannot be omitted, as it ensues from:

Example 7.15. Let $\mathcal{A}$ be false-singular, $\Sigma \triangleq \{\sim, \top\}$ with nullary $\top$ and $\top^{\mathfrak{A}} \triangleq \sim^{\mathfrak{A}}\frac{1}{2} \triangleq \frac{1}{2}$, in which case $2 \not\ni \frac{1}{2} = \top^{\mathfrak{A}}$ does not form a subalgebra of $\mathfrak{A}$, while $\langle \sim^{\mathfrak{A}}1, \sim^{\mathfrak{A}}\frac{1}{2}\rangle = \langle 0, \frac{1}{2}\rangle \notin \theta^{\mathcal{A}} \ni \langle 1, \frac{1}{2}\rangle$, in which case $\theta^{\mathcal{A}} \notin \mathrm{Con}(\mathfrak{A})$, whereas $L_4 \not\ni \langle \frac{1}{2}, \frac{1}{2}\rangle = \top^{\mathfrak{A}^2}$ does not form a subalgebra of $\mathfrak{A}^2$, and so, by Theorem[s] 4.12(i)⇒(v) [and 6.3], C is not $\sim$-[sub]classical. On the other hand, $L_3 \ni \langle \frac{1}{2}, \frac{1}{2}\rangle = \top^{\mathfrak{A}^2}$, being closed under $\sim^{\mathfrak{A}^2}$, forms a subalgebra of $\mathfrak{A}^2$, in which case, by Theorem 5.1(i)⇒(iv), C has a proper $\sim$-paraconsistent (and so inferentially consistent) extension, and so, by Corollary 5.4, C is not weakly conjunctive. □

8. Structural completeness, completions and extensions

8.1. **Paraconsistent logics.**

Theorem 8.1. *Suppose $\mathcal{A}$ is false-singular [while, providing C is $\sim$-subclassical, it is either $\sim$-paraconsistent or disjunctive]. Then, C is structurally complete if[f] the following hold:*

(i) *C has a theorem;*
(ii) *C has no proper $\sim$-paraconsistent extension;*
(iii) *$\mathcal{A}$ satisfies GC;*
(iv) *$\mathcal{A}$ satisfies DGC;*
(v) *L_4 does not form a subalgebra of $\mathfrak{A}^2$;*
(vi) *C is not $\sim$-subclassical, unless it is $\sim$-classical,*

in which case any three-valued expansion of C is structurally complete, unless C is $\sim$-classical. In particular, providing C is $\sim$-paraconsistent, it is structurally complete iff $\mathcal{A}$ satisfies both GC and DGC, while C is both maximally $\sim$-paraconsistent and neither $\sim$-subclassical nor purely-inferential, whereas L_4 does not form a subalgebra of $\mathfrak{A}^2$.

Proof. First, assume (i–vi) hold. Then, in case C is $\sim$-classical, by (i) and Corollary 3.7, it is structurally complete. Now, assume C is not $\sim$-classical, and so is not $\sim$-subclassical, in view of (vi). Let C' be any extension of C such that $T \triangleq C'(\varnothing) = C(\varnothing) \not\ni x_0$, in view of the consistency of $\mathcal{A}$, and so of C. Then, by (2.16), (i) and the structurality of C', $\mathcal{B} \triangleq \langle \mathfrak{Fm}^1_\Sigma, T \cap \mathrm{Fm}^1_\Sigma \rangle$ is a finitely-generated consistent truth-non-empty model of C' (in particular, of C), in which case, by Lemma 2.10, there are some finite set I, some $\overline{\mathcal{C}} \in \mathbf{S}_*(\mathcal{A})^I$ and some subdirect product $\mathcal{D} \in \mathbf{H}^{-1}(\mathbf{H}(\mathcal{B}))$ of it, in which case, by (2.16), $\mathcal{D}$ is a consistent truth-non-empty model of C', for $\mathcal{B}$ is so. Consider the following complementary cases:

- $\mathcal{D}$ is $\sim$-paraconsistent.
  Then, by (2.16), (ii), Lemma 4.15 and Theorem 5.1(i)$\Rightarrow$(iii), $\mathcal{A}$ is a model of C', for $\mathcal{D}$ is so.
- $\mathcal{D}$ is not $\sim$-paraconsistent.
  Then, as C is not $\sim$-subclassical, by (iii-v), Lemma 6.1(ii) and Theorem 6.3, $\mathcal{A}$ is a model of the logic of $\mathcal{D}$, and so of C', for $\mathcal{D}$ is a model of C'.

Thus, anyway, $\mathcal{A} \in \mathrm{Mod}(C')$, in which case C', being an extension of C, is equal to C, and so C is structurally complete. [Conversely, assume either of (i-vi) does not hold. Consider respective cases:

(i) does not hold.
Then, by Remark 2.4, C, being inferentially consistent, for $\mathcal{A}$ is both consistent and truth-non-empty, is not structurally complete.

(ii) does not hold.
Then, by Theorem 5.6(i), $C_{\frac{1}{2}}$ is a proper extension of C, and $\Delta_A \in \hom^{\mathrm{S}}(\mathcal{A}_{\frac{1}{2}}, \mathcal{A})$, in which case, by (2.17), $C_{\frac{1}{2}}(\varnothing) = C(\varnothing)$, and so C is not structurally complete.

(iii) does not hold.
Let $\mathcal{B}'$ be the submatrix of $\mathcal{A}^2$ generated by $\{\langle 1, \frac{1}{2}\rangle\}$. Then, by (2.16) and Claim 7.13, the logic C' of $\mathcal{B}'$ is a proper extension of C, while $(\pi_1 \upharpoonright B') \in \hom^{\mathrm{S}}(\mathcal{B}', \mathcal{A})$, for $\pi_1[M_2] = 2$, while $M_2 \subseteq B'$, in which case, by (2.17), $C'(\varnothing) = C(\varnothing)$, and so C is not structurally complete.

(iv) does not hold.
Let $\mathcal{B}'$ be the submatrix of $\mathcal{A}^2$ generated by $M_2 \cup \{\langle 1, \frac{1}{2}\rangle\}$ and C' the logic of $\mathcal{B}'$. Then, as $\langle 0, 0\rangle \notin B'$, while $\sim^{\mathfrak{A}} 1 = 0$, $\sim^{\mathfrak{A}} \frac{1}{2} \neq 0$, in which case $\mathcal{A}$ is $\sim$-paraconsistent, and so is C. Moreover, as $\langle \frac{1}{2}, \frac{1}{2}\rangle \notin B'$, $\mathcal{B}'$ is non-$\sim$-paraconsistent, and so is C', in which case, by (2.16), C' is a proper extension of C. And what is more, $(\pi_1 \upharpoonright B') \in \hom^{\mathrm{S}}(\mathcal{B}', \mathcal{A})$, for $\pi_1[M_2] = 2$, in which case, by (2.17), $C'(\varnothing) = C(\varnothing)$, and so C is not structurally complete.

(v) does not hold.
Let $\mathcal{B}' \triangleq (\mathcal{A}^2 \restriction L_4)$ and C' the logic of $\mathcal{B}'$. Then, as $\langle 0,0\rangle \notin L_4 \ni \langle \frac{1}{2},1\rangle$, while $\sim^{\mathfrak{A}}1 = 0$, $\sim^{\mathfrak{A}}\frac{1}{2} \neq 0$, in which case $\mathcal{A}$ is $\sim$-paraconsistent, and so is C. Moreover, as $\langle \frac{1}{2},\frac{1}{2}\rangle \notin L_4$, $\mathcal{B}'$ is non-$\sim$-paraconsistent, and so is C', in which case, by (2.16), C' is a proper extension of C. And what is more, $(\pi_0 \restriction L_4) \in \mathrm{hom}^{\mathrm{S}}(\mathcal{B}',\mathcal{A})$, for $\pi_0[L_4] = A$, in which case, by (2.17), $C'(\varnothing) = C(\varnothing)$, and so C is not structurally complete.

(vi) does not hold,
in which case C is $\sim$-subclassical but not $\sim$-classical. Let $\mathcal{B}' \triangleq \mathcal{A}_{\mathrm{PC}} \in \mathrm{Mod}(C)$. Then, $\mathcal{D} \triangleq (\mathcal{A} \times \mathcal{B}')$ is a model of C, in which case the logic C' of $\mathcal{D}$ is an extension of C, and so, as $(\pi_0 \restriction D) \in \mathrm{hom}^{\mathrm{S}}(\mathcal{D},\mathcal{A})$, by (2.17), we have $C'(\varnothing) = C(\varnothing)$. For proving the fact that $C' \neq C$, consider the following complementary cases:

- $\mathcal{A}$ is $\sim$-paraconsistent,
  and so is C. Then, by Remark 2.8(i)**d)**,(iii), C' is not $\sim$-paraconsistent, and so $C' \neq C$.
- $\mathcal{A}$ is not $\sim$-paraconsistent,
  in which case it is $\sim$-negative. Then, C, being both $\sim$-subclassical and non-$\sim$-paraconsistent, is $\veebar$-disjunctive, and so is $\mathcal{A}$, in view of Lemma 4.7, in which case, by Remark 2.8(i)**c)**, it is $\sqsupset$-implicative, while $\mathcal{D}$ is weakly $\veebar$-disjunctive, whereas, by Corollary 6.4, 2 forms a subalgebra of $\mathfrak{A}$, in which case $\mathcal{B}' = (\mathcal{A}\restriction 2)$. Moreover, by Corollary 4.13, $\mathcal{A}$ is hereditarily simple, in which case, by Theorem 3.4(i)$\Leftrightarrow$(iii), it has a unary equality determinant ε, and so $\{\phi \sqsupset \psi \mid (\phi \vdash \psi) \in \varepsilon\}$ is an axiomatic equality determinant for it, and so for $\mathcal{D}$, in view of Lemmas 3.3, 3.5, in which case it is hereditarily simple too. We prove that $C' \neq C$ by contradiction. For suppose $C' = C$, in which case $\mathcal{A}$ is a finite model of C', and so, by Corollary 2.12 and Remark 2.7(iii), there is some $h \in \mathrm{hom}^{\mathrm{S}}_{\mathrm{S}}(\mathcal{A},\mathcal{D})$. Then, $g \triangleq ((\pi_1 \restriction D) \circ h) \in \mathrm{hom}^{\mathrm{S}}(\mathcal{A},\mathcal{B}')$, in which case, as $D^{\mathcal{A}} = \{1,\frac{1}{2}\}$, we have $g(1) = 1 = g(\frac{1}{2})$, and so g is not injective, while $0 = \sim^{\mathfrak{B}}1 = \sim^{\mathfrak{B}}g(1) = g(\sim^{\mathfrak{A}}1) = g(0)$. Hence, g is strict. This contradicts to Remark 2.7(iii). Thus, $C' \neq C$.

Thus, anyway, $C' \neq C$, in which case C is not structurally complete.

Thus, in any case, C is not structurally complete.]

Finally, as expansions of $\mathcal{A}/C$ inherit (iii-v)/"both (i) and absence of $\sim$-classical models", respectively, Remark 2.8(i)**d)**, Corollary 4.18 and the last assertion of Theorem 5.1 complete the argument. □

Remark 2.4 and Theorem 8.1 inevitably raise the problem of finding the structural completion of C, whenever it is both $\sim$-paraconsistent and $\sim$-subclassical but not purely-inferential. Among other things, it is this case that covers all *already-known* instances of $\sim$-paraconsistent three-valued Σ-logics with subclassical negation $\sim$.

Lemma 8.2. *Let $i \in 2$, $\mathcal{K}'_{3,i}$ the submatrix of $\mathcal{A}^2$ generated by $K_{3,i} \triangleq (\Delta_2 \cup \{\langle \frac{1}{2}, i\rangle\})$. Suppose 2 forms a subalgebra of $\mathfrak{A}$, in which case C is $\sim$-subclassical, C^{PC} being defined by $\mathcal{A}\restriction 2$; cf. Theorem 6.3. Then, the following are equivalent:*

(i) $\langle 0,1\rangle \in K'_{3,i}$;
(ii) $\langle 1,0\rangle \in K'_{3,i}$;
(iii) $M_2 \subseteq K'_{3,i}$;
(iv) $(M_2 \cap K'_{3,i}) \neq \varnothing$;
(v) $K'_{3,i} \nsubseteq K_4 \triangleq (\bigcup_{j\in 2} K_{3,j})$;

(vi) *neither* $K_{3,i}$ *nor* K_4 *forms a subalgebra of* $\mathfrak{A}^2$.

Moreover, providing $\mathcal{A}$ *is ([both* $\bar{\wedge}$*-conjunctive and]* $\veebar$*-disjunctive as well as) false-singular {more specifically,* $\sim$*-paraconsistent},* **a)**$\Leftrightarrow$**b)**$\Rightarrow$ $(\Leftrightarrow)$**c)**$\Rightarrow$ $\{\Leftrightarrow\}$**d)**$\Leftarrow$ $([\Leftrightarrow$ $])$**e)**$\langle\Rightarrow$**b)**$\rangle$, *where:*

a) C^{PC} *is a proper axiomatic extension of* C*;*
b) $C^{\mathrm{PC}}(\varnothing) \neq C(\varnothing)$*;*
c) $\langle 0,1\rangle \in \bigcap_{j\in 2} K'_{3,j}$ *$\langle$while* C *is not* $\sim$*-classical$\rangle$;*
d) $\langle 0,1\rangle \in K'_{3,0}$ *$\langle$while* C *is not* $\sim$*-classical$\rangle$;*
e) $\mathcal{A}$ *is implicative $\langle$while* C *is not* $\sim$*-classical$\rangle$.*

In particular, the non-$\langle\rangle$-optional versions of **a)**–**e)** *are equivalent, whenever* C *is both conjunctive and disjunctive as well as* $\sim$*-subclassical, while* $\mathcal{A}$ *is false-singular (in particular,* $\sim$*-paraconsistent), whereas* C *is not* $\sim$*-classical (in particular,* $\sim$*-paraconsistent).*

Proof. First, (i)$\Leftrightarrow$(ii) is by the fact that $\sim^{\mathfrak{A}} j = (1-j)$, for all $j \in 2$, while (iii/iv) is the conjunction/disjunction of (i) and (ii), respectively. Next, (iii)$\Rightarrow$(v) is by the fact that $M_2 \nsubseteq K_4$. Further, (v)$\Rightarrow$(vi) is by the fact that $K_{3,i} \subseteq K_4$. The converse is by the fact that $K_4 = (K_{3,i} \cup \{\langle \frac{1}{2}, 1-i\rangle\})$, while $K_{3,i} \subseteq K'_{3,i}$. Furthermore, (v)$\Rightarrow$(iv) is by the fact that $K_4 = ((A \times 2) \setminus M_2)$, while $K'_{3,i} \subseteq (A \times 2)$, for $\pi_1[K_{3,i}] = 2$ forms a subalgebra of $\mathfrak{A}$.

Now, suppose $\mathcal{A}$ is ([both $\bar{\wedge}$-conjunctive and] $\veebar$-disjunctive, in which case $\mathcal{A}{\restriction}2$ is so, in view of Remark 2.8(ii), as well as) false-singular {more specifically, $\sim$-paraconsistent}.

First, **b)** is a particular case of **a)**. Conversely, assume **b)** holds. Then, $C^{\mathrm{PC}}(\varnothing) \nsubseteq C(\varnothing)$, for $C \subseteq C^{\mathrm{PC}}$, in which case there is some $\varphi \in (C^{\mathrm{PC}}(\varnothing) \setminus C(\varnothing)) \neq \varnothing$, and so φ is true in $\mathcal{A}{\restriction}2$ but is not true in $\mathcal{A}$. On the other hand, $\mathcal{A}{\restriction}2$ is the only proper consistent submatrix of $\mathcal{A}$. Hence, by Corollary 2.15, C^{PC} is the axiomatic extension of C relatively axiomatized by φ. Thus, **a)** holds.

Next, **d)** is a particular case of **c)**. {Conversely, assume $\langle 0,1\rangle \in K'_{3,0}$. Consider the following complementary cases:

- $\sim^{\mathfrak{A}}\frac{1}{2} = \frac{1}{2}$.
  Then, $\langle\frac{1}{2},0\rangle = \sim^{\mathfrak{A}^2}\langle\frac{1}{2},1\rangle \in K'_{3,1}$, for $K'_{3,1} \supseteq K_{3,1} \ni \langle\frac{1}{2},1\rangle$ forms a subalgebra of $\mathfrak{A}^2$, in which case $K_{3,0} = (\Delta_2 \cup \{\langle\frac{1}{2},0\rangle\}) \subseteq K'_{3,1}$, for $\Delta_2 \subseteq K_{3,1} \subseteq K'_{3,1}$, and so, $K'_{3,1}$, forming a subalgebra of $\mathfrak{A}^2$, includes $K'_{3,0} \ni \langle 0,1\rangle$.
- $\sim^{\mathfrak{A}}\frac{1}{2} \neq \frac{1}{2}$, in which case $\sim^{\mathfrak{A}}\frac{1}{2} = 1$,
  for $\mathcal{A}$ is $\sim$-paraconsistent, and so $\langle 0,1\rangle = \sim^{\mathfrak{A}^2}\sim^{\mathfrak{A}^2}\langle\frac{1}{2},1\rangle \in K'_{3,1}$, for $K'_{3,1} \supseteq K_{3,1} \ni \langle\frac{1}{2},1\rangle$ forms a subalgebra of $\mathfrak{A}^2$.

Thus, in any case, $\langle 0,1\rangle \in \bigcap_{j\in 2} K'_{3,j}$, and so **d)**$\Rightarrow$**c)** holds.}

(Further, assume $\langle 0,1\rangle \in \bigcap_{j\in 2} K'_{3,j}$. Then, there is some $\bar{\phi} \in (\mathrm{Fm}^3_\Sigma)^2$ such that, for each $j \in 2$, $\phi^{\mathfrak{A}}_j(0,\frac{1}{2},1) = 0$ and $\phi^{\mathfrak{A}}_j(0,j,1) = 1$. Moreover, by Remark 2.8(i)**d)**, $\varphi \triangleq (2.12) \in (C^{\mathrm{PC}}(\varnothing) \cap \mathrm{Fm}^1_\Sigma)$. Set $\psi \triangleq (\veebar\bar{\phi}[x_0/\sim\varphi, x_2/\varphi]) \in \mathrm{Fm}^2_\Sigma$. Then, since both $\mathcal{A}$ and $\mathcal{A}{\restriction}2$ are $\veebar$-disjunctive as well as false-singular, while the latter is also both $\sim$-negative and truth-singular, we have, for all $k \in 2$, $\psi^{\mathfrak{A}}(k,\frac{1}{2}) = 0$ as well as $\psi^{\mathfrak{A}}(k,l) = 1$, for all $l \in 2$, in which case ψ is not true in $\mathcal{A}$ under $[x_0/k, x_1/\frac{1}{2}]$ but is true in $\mathcal{A}{\restriction}2$, and so $\psi \in (C^{\mathrm{PC}}(\varnothing) \setminus C(\varnothing))$. Thus, **c)**$\Rightarrow$**b)** holds.) Conversely, if $\langle 0,1\rangle \notin K'_{3,j}$, for some $j \in 2$, then $(\pi_1{\restriction}K'_{3,j}) \in \hom^{\mathrm{S}}_{\mathrm{S}}(\mathcal{K}'_{3,j}, \mathcal{A}{\restriction}2)$, because $\pi_1[K_{3,j}] = 2$ forms a subalgebra of $\mathfrak{A}$, in which case, by (2.16), C^{PC} is defined by $\mathcal{K}'_{3,j}$, and so, by (2.17), $C^{\mathrm{PC}}(\varnothing) = C(\varnothing)$, for $(\pi_0{\restriction}K'_{3,j}) \in \hom^{\mathrm{S}}_{\mathrm{S}}(\mathcal{K}'_{3,j}, \mathcal{A})$, because $\pi_0[K_{3,j}] = A$. $\langle$Likewise, if C is $\sim$-classical, then, by Lemma 3.6, $C^{\mathrm{PC}} = C$, for $C \subseteq C^{\mathrm{PC}}$.$\rangle$ Thus, **b)**$\Rightarrow$**c)** holds.

⟨Furthermore, if **e)** holds, then **b)** is by Remark 2.8(ii) and the following claim:⟩

Claim 8.3. *Let C' be a finitary Σ-logic and C'' a 1-extension of C'. Suppose C' has DT with respect to $\sqsupset$, while* (2.8) *is satisfied in C''. Then, C'' is an extension of C'. In particular, any exiomatically-equivalent finitary weakly $\sqsupset$-implicative Σ-logics are equal.*

Proof. By induction on any $n \in \omega$, we prove that C'' is an n-extension of C'. For consider any $X \in \wp_n(\mathrm{Fm}^\omega_\Sigma)$, in which case $n \neq 0$, and any $\psi \in C'(X)$. Then, in case $X = \varnothing$, we have $X \in \wp_1(\mathrm{Fm}^\omega_\Sigma)$, and so $\psi \in C'(X) \subseteq C''(X)$, for C'' is a 1-extension of C'. Otherwise, take any $\phi \in X$, in which case $Y \triangleq (X \setminus \{\phi\}) \in \wp_{n-1}(\mathrm{Fm}^\omega_\Sigma)$, and so, by DT with respect to $\sqsupset$, that C' has, and the induction hypothesis, we have $(\phi \sqsupset \psi) \in C'(Y) \subseteq C''(Y)$. Therefore, by $(2.8)[x_0/\phi, x_1/\psi]$ satisfied in C'', in view of its structurality, we eventually get $\psi \in C''(Y \cup \{\phi\}) = C''(X)$. Hence, as $\omega = (\bigcup \omega)$, we eventually conclude that C'' is an ω-extension of C', and so an extension of C', for this is finitary. □

Finally, assume $\mathcal{A}$ is $\sqsupset$-implicative. Then, as $0 \notin D^{\mathcal{A}}$, we have both $(\frac{1}{2} \sqsupset^{\mathfrak{A}} 0) = 0$, for $\mathcal{A}$ is false-singular, and $(0 \sqsupset^{\mathfrak{A}} 0) = 1$, for 2 forms a subalgebra of $\mathfrak{A}$. Therefore, since $K'_{3,0} \supseteq K_{3,0} \ni \langle 0/\frac{1}{2}, 0\rangle$ forms a subalgebra of $\mathfrak{A}^2$, we get $\langle 0,1\rangle = (\langle \frac{1}{2}, 0\rangle \sqsupset^{\mathfrak{A}^2} \langle 0,0\rangle) \in K'_{3,0}$. Thus, **e)⇒d)** holds. ([Conversely, assume $\langle 0,1\rangle \in K'_{3,0}$. Then, there is some $\phi \in \mathrm{Fm}^3_\Sigma$ such that $\phi^{\mathfrak{A}}(\frac{1}{2},0,1) = 0$, while $\phi^{\mathfrak{A}}(0,0,1) = 1$, in which case $\psi \triangleq (\phi[x_2/{\sim}x_1]) \in \mathrm{Fm}^2_\Sigma$, while $\psi^{\mathfrak{A}}(\frac{1}{2},0) = 0$, whereas $\psi^{\mathfrak{A}}(0,0) = 1$, and so $\varphi \triangleq (\psi \mathbin{\bar\wedge} {\sim}x_0) \in \mathrm{Fm}^2_\Sigma$, while $\varphi^{\mathfrak{A}}(a,0) = (1 - \chi^{\mathcal{A}}(a))$, for all $a \in A$, for $\mathcal{A}$ is both $\bar\wedge$-conjunctive and false-singular, while 2 forms a subalgebra of $\mathfrak{A}$. In this way, by the following claim, $\mathcal{A}$, being $\veebar$-disjunctive, is implicative:

Claim 8.4. *Let $\mathcal{N}'_2$ be the submatrix of $\mathcal{A}^3$ generated by $N_2 \triangleq \{\langle 0,1,\frac{1}{2}\rangle, \langle 0,0,0\rangle\}$. Suppose $\mathcal{A}$ is false-singular, while 2 forms a subalgebra of $\mathfrak{A}$. Then, $\mathcal{A}$ is implicative iff it is disjunctive, while $\langle 1,0,0\rangle \in N'_2$.*

Proof. First, if $\mathcal{A}$ is $\sqsupset$-implicative, then it is $\uplus_\sqsupset$-disjunctive, while $N'_2 \ni (\langle 0,1,\frac{1}{2}\rangle \sqsupset^{\mathfrak{A}^3} \langle 0,0,0\rangle) = \langle 1,0,0\rangle$, for $N'_2 \supseteq N_2$ forms a subalgebra of $\mathfrak{A}^3$, while 2 forms a subalgebra of $\mathfrak{A}$, whereas $\mathcal{A}$ is false-singular. Conversely, assume $\mathcal{A}$ is $\veebar$-disjunctive, while, $\langle 1,0,0\rangle \in N'_2$, in which case there is some $\phi \in \mathrm{Fm}^2_\Sigma$ such that $\phi^{\mathfrak{A}}(a,0) = (1 - \chi^{\mathcal{A}}(a))$, for all $a \in A$, and so $\psi \triangleq (\phi \veebar x_1) \in \mathrm{Fm}^2_\Sigma$, while $\mathcal{A}$ is ψ-implicative. □

In this way, **d)⇒e)** holds.])

Thus, Remark 2.8(i)**d)**, Lemmas 4.7, 4.8 and Corollary 6.4 complete the argument. □

By Remark 2.8(i)**d)**, Corollary 6.4 and Lemma 8.2, we immediately get:

Corollary 8.5. *Suppose C is [both conjunctive and] disjunctive as well as ${\sim}$-subclassical, while $\mathcal{A}$ is false-singular (more specifically, ${\sim}$-paraconsistent). Then, C^{PC} is the structural completion of C iff [either C is ${\sim}$-classical or] either K_4 or $K_{3,i}$, for some $i \in (2(\cap 1)[\cap 1])$, forms a subalgebra of $\mathfrak{A}^2$ [if and] only if C is either ${\sim}$-classical or non-implicative. In particular, providing C is ${\sim}$-paraconsistent, C^{PC} is the structural completion of it iff either K_4 or $K_{3,0}$ forms a subalgebra of $\mathfrak{A}^2$ [if and] only if it is not implicative.*

The opposite case is analyzed in Subsubsection 8.1.1 below within the framework of ${\sim}$-paraconsistent three-valued Σ-logics with subclassical negation ${\sim}$ as well as lattice conjunction and disjunction. On the other hand, the []-optional stipulation of conjunctivity cannot be omitted in the formulations of Lemma 8.2 and Corollary 8.5, even if C is ${\sim}$-paraconsistent, in view of:

Example 8.6. Let $\Sigma \triangleq \{\diamond, \vee, \sim\}$ with binary $\diamond$ and $\mathcal{A}$ false-singular with $\sim^{\mathfrak{A}}\frac{1}{2} \triangleq \frac{1}{2}$ and

$$(a(\vee|\diamond)^{\mathfrak{A}}b) \triangleq \begin{cases} \frac{1}{2}|0 & \text{if } \frac{1}{2} \in \{a,b\}, \\ \max(a,b)|1 & \text{otherwise,} \end{cases}$$

for all $a, b \in A$. Then, 2 forms a subalgebra of $\mathfrak{A}$, while $\mathcal{A}$ is both $\sim$-paraconsistent and $\vee$-disjunctive, whereas $\langle 0,1\rangle = (\langle \frac{1}{2}, 0\rangle \diamond^{\mathfrak{A}^2} \langle 0,0\rangle) \in K'_{3,0}$, for $K'_{3,0} \supseteq K_{3,0} \supseteq \{\langle \frac{1}{2}, 0\rangle, \langle 0,0\rangle\}$ forms a subalgebra of $\mathfrak{A}^2$. On the other hand, $((2^2 \times \{\frac{1}{2}\}) \cup (\Delta_2 \times 2)) \supseteq N_2$ forms a subalgebra of $\mathfrak{A}^3$ but does not contain $\langle 1,0,0\rangle$, for $1 \neq 0 \neq \frac{1}{2}$, in which case, by Claim 8.4, $\mathcal{A}$ is not implicative, and so is not conjunctive, in view of Lemma 8.2**d)**$\Rightarrow$**e)**. $\Box$

Remark 8.7. Let φ be a binary semi-conjunction for $\mathcal{A}$. Then, $\varphi^{\mathfrak{A}^2}(\langle 0,1\rangle, \langle 1,0\rangle) = \langle 0,0\rangle$, so $\mathcal{A}$ satisfies DGC. $\Box$

Remark 8.8. Suppose $\mathcal{A}$ is both false-singular and weakly $\overline{\wedge}$-conjunctive (viz., C is so). Then, as 0 is the only non-distinguished value of $\mathcal{A}$, we have $(0 \mathbin{\overline{\wedge}^{\mathfrak{A}}} a) = 0 = (a \mathbin{\overline{\wedge}^{\mathfrak{A}}} 0)$, for all $a \in A$, in which case we get $(\langle 0,a\rangle \mathbin{\overline{\wedge}^{\mathfrak{A}^2}} \langle a,0\rangle) = \langle 0,0\rangle \notin L_4 \supseteq \{\langle 0, \frac{1}{2}\rangle, \langle \frac{1}{2}, 0\rangle\}$, and so, in particular, L_4 does not form a subalgebra of $\mathfrak{A}^2$, while, in case $\sim^{\mathfrak{A}}\frac{1}{2} = 1$, we have $\langle 0,0\rangle = (\sim^{\mathfrak{A}^2}\langle 1, \frac{1}{2}\rangle \mathbin{\overline{\wedge}^{\mathfrak{A}^2}} \sim^{\mathfrak{A}^2}\sim^{\mathfrak{A}^2}\langle 1, \frac{1}{2}\rangle)$, whereas, otherwise, we have $\sim^{\mathfrak{A}^2}\langle 1, \frac{1}{2}\rangle \in \{\langle 0,0\rangle, \langle 0, \frac{1}{2}\rangle\}$. Thus, in addition, $\mathcal{A}$ satisfies GC. $\Box$

Combining Theorems 5.1(iii)$\Rightarrow$(i), 7.7(iii)$\Rightarrow$(i), 8.1 with Remarks 5.3, 8.7 and 8.8, we immediately get:

Corollary 8.9. *Suppose $\mathcal{A}$ is false-singular (in particular, $\sim$-paraconsistent) and weakly conjunctive. Then, C is structurally complete iff it is either $\sim$-classical or non-$\sim$-subclassical.*

Further, $\mathcal{A}$ is said to be *classically-hereditary*, provided 2 forms a subalgebra of $\mathfrak{A}$. Likewise, $\mathcal{A}$ is said to be *classically-valued*, provided, for each $\varsigma \in \Sigma$, $(\operatorname{img} \varsigma^{\mathfrak{A}}) \subseteq 2$, in which case it is classically-hereditary.

Remark 8.10. Suppose $\mathcal{A}$ is both classically-valued and $\veebar$-disjunctive. Then, as $1 \in D^{\mathcal{A}} \not\ni 0$, we have $(a \mathbin{\veebar^{\mathfrak{A}}} a) = \chi^{\mathcal{A}}(a)$, for all $a \in A$, in which case, since $\sim^{\mathfrak{A}} i = (1 - i)$, for all $i \in 2$, $\mathcal{A}$ is $\neg$-negative, where $(\neg x_0) \triangleq \sim(x_0 \veebar x_0)$, and so both $\veebar^{\neg}$-conjunctive and $\sqsupset_{\veebar}^{\neg}$-implicative, in view of Remark 2.8(i)**a)**,**c)**. $\Box$

Combining Remarks 2.8(i)**d)** and 8.10 with Corollaries 4.6, 5.4, 6.4 and 8.9, we also have:

Corollary 8.11. *Let $c \notin \Sigma$ be a nullary connective, $\Sigma' \triangleq (\Sigma \cup \{c\})$, $\mathcal{A}'$ the Σ'-expansion of $\mathcal{A}$ with $c^{\mathfrak{A}'} \triangleq \frac{1}{2}$ and C' the logic of $\mathcal{A}'$. Suppose $\mathcal{A}$ is $\sim$-paraconsistent as well as both classically-hereditary and weakly conjunctive (in particular, both classically-valued and implicative [i.e., disjunctive]). Then, C' is structurally complete, while C is not so, whereas both C and C' are maximally $\sim$-paraconsistent.*

This covers, in particular, both LP, LA, HZ (recall that this is $\vee^{\sim}$-conjunctive) — as non-classically-valued conjunctive classically-hereditary instances — and P^1 — as a term-wise definitionally minimal classically-valued disjunctive instance; cf. Remark 8.10 — as well as their bounded expansions by classical constants $\perp$ and $\top$ interpreted as 0 and 1, respectively. (In this connection, recall that the fact that LP is "maximally $\sim$-paraconsistent"/"not structurally complete" has been due to [14]/[16], respectively, proved *ad hoc* therein.) Thus, in view of Remark 2.8(i)**d)**, Corollaries 4.6, 8.11 and Theorem 8.1, any $\sim$-paraconsistent three-valued $\sim$-paraconsistent Σ-logic with subclassical negation $\sim$ is maximally so, whenever it is structurally complete, while the converse does not, generally speaking, hold, whereas the structural completeness of such a logic subsumes absence of its $\sim$-classical extensions. On the other hand, the situation with paracompleteness is quite different, as we show in Subsection 8.2 below.

8.1.1. *Extensions of logics with lattice conjunction and disjunction.* Throughout this subsubsection, it is supposed that:

- $\mathfrak{A}$ is a $(\bar{\wedge}, \veebar)$-lattice, in which case $\langle A, \leq^{\mathfrak{A}} \rangle$ is a chain poset for $|A| = 3$, and so $\mathfrak{A}$, being finite, is a distributive $(\bar{\wedge}, \veebar)$-lattice with zero and unit;
- $\mathcal{A}$ is $\sim$-paraconsistent (and so false-singular) and $\bar{\wedge}$-conjunctive, in which case $\flat^{\mathfrak{A}}_{\bar{\wedge}} = 0$, and so $\mathcal{A}$ is $\veebar$-disjunctive (in particular, C is maximally $\sim$-paraconsistent [cf. Corollary 5.4], while it is $\sim$-subclassical iff 2 forms a subalgebra of $\mathfrak{A}$, in which case C^{PC} is defined by $\mathcal{A}\restriction 2$ [cf. Corollary 6.4]);
- *unless otherwise specified*, $\sqsupset$ is the *material implication* $\sqsupset^{\sim}_{\veebar}$, in which case, by (2.3) satisfied in C, in view of its $\veebar$-disjunctivity, we have $C^{\mathrm{NP}} \subseteq C^{\mathrm{MP}}$, and so C, being $\sim$-paraconsistent, is not (weakly) $\sqsupset$-implicative.

Lemma 8.12. *Let $\mathcal{B}$ be a three-valued $\sim$-super-classical Σ-matrix, I a finite set, $\overline{\mathcal{C}} \in \mathbf{S}_*(\mathcal{B})^I$, $\mathcal{D}$ a subdirect product of it and $J \triangleq \{i \in I \mid \frac{1}{2} \in \pi_i[D]\}$. Suppose $\mathfrak{B}$ is a $(\bar{\wedge}, \veebar)$-lattice with $0 \leq^{\mathfrak{B}}_{\bar{\wedge}} 1$ and $\frac{1}{2}(\leq \mid \not\leq)^{\mathfrak{B}}_{\bar{\wedge}} \sim^{\mathfrak{B}} \frac{1}{2}$, while $\mathcal{A}$ is weakly conjunctive, whenever it is $\sim$-paraconsistent, whereas $\mathcal{D}$ is truth-non-empty, otherwise. Then, there is some $a \in (D \cap \{\frac{1}{2}, 0|1\}^I)$ including $J \times \{\frac{1}{2}\}$.*

Proof. Then, by Claim 4.17, for each $j \in 2$, $(I \times \{j\}) \in D$. Moreover, $\langle B, \leq^{\mathfrak{B}}_{\bar{\wedge}} \rangle$ is a chain, for $|B| = 3$, in which case $\frac{1}{2}(\leq \mid \geq)^{\mathfrak{B}}_{\bar{\wedge}} \sim^{\mathfrak{B}} \frac{1}{2}$, while $\frac{1}{2}(\leq / \geq)^{\mathfrak{B}}_{\bar{\wedge}}(0|1)$. By induction on the cardinality of any $K \subseteq J$, let us prove that there is some $a \in (D \cap \{\frac{1}{2}, 0|1\}^I)$ including $K \times \{\frac{1}{2}\}$. In case $K = \varnothing$, we have $j \triangleq (0|1) \in 2$, while $(K \times \{\frac{1}{2}\}) = \varnothing \subseteq (I \times \{j\}) \in (D \cap \{\frac{1}{2}, 0|1\}^I)$. Now, assume $K \neq \varnothing$. Take any $j \in K \subseteq J$, in which case $L \triangleq (K \setminus \{j\}) \subseteq J$, while $|L| < |K|$, and so, by induction hypothesis, there is some $b \in (D \cap \{\frac{1}{2}, 0|1\}^I)$ including $L \times \{\frac{1}{2}\}$. Moreover, since $\frac{1}{2} \in \pi_j[D]$, there is some $c \in D$ such that $\pi_j(c) = \frac{1}{2}$. Let $d \triangleq (c(\bar{\wedge}|\veebar)^{\mathfrak{D}} \sim^{\mathfrak{D}} c) \in D$ and $a \triangleq (b(\bar{\wedge}/\veebar)^{\mathfrak{D}} d) \in D$. Then, as $0 \leq^{\mathfrak{B}}_{\bar{\wedge}} 1$, while $\frac{1}{2}(\leq \mid \geq)^{\mathfrak{B}}_{\bar{\wedge}} \sim^{\mathfrak{B}} \frac{1}{2}$, for each $i \in I$, $\pi_i(d)$ is equal to $\frac{1}{2}$, if $\pi_i(c)$ is so, and is equal to $0|1$, otherwise, in which case, as $b \in \{\frac{1}{2}, 0|1\}^I$, while $\frac{1}{2}(\leq / \geq)^{\mathfrak{B}}_{\bar{\wedge}}(0|1)$, $\pi_i(a)$ is equal to $\frac{1}{2}$, if either $\pi_i(b)$ or $\pi_i(d)$ is so, and is equal to $0|1$, otherwise, and so $a \in (D \cap \{\frac{1}{2}, 0|1\}^I)$ includes $K \times \{\frac{1}{2}\}$, for $K = (L \cup \{j\})$. Thus, the case, when $K = J$, completes the argument. □

Corollary 8.13. *Let I be a finite set, $\overline{\mathcal{C}} \in \mathbf{S}_*(\mathcal{A})^I$ and $\mathcal{D}$ a consistent non-$\sim$-paraconsistent subdirect product of $\overline{\mathcal{C}}$. Then, 2 forms a subalgebra of $\mathfrak{A}$ and $\hom(\mathcal{D}, \mathcal{A}\restriction 2) \neq \varnothing$.*

Proof. First, by Lemma 8.12 with $J = I$, if $\frac{1}{2}$ was in $\pi_i[D] = C_i$, for each $i \in I$, then $a \triangleq (I \times \{\frac{1}{2}\})$ would be in D, in which case (2.10) would not be true in $\mathcal{D}$ under $[x_0/a, x_1/b]$, where $b \in (D \setminus D^{\mathcal{D}}) \neq \varnothing$, for $\mathcal{D}$ is consistent, and so $\mathcal{D}$ would be $\sim$-paraconsistent. Hence, there is some $i \in I$ such that $\frac{1}{2} \notin B \triangleq \pi_i[D] = C_i \neq \varnothing$, in which case $B \subseteq 2$ forms a subalgebra of $\mathfrak{A}$, and so $B = 2$, while $(\pi_i \restriction D) \in \hom(\mathcal{D}, \mathcal{A}\restriction B)$. □

Theorem 8.14. *Suppose C is $\sim$-subclassical (i.e., 2 forms a subalgebra of $\mathfrak{A}$; cf. Corollary 6.4). Then, C^{NP} is defined by $\mathcal{L}_6 \triangleq (\mathcal{A} \times (\mathcal{A}\restriction 2))$, in which case $C^{\mathrm{NP}}(\varnothing) = C(\varnothing)$.*

Proof. Then, by Theorem 2.14 with $\mathsf{M} \triangleq \{\mathcal{A}\}$ and $\mathsf{K} \triangleq \mathbf{P}^{\mathrm{SD}}_{\omega}(\mathbf{S}_*(\mathsf{M}))$, C^{NP} is finitely-defined by the class S of all consistent members of $\mathsf{K} \cap \mathrm{Mod}(C^{\mathrm{NP}})$. Consider any $\mathcal{D} \in \mathsf{S} \subseteq \mathrm{Mod}\,(2.10)$, in which case there are some finite set I and some

$\overline{\mathcal{C}} \in \mathbf{S}_*(\mathcal{A})^I$ such that $\mathcal{D}$ is a subdirect product of it, and so, by Corollary 8.13, $\hom(\mathcal{D}, \mathcal{A}{\restriction}2) \neq \varnothing$. Take any $g \in \hom(\mathcal{D}, \mathcal{A}{\restriction}2)$. Consider any $a \in (D \setminus D^{\mathcal{D}})$. Then, there is some $i \in I$ such that $\pi_i(a) \notin D^{\mathcal{A}}$, while $f \triangleq (\pi_i{\restriction}D) \in \hom(\mathcal{D}, \mathcal{A})$, in which case $h \triangleq (f \times g) \in J \triangleq \hom(\mathcal{D}, \mathcal{L}_6)$, while $h(a) \notin D^{\mathcal{L}_6}$, and so $(\prod J) \in \hom_{\mathrm{S}}(\mathcal{D}, \mathcal{L}_6^J)$. Thus, by (2.16), C^{NP} is finitely-defined by the finite $\mathcal{L}_6$, in which case it, being finitary, for (2.10) is so, while $\mathcal{A}$ is finite, is defined by $\mathcal{L}_6$, and so (2.17) and the fact that $(\pi_0{\restriction}L_6) \in \hom^{\mathrm{S}}(\mathcal{L}_6, \mathcal{A})$ complete the argument. □

Theorem 8.15. *C^{MP} is consistent iff C is $\sim$-subclassical, in which case $C^{\mathrm{NP}} \subsetneq C^{\mathrm{MP}} = C^{\mathrm{PC}}$, and so C^{NP} is not $\veebar$-disjunctive.*

Proof. First, if C^{MP} is consistent, then so is its sublogic C^{NP} (in view of (2.3) satisfied in C), in which case C is $\sim$-subclassical, by Theorem 7.11. Conversely, assume C is $\sim$-subclassical, in which case, by Corollary 6.4, 2 forms a subalgebra of $\mathfrak{A}$, while C^{PC} is defined by $\mathcal{A}{\restriction}2$. Then, by Remark 2.8(i)**c)**,(ii), $\mathcal{A}{\restriction}2$ is $\sqsupset^{\sim}_{\veebar}$-implicative, and so is C^{PC}, in which case $C^{\mathrm{MP}} \subseteq C^{\mathrm{PC}}$. For proving the converse, consider the following complementary cases:

- $C^{\mathrm{PC}}(\varnothing) = C(\varnothing)$.
  Then, Claim 8.3 implies that $C^{\mathrm{PC}} \subseteq C^{\mathrm{MP}}$.
- $C^{\mathrm{PC}}(\varnothing) \neq C(\varnothing)$.

  1st argument. Then, by Lemma 8.2**b)⇒e)**, $\mathcal{A}$ is implicative. Hence, by the following claim, there is some $\varphi \in (\mathrm{Fm}^1_\Sigma \cap C(\varnothing)) \subseteq C^{\mathrm{MP}}(\varnothing)$ such that $\varphi^{\mathfrak{A}}(\frac{1}{2}) = \frac{1}{2}$:

  Claim 8.16. *Suppose $\mathcal{A}$ is implicative. Then, there is some $\varphi \in (\mathrm{Fm}^1_\Sigma \cap C(\varnothing))$ such that $\varphi^{\mathfrak{A}}(\frac{1}{2}) = \frac{1}{2}$.*

  Proof. By contradiction. For suppose, for all $\varphi \in (\mathrm{Fm}^1_\Sigma \cap C(\varnothing))$, $\varphi^{\mathfrak{A}}(\frac{1}{2}) \neq \frac{1}{2}$, in which case $\varphi^{\mathfrak{A}}(\frac{1}{2}) \in D^{\mathcal{A}} = \{\frac{1}{2}, 1\}$, and so $\varphi^{\mathfrak{A}}(\frac{1}{2}) = 1$. In particular, since $\mathcal{A}$ is both $\veebar$-disjunctive and, being false-singular, weakly $\sim$-negative, it is not $(\veebar, \sim)$-paracomplete, in view of Remark 2.8(i)**d)**, in which case $(\frac{1}{2} \veebar^{\mathfrak{A}} \sim^{\mathfrak{A}} \frac{1}{2}) = 1$, and so

  $$\tfrac{1}{2} \leq^{\mathfrak{A}}_{\barwedge} 1 = \sim^{\mathfrak{A}} \tfrac{1}{2}, \tag{8.1}$$

  in view of the linearity of the poset $\langle A, \leq^{\mathfrak{A}}_{\barwedge} \rangle$. Consider any $\phi \in C(\varnothing)$ and any $h \in \hom(\mathfrak{Fm}^\omega_\Sigma, \mathfrak{A})$. Let $U_a \triangleq (V_\omega \cap h^{-1}[\{a\}])$, where $a \in A$, and σ the Σ-substitution extending $(U_{\frac{1}{2}} \times \{x_0\}) \cup (U_1 \times \{\sim x_0\}) \cup (U_0 \times \{\sim\sim x_0\})$, in which case, by the structurality of C, we have $\psi \triangleq \sigma(\phi) \in (\mathrm{Fm}^1_\Sigma \cap C(\varnothing))$, and so, by (8.1), we get $h(\phi) = \psi^{\mathfrak{A}}(\frac{1}{2}) = 1$. Hence, $\mathcal{B} \triangleq \langle \mathfrak{A}, \{1\} \rangle \in \mathrm{Mod}_1(C)$. Let $\supset$ be any (possibly, secondary) binary connective of Σ, such that $\mathcal{A}$ is $\supset$-implicative, and $(x_0 \sqsupset x_1) \triangleq ((x_0 \supset x_1) \barwedge (x_0 \sqsupset^{\sim}_{\veebar} x_1))$, in which case $\mathcal{A}$ is $\sqsupset$-implicative, for it is $\supset$-implicative, $\barwedge$-conjunctive, $\veebar$-disjunctive and false-singular, and so $(1 \sqsupset^{\mathfrak{A}} 0) = 0$. Moreover, $(1 \supset^{\mathfrak{A}} \frac{1}{2}) \in D^{\mathcal{A}} = \{\frac{1}{2}, 1\}$, in which case, by (8.1), we have $\frac{1}{2} \leq^{\mathfrak{A}}_{\barwedge} (1 \supset^{\mathfrak{A}} \frac{1}{2})$, and so we get $(1 \sqsupset^{\mathfrak{A}} \frac{1}{2}) = \frac{1}{2}$, for $\sim^{\mathfrak{A}} 1 = 0 \leq^{\mathfrak{A}}_{\barwedge} \frac{1}{2}$. Therefore, (2.8) is true in $\mathcal{B} \in \mathrm{Mod}_1(C)$, in which case, by Claim 8.3, $\mathcal{B} \in \mathrm{Mod}(C)$ is both finite and, by (8.1), $\veebar$-disjunctive, and so, by Remarks 2.7(iii), 2.8(i)**d)** and Corollaries 2.12 and 4.13, there is some $h \in \hom_{\mathrm{S}}(\mathcal{B}, \mathcal{A})$. Then, $h(0) = h(\frac{1}{2}) = 0$, in which case $0 = h(0) = h(\frac{1}{2} \supset^{\mathfrak{A}} 0) = (h(\frac{1}{2}) \supset^{\mathfrak{A}} h(0)) = (0 \supset^{\mathfrak{A}} 0) \in D^{\mathcal{A}}$, and so this contradiction completes the argument. □

Hence, $\sim^{\mathfrak{A}}\varphi^{\mathfrak{A}}(\frac{1}{2}) = \sim^{\mathfrak{A}}\frac{1}{2} \in D^{\mathcal{A}}$, for $\mathcal{A}$ is $\sim$-paraconsistent, in which case $\sim\varphi\veebar(2.11)$ is true in $\mathcal{A}$ under any $[x_0/\frac{1}{2}, x_1/a]$, where $a \in A$, for $\mathcal{A}$ is $\veebar$-disjunctive, and so, since (2.11) is true in $\mathcal{A}$ under any $[x_0/i, x_1/a]$, where $i \in 2$, $\sim\varphi\veebar(2.11)$ is true in $\mathcal{A}$. Thus, $(\sim\varphi\veebar(2.11)) \in C(\varnothing) \subseteq C^{\mathrm{MP}}(\varnothing)$, in which case, by the structurality of C^{MP} and $(2.8)[x_0/\varphi, x_1/(2.11)]$, (2.11) is satisfied in C^{MP}, and so, by Corollary 6.6, $C^{\mathrm{PC}} \subseteq C^{\mathrm{MP}}$.

2nd argument. Then, by Lemma 8.2**b)⇒c)**, $\langle 0, 1\rangle \in K'_{3,i}$, for each $i \in 2$, in which case there is some $\varphi_i \in \mathrm{Fm}^3_\Sigma$ such that $\varphi_i^{\mathfrak{A}}(0, \frac{1}{2}[-\frac{1}{2}+i], 1) = (0[+1])$. Moreover, by Theorem 2.14 with $\mathsf{M} \triangleq \{\mathcal{A}\}$ and $\mathsf{K} \triangleq \mathbf{P}^{\mathrm{SD}}_\omega(\mathbf{S}_*(\mathsf{M}))$, C^{MP} is finitely-defined by $\mathsf{S} \triangleq (\mathsf{K} \cap \mathrm{Mod}(C^{\mathrm{MP}}))$. Consider any $\mathcal{D} \in \mathsf{S} \subseteq \mathrm{Mod}\,(2.8)$, in which case there are some finite set I and some $\overline{\mathcal{C}} \in \mathbf{S}_*(\mathcal{A})^I$ such that $\mathcal{D}$ is a subdirect product of it. Let $J \triangleq \{i \in I \mid \frac{1}{2} \in \pi_i[D]\}$. Given any $\bar{a} \in A^2$, set $(a_0\|a_1) \triangleq ((J \times \{a_0\}) \cup ((I \setminus J) \times \{a_1\})) \in A^I$. Then, by Claim 4.17, $D \ni (a/b) \triangleq (0/1\|0/1)$. Moreover, by Lemma 8.12, $D \ni c \triangleq (\frac{1}{2}\|0|1)$, whenever $\frac{1}{2}(\leq \mid \not\leq)^{\mathfrak{A}}_{\barwedge}\sim^{\mathfrak{A}}\frac{1}{2}$. Then, $D \ni d \triangleq \varphi^{\mathfrak{D}}_{0|1}(a, c, b) = (0\|1)$, in which case $D \ni e \triangleq (c\veebar^{\mathfrak{D}} d) = (\frac{1}{2}\|1)$, and so $(\sim^{\mathfrak{D}} e\veebar^{\mathfrak{D}} d) = (\sim^{\mathfrak{A}}\frac{1}{2}\|1) \in D^{\mathcal{D}} \ni e$, for $\mathcal{A}$, being $\sim$-paraconsistent, is false-singular. Hence, by (2.8) true in $\mathcal{D}$, we have $d \in D^{\mathcal{D}}$, in which case $J = \varnothing$, and so $\mathcal{D}$ is a subdirect I-power of $\mathcal{A}{\upharpoonright}2$. Therefore, by (2.16), $\mathcal{D} \in \mathrm{Mod}(C^{\mathrm{PC}})$. In this way, $\mathsf{S} \subseteq \mathrm{Mod}(C^{\mathrm{PC}})$, in which case, for all $X \in \wp_\omega(\mathrm{Fm}^\omega_\Sigma)$, it holds that $C^{\mathrm{PC}}(X) \subseteq \mathrm{Cn}^\omega_{\mathsf{S}}(X) = C^{\mathrm{MP}}(X)$, and so C^{PC}, being finitary, for it is two-valued, is a sublogic of C^{MP}.

Thus, $C^{\mathrm{MP}} = C^{\mathrm{PC}}$ is consistent. Moreover, by Theorem 8.14, C^{NP} is defined by $\mathcal{L}_6$, in which (2.8) is not true under $[x_0/\langle\frac{1}{2}, 1\rangle, x_1/\langle 0, 1\rangle]$. Finally, the following claim completes the argument:

Claim 8.17. *Any $\veebar$-disjunctive extension C' of C^{NP} is an extension of C^{MP}.*

Proof. In that case, we have $x_1 \in (C'(\{x_0, \sim x_0\}) \cap C'(\{x_0, x_1\})) = C'(\{x_0, \sim x_0 \veebar x_1\})$, as required. □

Next, by C^{DMP} we denote the extension of C relatively axiomatized by the *Dual Modens Ponens* rule:

$$\{\sim x_0, x_0 \veebar x_1\} \vdash x_1, \tag{8.2}$$

being actually dual to (2.8) for material implication. Clearly, by (2.3) satisfied in C, in view of its $\veebar$-disjunctivity, C^{DMP} is an extension of C^{NP}.

Lemma 8.18. *Suppose C is $\sim$-subclassical (i.e., 2 forms a subalgebra of $\mathfrak{A}$, C^{PC} being defined by $\mathcal{A}{\upharpoonright}2$; cf. Corollary 6.4). Then, the following hold:*

(i) *C^{DMP} is a proper extension of C^{NP};*

(ii) *$(\mathcal{A}{\upharpoonright}2) \in \mathrm{Mod}(C^{\mathrm{DMP}})$;*

(iii) *providing $L_5 \triangleq (K_{3,1} \cup M_2)$ forms a subalgebra of $\mathfrak{A}^2$, the following hold:*

- **a)** *$\sim^{\mathfrak{A}}\frac{1}{2} = 1 \leq^{\mathfrak{A}}_{\barwedge} \frac{1}{2}$, that is, $\sim(x_0 \barwedge \sim x_0) \notin C(\varnothing)$;*
- **b)** *$\mathcal{A}$ is generated by $\{\frac{1}{2}\}$;*
- **c)** *$\mathcal{L}_6$ is generated by $L_6 \setminus L_5$;*
- **d)** *$\mathcal{A}$ is implicative;*
- **e)** *$\mathcal{L}_5 \triangleq (\mathcal{A}^2{\upharpoonright}L_5) \in \mathrm{Mod}(C^{\mathrm{DMP}})$;*
- **f)** *the logic of $\mathcal{L}_5$ is an axiomatically-equivalent to C (and so proper) sublogic of C^{PC}, and so is its sulogic C^{DMP}.*

Proof. (i) Then, by Theorem 8.14, C^{NP} is defined by $\mathcal{L}_6$, in which (8.2) is not true under $[x_0/\langle\frac{1}{2}, 0\rangle, x_1/\langle 0, 1\rangle]$, for $\mathcal{A}$ is both $\veebar$-disjunctive and $\sim$-paraconsistent, and so false-singular.

(ii) Since $\sim^{\mathfrak{A}}\sim^{\mathfrak{A}}i = i$, for all $i \in 2$, the Σ-rule $(x_0 \veebar x_1) \vdash (\sim\sim x_0 \veebar x_1)$ is true in $\mathcal{A}{\restriction}2$, and so is (8.2), for (2.8) for the material implication is so, in view of Theorem 8.15.

(iii) **a)** If it did hold that $(\sim^{\mathfrak{A}}\frac{1}{2} = \frac{1}{2})|(\frac{1}{2} \leq^{\mathfrak{A}}_{\wedge} 1)$, then we would have $(\sim^{\mathfrak{A}^2}\langle\frac{1}{2},1\rangle|$ $(\langle\frac{1}{2},1\rangle \,\overline{\wedge}^{\mathfrak{A}^2}\, \langle 1,0\rangle)) = \langle\frac{1}{2},0\rangle \notin L_5$, in which case $L_5 \supseteq \{\langle\frac{1}{2},1\rangle, \langle 1,0\rangle\}$ would not form a subalgebra of $\mathfrak{A}^2$, and so the $\sim$-paraconsistency of $\mathcal{A}$ and the linearity of the poset $\langle A, \leq^{\mathfrak{A}}_{\wedge}\rangle$ complete the argument.

b) Then, by **a)**, we have $(\sim^{\mathfrak{A}})^{2-i}\frac{1}{2} = i$, for all $i \in 2$.

c) Likewise, by **a)**, we have $(\sim^{\mathfrak{A}^2})^{2-i}\langle\frac{1}{2},0\rangle = \langle i,i\rangle$, for all $i \in 2$, while $(\langle\frac{1}{2},0\rangle \veebar^{\mathfrak{A}^2} \langle 1,1\rangle) = \langle\frac{1}{2},1\rangle$, whereas $(\sim^{\mathfrak{A}^2})^{2-i}\langle\frac{1}{2},1\rangle = \langle i,1-i\rangle$, for all $i \in 2$.

d) Then, as $(L_6 \setminus L_5) \subseteq K_{3,0}$, by **c)**, we have $K'_{3,0} \supseteq L_6 \ni \langle 0,1\rangle$, and so Lemma 8.2**d)**$\Rightarrow$**e)** completes the argument.

e) Then, by (ii), (8.2) is true in $(\mathcal{A}{\restriction}2)^2 = (\mathcal{L}_5{\restriction}2^2)$, while $(L_5 \setminus 2^2) = \{\langle\frac{1}{2},1\rangle\} \subseteq D^{\mathcal{L}_5}$, whereas $\sim^{\mathfrak{L}_5}\langle\frac{1}{2},1\rangle = \langle\sim^{\mathfrak{A}}\frac{1}{2},0\rangle \notin D^{\mathcal{L}_5}$, in which case (8.2) is true in $\mathcal{L}_5$, and so (2.16), due to which $\mathcal{L}_5$ is a model of C, for $\mathcal{A}^2$ is so, completes the argument.

f) Then, as $\Delta_2 \subseteq K_{3,1} \subseteq L_5$, $\Delta_2 \times \Delta_2$ is an embedding of $\mathcal{A}{\restriction}2$ into $\mathcal{L}_5$, while $(\pi_0{\restriction}L_5) \in \hom_{\mathrm{S}}(\mathcal{L}_5, \mathcal{A})$, for $A = \pi_0[A^2] \supseteq \pi_0[L_5] \supseteq \pi_0[K_{3,1}] = A$. In this way, **d)**, **e)**, (2.16), (2.17) and Lemma 8.2**e)**$\Rightarrow$**b)** complete the argument. □

Lemma 8.19. *Let C' be an extension of C and $\mathcal{L}'_5$ the submatrix of $\mathcal{A}^2$ generated by L_5. Suppose C is $\sim$-subclassical (i.e., 2 forms a subalgebra of $\mathfrak{A}$, C^{PC} being defined by $\mathcal{A}{\restriction}2$; cf. Corollary 6.4), while* (2.8) *is not satisfied in C'. Then, $\mathcal{L}'_5 \in \mathrm{Mod}(C')$. In particular, $C^{\mathrm{DMP}} = C^{\mathrm{PC}}$, unless L_5 forms a subalgebra of $\mathfrak{A}^2$.*

Proof. Then, by Theorem 8.14, C^{NP} is defined by $\mathcal{L}_6$. On the other hand, as C' does not satisfy the finitary (2.8), by Theorem 2.14, there are some finite set I, some $\overline{\mathcal{C}} \in \mathbf{S}_*(\mathcal{A})^I$ and some subdirect product $\mathcal{D} \in \mathrm{Mod}(C')$ of it not being a model of (2.8), in which case there are some $a \in D^{\mathcal{D}} \subseteq \{\frac{1}{2},1\}^I$ and some $b \in (D \setminus D^{\mathcal{D}})$ such that $(\sim^{\mathfrak{D}}a \veebar^{\mathfrak{D}} b) \in D^{\mathcal{D}}$, and so $J \triangleq \{i \in I \mid \pi_i(a) = \frac{1}{2}\} \supseteq K \triangleq \{i \in I \mid \pi_i(b) = 0\} \neq \varnothing$. Put $L \triangleq \{i \in I \mid \pi_i(b) = 1\}$. Then, given any $\bar{a} \in A^5$, set $(a_0\|a_1\|a_2\|a_3\|a_4) \triangleq ((((I \setminus (L \cup K)) \cap J) \times \{a_0\}) \cup ((I \setminus (L \cup J)) \times \{a_1\}) \cup ((L \setminus J) \times \{a_2\}) \cup ((L \cap J) \times \{a_3\}) \cup (K \times \{a_4\})) \in A^I$. In this way:

$$D \ni a = (\tfrac{1}{2}\|1\|1\|\tfrac{1}{2}\|\tfrac{1}{2}), \tag{8.3}$$

$$D \ni b = (\tfrac{1}{2}\|\tfrac{1}{2}\|1\|1\|0). \tag{8.4}$$

Moreover, by Claim 4.17, we also have:

$$D \ni f \triangleq (0\|0\|0\|0\|0), \tag{8.5}$$

$$D \ni t \triangleq (1\|1\|1\|1\|1). \tag{8.6}$$

Consider the following exhaustive (as $\sim^{\mathfrak{A}}\frac{1}{2} \in D^{\mathcal{A}} = \{\frac{1}{2},1\}$) cases:

- $\sim^{\mathfrak{A}}\frac{1}{2} = \frac{1}{2}$.
  Then, in case $\frac{1}{2} \leq^{\mathfrak{A}}_{\wedge} 1$, by (8.3) and (8.4), we have:

$$D \ni e \triangleq (a \,\overline{\wedge}^{\mathfrak{D}}\, b) = (\tfrac{1}{2}\|\tfrac{1}{2}\|1\|\tfrac{1}{2}\|0), \tag{8.7}$$

$$D \ni \sim^{\mathfrak{D}}e = (\tfrac{1}{2}\|\tfrac{1}{2}\|0\|\tfrac{1}{2}\|1), \tag{8.8}$$

$$D \ni c \triangleq (e \veebar^{\mathfrak{D}} \sim^{\mathfrak{D}}b) = (\tfrac{1}{2}\|\tfrac{1}{2}\|1\|\tfrac{1}{2}\|1), \tag{8.9}$$

$$D \ni \sim^{\mathfrak{D}}c = (\tfrac{1}{2}\|\tfrac{1}{2}\|0\|\tfrac{1}{2}\|0). \tag{8.10}$$

Likewise, in case $\frac{1}{2}(\leq_{\bar{\wedge}} / \geq)^{\mathfrak{A}}1$, by (8.3) and (8.7)/(8.4), we have:

$$D \ni d \triangleq ((e/b) \veebar^{\mathfrak{D}} \sim^{\mathfrak{D}} a) = (\tfrac{1}{2}\|\tfrac{1}{2}\|1\|\tfrac{1}{2}\|\tfrac{1}{2}), \tag{8.11}$$

$$D \ni \sim^{\mathfrak{D}} d = (\tfrac{1}{2}\|\tfrac{1}{2}\|0\|\tfrac{1}{2}\|\tfrac{1}{2}). \tag{8.12}$$

Consider the following complementary subcases:

– $L \subseteq J$.
  Then, since $I \supseteq K \neq \varnothing = (L \setminus J)$, by (8.5), (8.6) and (8.11), $\langle g, I \times \{g\}\rangle \mid g \in A\}$ is an embedding of $\mathcal{A}$ into $\mathcal{D}$, in which case, by (2.16), $\mathcal{A}$ is a model of C', for $\mathcal{D}$ is so, and so is $\mathcal{L}'_5$.

– $L \nsubseteq J$.
  Then, consider the following complementary subsubcases:
  * there is some $\varphi \in \mathrm{Fm}^2_\Sigma$ such that $\varphi^{\mathfrak{A}}(\frac{1}{2}, 0) = 0$ and $\varphi^{\mathfrak{A}}(0,0) = 1$,
    in which case, by (8.5) and (8.12), we have:

$$D \ni \varphi^{\mathfrak{D}}(\sim^{\mathfrak{D}} d, f) = (0\|0\|1\|0\|0), \tag{8.13}$$

$$D \ni \sim^{\mathfrak{D}} \varphi^{\mathfrak{D}}(\sim^{\mathfrak{D}} d, f) = (1\|1\|0\|1\|1). \tag{8.14}$$

    Then, since $(L \setminus J) \neq \varnothing \neq K$, taking (8.5), (8.6), (8.11), (8.12), (8.13) and (8.14) into account, we see that $\{\langle\langle g, h\rangle, (g\|g\|h\|g\|g)\rangle \mid \langle g, h\rangle \in L_6\}$ is an embedding of $\mathcal{L}_6$ into $\mathcal{D}$, and so, by (2.16), $\mathcal{L}_6$ is a model of C', for $\mathcal{D}$ is so, and so is its submatrix $\mathcal{L}'_5$, for $L_6 \supseteq L_5$ forms a subalgebra of $\mathfrak{A}^2$, because 2 forms a subalgebra of $\mathfrak{A}$.
  * there is no $\varphi \in \mathrm{Fm}^2_\Sigma$ such that $\varphi^{\mathfrak{A}}(\frac{1}{2}, 0) = 0$ and $\varphi^{\mathfrak{A}}(0,0) = 1$,
    Then, $\frac{1}{2} \leq^{\mathfrak{A}}_{\bar{\wedge}} 1$, for, otherwise, we would have $1 \leq^{\mathfrak{A}}_{\bar{\wedge}} \frac{1}{2}$, in which case we would get $\varphi^{\mathfrak{A}}(\frac{1}{2}, 0) = 0$ and $\varphi^{\mathfrak{A}}(0,0) = 1$, where $\varphi \triangleq \sim(x_0 \mathbin{\bar{\wedge}} \sim x_1) \in \mathrm{Fm}^2_\Sigma$. Consider the following complementary subsubsubcases:
    · $(((I \setminus (L \cup K)) \cap J) \cup (I \setminus (L \cup J)) \cup (L \cap J)) = \varnothing$.
      Then, taking (8.7), (8.8), (8.9), (8.10), (8.11) and (8.12) into account, as $K \neq \varnothing \neq (L \setminus J)$, we conclude that $\{\langle\langle g, h\rangle, (\frac{1}{2}\|\frac{1}{2}\|h\|\frac{1}{2}\|g)\rangle \mid \langle g, h\rangle \in L_6\}$ is an embedding of $\mathcal{L}_6$ into $\mathcal{D}$, and so, by (2.16), $\mathcal{L}_6$ is a model of C', for $\mathcal{D}$ is so, and so is its submatrix $\mathcal{L}'_5$, for $L_6 \supseteq L_5$ forms a subalgebra of $\mathfrak{A}^2$, because 2 forms a subalgebra of $\mathfrak{A}$.
    · $(((I \setminus (L \cup K)) \cap J) \cup (I \setminus (L \cup J)) \cup (L \cap J)) \neq \varnothing$.
      Let $\mathfrak{G}$ be the subalgebra of $\mathfrak{L}_6 \times \mathfrak{A}$ generated by $((L_6 \times \{\frac{1}{2}\}) \cup \{\langle\langle i, i\rangle, i\rangle \mid i \in 2\})$. Then, as $(((I \setminus (L \cup K)) \cap J) \cup (I \setminus (L \cup J)) \cup (L \cap J)) \neq \varnothing \notin \{K, L \setminus J\}$, by (8.5), (8.6), (8.7), (8.8), (8.9), (8.10), (8.11) and (8.12), we see that $\{\langle\langle\langle g, h\rangle, j\rangle, (j\|j\|h\|j\|g)\rangle \mid \langle\langle g, h\rangle, j\rangle \in G\}$ is an embedding of $\mathcal{G} \triangleq ((\mathcal{L}_6 \times \mathcal{A}){\upharpoonright}G)$ into $\mathcal{D}$, in which case, by (2.16), $\mathcal{G}$ is a model of C', for $\mathcal{D}$ is so. Let us prove, by contradiction, that $((D^{\mathcal{L}_6} \times \{0\}) \cap G) = \varnothing$. For suppose $((D^{\mathcal{L}_6} \times \{0\}) \cap G) \neq \varnothing$. Then, there is some $\psi \in \mathrm{Fm}^8_\Sigma$ such that $\psi^{\mathfrak{A}}(1, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0) = 0$ and $\psi^{\mathfrak{A}}(1,1,1,1,0,0,0,0) = 1$, for $\pi_1[D^{\mathcal{L}_6}] = \{1\}$. Let $\varphi \triangleq \psi(\sim x_1, \sim x_0, \sim x_0, \sim x_0, x_0, x_0, x_0, x_1) \in \mathrm{Fm}^2_\Sigma$. Then, $\varphi^{\mathfrak{A}}(\frac{1}{2}, 0) = 0$ and $\varphi^{\mathfrak{A}}(0,0) = 1$. This contradiction shows that $((D^{\mathcal{L}_6} \times \{0\}) \cap G) = \varnothing$, in which case $(\pi_0{\upharpoonright}G) \in \hom^{\mathrm{S}}_{\mathrm{S}}(\mathcal{G}, \mathcal{L}_6)$, and so, by (2.16), $\mathcal{L}_6$ is a model of C', for $\mathcal{G}$ is so, and so is its submatrix $\mathcal{L}'_5$, for $L_6 \supseteq L_5$

forms a subalgebra of $\mathfrak{A}^2$, because 2 forms a subalgebra of $\mathfrak{A}$.

- $\sim^{\mathfrak{A}} \frac{1}{2} = 1$,
  Consider the following exhaustive (as $\langle A, \leq^{\mathfrak{A}}_{\bar\wedge} \rangle$ is a chain poset) subcases:
  - $\frac{1}{2} \leq^{\mathfrak{A}}_{\bar\wedge} 1$.
    Then, by (8.3) and (8.4), we get:
    $$D \ni c' \triangleq (a \veebar^{\mathfrak{D}} b) = (\tfrac{1}{2}\|1\|1\|1\|\tfrac{1}{2}), \tag{8.15}$$
    $$D \ni d' \triangleq \sim^{\mathfrak{D}} c' = (1\|0\|0\|0\|1), \tag{8.16}$$
    $$D \ni e' \triangleq \sim^{\mathfrak{D}} d' = (0\|1\|1\|1\|0), \tag{8.17}$$
    $$D \ni f' \triangleq (c' \bar\wedge^{\mathfrak{D}} d') = (\tfrac{1}{2}\|0\|0\|0\|\tfrac{1}{2}). \tag{8.18}$$
    Consider the following complementary subsubcases:
    * $((I \setminus (L \cup J)) \cup (L \setminus J) \cup (L \cap J)) = \varnothing$.
      Then, since $I \supseteq K \neq \varnothing$, by (8.5), (8.6) and (8.15), we see that $\{\langle g, I \times \{g\}\rangle \mid g \in A\}$ is an embedding of $\mathcal{A}$ into $\mathcal{D}$, in which case, by (2.16), $\mathcal{A}$ is a model of C', for $\mathcal{D}$ is so, and so is $\mathcal{L}'_5$.
    * $((I \setminus (L \cup J)) \cup (L \setminus J) \cup (L \cap J)) \neq \varnothing$.
      Then, as $K \neq \varnothing$, by (8.5), (8.6), (8.15), (8.16), (8.17) and (8.18), we conclude that $\{\langle\langle g, h\rangle, (g\|h\|h\|h\|g)\rangle \mid \langle g, h\rangle \in L_6\}$ is an embedding of $\mathcal{L}_6$ into $\mathcal{D}$, in which case, by (2.16), $\mathcal{L}_6$ is a model of C', for $\mathcal{D}$ is so, and so is its submatrix $\mathcal{L}'_5$, for $L_6 \supseteq L_5$ forms a subalgebra of $\mathfrak{A}^2$, because 2 forms a subalgebra of $\mathfrak{A}$.
  - $1 \leq^{\mathfrak{A}}_{\bar\wedge} \frac{1}{2}$.
    Then, by (8.3) and (8.4), we get:
    $$D \ni c'' \triangleq (a \veebar^{\mathfrak{D}} b) = (\tfrac{1}{2}\|\tfrac{1}{2}\|1\|\tfrac{1}{2}\|\tfrac{1}{2}), \tag{8.19}$$
    $$D \ni d'' \triangleq \sim^{\mathfrak{D}} c'' = (1\|1\|0\|1\|1), \tag{8.20}$$
    $$D \ni e'' \triangleq \sim^{\mathfrak{D}} d'' = (0\|0\|1\|0\|0). \tag{8.21}$$
    Consider the following complementary subsubcases:
    * $L \subseteq J$.
      Then, as $I \supseteq K \neq \varnothing = (L \setminus J)$, taking (8.5), (8.6) and (8.19) into account, we see that $\{\langle g, I \times \{g\}\rangle \mid g \in A\}$ is an embedding of $\mathcal{A}$ into $\mathcal{D}$, in which case, by (2.16), $\mathcal{A}$ is a model of C', for $\mathcal{D}$ is so, and so is $\mathcal{L}'_5$.
    * $L \nsubseteq J$.
      Then, as $K \neq \varnothing \neq (L \setminus J)$, taking (8.5), (8.6), (8.19), (8.20) and (8.21) into account, we see that $\{\langle\langle g, h\rangle, (g\|g\|h\|g\|g)\rangle \mid \langle g, h\rangle \in L'_5\}$ is an embedding of $\mathcal{L}'_5$ into $\mathcal{D}$, in which case, by (2.16), $\mathcal{L}'_5$ is a model of C', for $\mathcal{D}$ is so.

In this way, Theorem 8.15 and Lemma 8.18(i,ii) complete the argument, for $\mathcal{L}'_5 = \mathcal{L}_6$, unless L_5 forms a subalgebra of $\mathfrak{A}^2$, because $(L_6 \setminus L_5) = \{\langle \frac{1}{2}, 0\rangle\}$ is a singleton, while $L_6 \supseteq L_5$ forms a subalgebra of $\mathfrak{A}^2$, since 2 forms a subalgebra of $\mathfrak{A}$. □

Corollary 8.20. *Let C' be an extension of C. Suppose* (8.2) *is not satisfied in C'. Then, $C' \subseteq C^{\mathrm{NP}}$.*

Proof. The case, when C^{NP} is inconsistent, is evident. Now, assume C^{NP} is consistent. Then, by Theorem 7.11, C is $\sim$-subclassical (i.e., 2 forms a subalgebra of $\mathfrak{A}$, C^{PC} being defined by $\mathcal{A}{\restriction}2$; cf. Corollary 6.4), in which case, by Theorem 8.14, C^{NP} is defined by $\mathcal{L}_6$. Consider the following complementary cases:

- L_5 forms a subalgebra of $\mathfrak{A}^2$.
  Then, as C' does not satisfy the finitary (8.2), by Theorem 2.14, there are some finite set I, some $\overline{\mathcal{C}} \in \mathbf{S}_*(\mathcal{A})^I$ and some subdirect product $\mathcal{D} \in \mathrm{Mod}(C')$ of it not being a model of (8.2), in which case there are some $a \in D$ and some $b \in (D \setminus D^{\mathcal{D}})$ such that $(a \veebar^{\mathfrak{D}} b) \in D^{\mathcal{D}} \ni \sim^{\mathfrak{D}} a$, in which case $a \in \{\frac{1}{2}, 0\}^I$, and so $J \triangleq \{i \in I \mid \pi_i(a) = \frac{1}{2}\} \supseteq \{i \in I \mid \pi_i(b) = 0\} \neq \varnothing$. Then, given any $\bar{a} \in A^2$, set $(a_0 \| a_1) \triangleq ((J \times \{a_0\}) \cup ((I \setminus J) \times \{a_1\})) \in A^I$. In this way, $D \ni a = (\frac{1}{2} \| 0)$. Consider the following complementary subcases:
  - $J = I$,
    Then, $D \ni a = (I \times \{\frac{1}{2}\})$, in which case, as $I = J \neq \varnothing$, by Lemma 8.18(iii)**b)**, $\{\langle x, I \times \{x\}\rangle \mid x \in A\}$ is an embedding of $\mathcal{A}$ into $\mathcal{D}$, and so, by (2.16), $\mathcal{A}$ is a model of C', for $\mathcal{D}$ is so. In this way, $C' \subseteq C \subseteq C^{\mathrm{NP}}$.
  - $J \neq I$,
    Then, as $J \neq \varnothing \neq (I \setminus J)$, by Lemma 8.18(iii)**c)**, $\{\langle\langle x, y\rangle, (x \| y)\rangle \mid \langle x, y\rangle \in L_6\}$ is an embedding of $\mathcal{L}_6$ into $\mathcal{D}$, in which case, by (2.16), $\mathcal{L}_6$ is a model of C', for $\mathcal{D}$ is so, and so $C' \subseteq C^{\mathrm{NP}}$.
- L_5 does not form a subalgebra of $\mathfrak{A}^2$.
  Then, $\mathcal{L}'_5 = \mathcal{L}_6$, for $(L_6 \setminus L_5) = \{\langle \frac{1}{2}, 0\rangle\}$ is a singleton, while $L_6 \supseteq L_5$ forms a subalgebra of $\mathfrak{A}^2$, because 2 forms a subalgebra of $\mathfrak{A}$. And what is more, by Theorem 8.15 and Lemma 8.18(ii), we have $C^{\mathrm{DMP}} \subseteq C^{\mathrm{PC}} = C^{\mathrm{MP}}$, in which case (2.8) is not satisfied in C', and so, by Lemma 8.19, we get $C' \subseteq C^{\mathrm{NP}}$. □

Finally, by Lemmas 4.7, 4.8, 8.2, 8.18, 8.19, Corollaries 3.7, 5.4, 6.4, 6.6, 8.20, Theorems 7.11, 8.14, 8.15 and Remark 2.8(i)**d)**, we eventually get:

Theorem 8.21. *Suppose C is [not] non-$\sim$-subclassical — i.e., 2 is [not] non-$\mathfrak{A}$-closed — and (not) non-implicative [i.e., (n)either $K_{3,0}$ (n)or K_4 forms a subalgebra of $\mathfrak{A}^2$, while L_5 is ({not}) non-$\mathfrak{A}^2$-closed ({whereas C^{DMP} is ⟨not⟩ defined by $\mathcal{L}_5$})]. Then, the following hold:*

(i) *[({⟨some of⟩})] extensions of C form the* $(2[+2(\{+1\langle+1\rangle\})])$*-element chain* $C \subsetneq C^{\mathrm{NP}} = [\mathrm{Cn}^{\omega}_{\mathcal{L}_6} \subsetneq] C^{\mathrm{DMP}} = [(\{\langle\subsetneq\rangle\, \mathrm{Cn}^{\omega}_{\mathcal{L}_5} \subsetneq\})](C^{\mathrm{INP}} =) C^{\mathrm{MP}} = [C^{\mathrm{PC}} = \mathrm{Cn}^{\omega}_{\mathcal{A}\restriction 2} \subsetneq]\, \mathrm{IC}$ *[({⟨others being simultaneously extensions of C^{DMP} and sublogics of $\mathrm{Cn}^{\omega}_{\mathcal{L}_5}$⟩})];*

(ii) $C[\cup(C^{\mathrm{PC}}(\cap(C^{\mathrm{NP}}\{\cup\, \mathrm{Cn}^{\omega}_{\mathcal{L}_5}\})))]$ *is the structural completion of C.*

In view of Corollary 4.6, the item (ii) of this theorem exhausts the issue of finite matrix semantics of the structural completions of $\sim$-paraconsistent three-valued Σ-logics with subclassical negation $\sim$ as well as lattice conjunction and disjunction. And what is more, its item (i) subsumes the particular results, thus providing a generic insight into these, obtained *ad hoc* for LP in [16] as well as for arbitrary three-valued expansions (cf. Corollary 4.18) of both LA and HZ in [20] (cf. [17] for HZ as such) — in this connection, recall that the underlying algebra of the characteristic $\Sigma_{+,\sim}$-matrix $\mathcal{HZ}$ of HZ is a $(\wedge, \vee)$-lattice with zero $\frac{1}{2}$ and unit 1 as well as $\sim^{\mathfrak{HZ}} \frac{1}{2} = \frac{1}{2}$, in which case it is a $(\vee^{\sim}, \wedge^{\sim})$-lattice with zero 0 and unit $\frac{1}{2}$, and so $\mathcal{HZ}$, being $\diamond$- neither conjunctive nor disjunctive, for any $\diamond \in \Sigma_+$, is still both $\vee^{\sim}$-conjunctive and $\wedge^{\sim}$-disjunctive, thus becoming a *non-artificial* instance of a $\sim$-paraconsistent three-valued Σ-logic with subclassical negation $\sim$ as well as lattice conjunction and disjunction but with unit rather $\frac{1}{2}$ than 1, and, in this way, justifying regarding such an extraordinary (at the first sight) case.

As a matter of fact, the condition of having *lattice* conjunction and disjunction is essential for the above advanced results to hold, as it is demonstrated in Subsubsection 8.3.1 covering P^1.

8.2. **Paracomplete disjunctive logics.** In general, we have:

Lemma 8.22. *Suppose C is maximally $(\veebar, \sim)$-paracomplete. Then, it is structurally complete.*

Proof. In that case, any extension C' of C such that $C'(\varnothing) = C(\varnothing)$ is $(\veebar, \sim)$-paracomplete as well, and so equal to C, as required. □

Lemma 8.23. *Let $\mathcal{K}'_3$ be the submatrix of $\mathcal{A}^2$ generated by $K_3 \triangleq K_{3,1}$ and C' the logic of $\mathcal{K}'_3$. Suppose C is both $\veebar$-disjunctive and $(\veebar, \sim)$-paracomplete (viz. $\mathcal{A}$ is so; cf. Lemma 4.7) as well as $\sim$-subclassical. Then, C' is an axiomatically-equivalent extension of C, in which case it is $(\veebar, \sim)$-paracomplete, and so inferentially so(in particular, C' is a proper sublogic of C^{PC}). Moreover, (i)⇒ [⇔](ii)⇔(iii)⇔(iv)⇔(v), where:*

(i) *$\mathcal{A}$ is implicative;*
(ii) *$\langle 1, 0\rangle \in K'_3$ [and C has a theorem];*
(iii) *$K'_3 \nsubseteq K_4$ [and C has a theorem];*
(iv) *[both] neither K_3 nor K_4 forms a subalgebra of $\mathfrak{A}^2$ [and C has a theorem];*
(v) *$C \neq C'$ [has a theorem].*

Proof. In that case, $\mathcal{A}$ is truth-singular, while, by Remark 2.8(i)**d)**, C is not $\sim$-classical, and so, by Corollary 6.4, 2 forms a subalgebra of $\mathfrak{A}$, while, by (2.16), C' is an extension of C. And what is more, as $\pi_0[K_3] = A$, $(\pi_0 \restriction K'_3) \in \hom^{\mathrm{S}}(\mathcal{K}'_3, \mathcal{A})$, in which case, by (2.17), $C'(\varnothing) = C(\varnothing)$, and so C' (viz., $\mathcal{K}'_3$) is $(\veebar, \sim)$-paracomplete. Hence, as $\mathcal{K}'_3$ is truth-non-empty, for $\langle 1, 1\rangle \in K_3$, it (viz., C') is inferentially $(\veebar, \sim)$-paracomplete, in which case C' is inferentially consistent, and so, by Remark 2.8(i)**d)** and Theorem 7.12, is a proper sublogic of C^{PC}.

Next, assume $\mathcal{A}$ is $\sqsupset$-implicative, in which case, since $D^{\mathcal{A}} = \{1\}$, $(\frac{1}{2} \sqsupset^{\mathfrak{A}} 0) = 1$ and, as 2 forms a subalgebra of $\mathfrak{A}$, $(1 \sqsupset^{\mathfrak{A}} 0) = 0$, and so $\langle 1, 0\rangle = (\langle \frac{1}{2}, 1\rangle \sqsupset^{\mathfrak{A}^2} \langle 0, 0\rangle) \in K'_3$, for $K'_3 \supseteq K_3 \supseteq \{\langle \frac{1}{2}, 1\rangle, \langle 0, 0\rangle\}$ forms a subalgebra of $\mathfrak{A}^2$. Thus, (i)⇒(ii) holds [in view of (2.6)].

[Conversely, assume (ii) holds, in which case, by the following claim, there is some $\phi \in \mathrm{Fm}^1_\Sigma$ such that $\phi^{\mathfrak{A}}(\frac{1}{2}) = 1$, while $\phi^{\mathfrak{A}}(1) = 0$:

Claim 8.24. *Suppose $\mathcal{A}$ is truth-singular, while C has a theorem, whereas $\langle 1, 0\rangle \in K'_3$. Then, there is some $\phi \in \mathrm{Fm}^1_\Sigma$ such that $\phi^{\mathfrak{A}}(\frac{1}{2}) = 1$, while $\phi^{\mathfrak{A}}(1) = 0$.*

Proof. In that case, there is some $\varphi \in \mathrm{Fm}^3_\Sigma$ such that $\varphi^{\mathfrak{A}}(\frac{1}{2}, 1, 0) = 1$, while $\varphi^{\mathfrak{A}}(1, 1, 0) = 0$, and so we have $\psi \triangleq \varphi[x_2/{\sim}x_1] \in \mathrm{Fm}^2_\Sigma$ such that $\psi^{\mathfrak{A}}(\frac{1}{2}, 1) = 1$, while $\psi^{\mathfrak{A}}(1, 1) = 0$. Take any $\zeta \in (\mathrm{Fm}^1_\Sigma \cap C(\varnothing)) \neq \varnothing$, in view of the structurality of C. Then, $\phi \triangleq \psi[x_1/\zeta] \in \mathrm{Fm}^1_\Sigma$, while $\phi^{\mathfrak{A}}(\frac{1}{2}) = 1$, whereas $\phi^{\mathfrak{A}}(1) = 0$, for $\mathcal{A}$ is truth-singular. □

Then, $\xi \triangleq (\phi \veebar {\sim}x_0) \in \mathrm{Fm}^1_\Sigma$, in which case $\mathcal{A}$, being truth-singular and $\veebar$-disjunctive, is ξ-negative, and so (i) is by Remark 2.8(i)**c)**.]

Further, (ii)⇔(iii)⇔(iv) is by Lemma 8.2(ii)⇔(v)⇔(vi) with $i = 1$.

Finally, assume (ii) holds. We prove that $C' \neq C$, by contradiction. For suppose $C' = C$, in which case $\mathcal{A}$ is a finite consistent truth-non-empty $\veebar$-disjunctive simple (in view of Theorem 4.12(iv)⇒(i)) model of $C' \supseteq C$, being, in its turn, weakly $\veebar$-disjunctive, and so being $\mathcal{K}'_3$. Then, by Corollary 2.12 and Remark 2.7(iii), there is some submatrix $\mathcal{D}$ of $\mathcal{K}'_3$, being a strict surjective homomorphic counter-image of $\mathcal{A}$, in which case, by (2.16) and Remark 2.8(ii), it is both truth-non-empty, $(\veebar, \sim)$-paracomplete and $\veebar$-disjunctive, for $\mathcal{A}$ is so, and so $D^{\mathcal{D}} = \{\langle 1, 1\rangle\}$, while there is some $a \in D$ such that $D \ni b \triangleq (a \veebar^{\mathfrak{A}^2} {\sim}^{\mathfrak{A}^2} a) \notin D^{\mathcal{D}} = \{\langle 1, 1\rangle\}$. On the other hand, since $\pi_1[K_3] = 2$ forms a subalgebra of $\mathfrak{A}$, in which case $\pi_1[D] \subseteq \pi_1[K'_3] \subseteq 2$, by the truth-singularity and $\veebar$-disjunctivity of $\mathcal{A}$, we have $\pi_1(b) = 1$, in which case $\pi_0(b) \neq 1$, and so we have the following two exhaustive cases:

- $\pi_0(b) = \frac{1}{2}$.
  Then, as $\langle 0, 0\rangle = \sim^{\mathfrak{A}^2}\langle 1, 1\rangle \in D$, we have $K_3 \subseteq D$, in which case, by (ii), we get $\langle 1, 0\rangle \in D$, and so $\langle 0, 1\rangle = \sim^{\mathfrak{A}^2}\langle 1, 0\rangle \in D$.
- $\pi_0(b) = 0$.
  Then, we also have $\langle 1, 0\rangle = \sim^{\mathfrak{A}^2}\langle 0, 1\rangle \in D$.

Thus, anyway, $M_2 \subseteq (D \setminus D^{\mathcal{D}})$, while, by the $\veebar$-disjunctivity of $\mathcal{A}$, $(\langle 0, 1\rangle \veebar^{\mathfrak{A}^2} \langle 1, 0\rangle) = \langle 1, 1\rangle \in D^{\mathcal{D}}$. This contradicts to the $\veebar$-disjunctivity of $\mathcal{D}$. Thus, (v) holds. Conversely, assume $\langle 1, 0\rangle \notin K'_3$, in which case $(\pi_0 \restriction B) \in \hom^{\mathrm{S}}_{\mathrm{S}}(\mathcal{K}'_3, \mathcal{A})$, and so $C' = C$, by (2.16), as required. $\Box$

Lemma 8.25. *Suppose C is $\veebar$-disjunctive (viz. $\mathcal{A}$ is so; cf. Lemma 4.7), while, providing C is $\sim$-subclassical, either K_3 or K_4 forms a subalgebra of $\mathfrak{A}^2$. Then, C has no proper inferentially $(\veebar, \sim)$-paracomplete extension.*

Proof. Let C' be an inferentially $(\veebar, \sim)$-paracomplete (and so inferentially consistent) extension of C, in which case $(x_1 \veebar \sim x_1) \notin T \triangleq C'(x_0) \ni x_0$, while, by the structurality of C', $\langle \mathfrak{Fm}^{\omega}_{\Sigma}, T\rangle$ is a model of C' (in particular, of C), and so is its $(\veebar, \sim)$-paracomplete truth-non-empty finitely-generated submatrix $\mathcal{B} \triangleq \langle \mathfrak{Fm}^2_{\Sigma}, \mathrm{Fm}^2_{\Sigma} \cap T\rangle$, in view of (2.16), whereas C is [inferentially] $(\veebar, \sim)$-paracomplete (viz., $\mathcal{A}$ is so), in which case, since $\mathcal{A}$ is weakly $\veebar$-disjunctive and $1 \in D^{\mathcal{A}}$, and so $((1/0) \veebar^{\mathfrak{A}} \sim^{\mathfrak{A}}(1/0)) = ((1/0) \veebar^{\mathfrak{A}} (0/1)) \in D^{\mathcal{A}}$, we have $(\frac{1}{2} \veebar^{\mathfrak{A}} \sim^{\mathfrak{A}} \frac{1}{2}) \notin D^{\mathcal{A}}$, and so $\mathcal{A}$ is truth-singular.

Then, in case C is not $\sim$-subclassical, by Theorem 7.14, we have $C' = C$. Now, assume C is $\sim$-subclassical, in which case either K_3 or K_4 forms a subalgebra of $\mathfrak{A}^2$, and so $(\frac{1}{2} \veebar^{\mathfrak{A}} \sim^{\mathfrak{A}} \frac{1}{2}) = \frac{1}{2}$, for, otherwise, we would have $(\frac{1}{2} \veebar^{\mathfrak{A}} \sim^{\mathfrak{A}} \frac{1}{2}) = 0$, in which case we would get $(\langle \frac{1}{2}, 1\rangle \veebar^{\mathfrak{A}^2} \sim^{\mathfrak{A}^2}\langle \frac{1}{2}, 1\rangle) = \langle 0, 1\rangle \notin K_4 \supseteq K_3$, and so neither $K_3 \ni \langle \frac{1}{2}, 1\rangle$ nor K_4 would form a subalgebra of $\mathfrak{A}^2$. Further, by Lemma 2.10, there are some finite set I, some $\overline{\mathcal{C}} \in \mathbf{S}(\mathcal{A})^I$ and some subdirect product $\mathcal{D}$ of it, being a strict homomorphic counter-image of a strict homomorphic image of $\mathcal{B}$, and so a $(\veebar, \sim)$-paracomplete (in particular, consistent, in which case $I \neq \varnothing$) truth-non-empty model of C', in view of (2.16), for $\mathcal{B}$ is so. Hence, $C' \subseteq C$, by (2.16), Lemma 8.23(v)⇒(iv) and the following claim:

Claim 8.26. *Let I be a finite set, $\overline{\mathcal{C}} \in \mathbf{S}(\mathcal{A})^I$ and $\mathcal{D}$ a truth-non-empty $(\veebar, \sim)$-paracomplete subdirect product of it. Suppose both C is $\veebar$-disjunctive (viz., $\mathcal{A}$ is so; cf. Lemma 4.7) and either $(\frac{1}{2} \veebar^{\mathfrak{A}} \sim^{\mathfrak{A}} \frac{1}{2}) = \frac{1}{2}$ or $(I \times \{\frac{1}{2}\}) \in D$. Then, $\mathcal{A}$ is embeddable into $\mathcal{D}$, if $(I \times \{\frac{1}{2}\}) \in D$, and $\mathcal{K}'_3$ is embeddable into $\mathcal{D}$, otherwise.*

Proof. Then, by (2.16), $\mathcal{D} \in \mathrm{Mod}(C)$, in which case C is $(\veebar, \sim)$-paracomplete, for $\mathcal{D}$ is so, and so is $\mathcal{A}$. Therefore, $\mathcal{A}$, being $\veebar$-disjunctive with $1 \in D^{\mathcal{A}}$, is truth-singular, and so not $\sim$-paraconsistent, in which case, by Claim 4.17, D contains both $a \triangleq (I \times \{1\})$ and $b \triangleq (I \times \{0\})$. Consider the following complementary cases:

- $(I \times \{\frac{1}{2}\}) \in D$,
  in which case, as $I \neq \varnothing$, for $\mathcal{D}$, being $(\veebar, \sim)$-paracomplete, is consistent, $\{\langle e, I \times \{e\}\rangle \mid e \in A\}$ is an embedding of $\mathcal{A}$ into $\mathcal{D}$.
- $(I \times \{\frac{1}{2}\}) \notin D$,
  in which case $(\frac{1}{2} \veebar^{\mathfrak{A}} \sim^{\mathfrak{A}} \frac{1}{2}) = \frac{1}{2}$, and so $((1/0/\frac{1}{2}) \veebar^{\mathfrak{A}} \sim^{\mathfrak{A}}(1/0/\frac{1}{2})) = (1/1/\frac{1}{2})$, for $\mathcal{A}$ is $\veebar$-disjunctive and $D^{\mathcal{A}} = \{1\}$. Hence, as $\mathcal{D}$ is $(\veebar, \sim)$-paracomplete, there is some $c \in D$ such that $d \triangleq (c \veebar^{\mathfrak{D}} \sim^{\mathfrak{D}} c) \notin D^{\mathcal{D}}$, in which case $d \in (D \cap \{\frac{1}{2}, 1\}^I) \subseteq D \not\ni (I \times \{\frac{1}{2}\})$, and so $I \neq J \triangleq \{i \in I \mid \pi_i(d) = \frac{1}{2}\} \neq \varnothing$.

Given any $\bar{e} \in A^2$, set $(e_0\|e_1) \triangleq ((J \times \{e_0\}) \cup ((I \setminus J) \times \{e_1\})) \in A^I$. In this way, $D \ni a = (1\|1)$, $D \ni b = (0\|0)$ and $D \ni d = (\frac{1}{2}\|1)$. Then, as $J \neq \varnothing \neq (I \setminus J)$ and $\{(x\|y) \mid \langle x, y\rangle \in K_3\} \subseteq D$, $\{\langle\langle x, y\rangle, (x\|y)\rangle \mid \langle x, y\rangle \in K'_3\}$ is an embedding of $\mathcal{K}'_3$ into $\mathcal{D}$. □

Thus, $C' = C$, as required. □

By Lemmas 8.23(vi)⇒(v), 8.25, Corollaries 2.15, 2.13(ii)⇒(i), 6.4 and Remark 2.8(i)**d)**, we first get the following effective algebraic criterion of the maximal inferential $(\veebar, \sim)$-paracompleteness of $\veebar$-disjunctive $(\veebar, \sim)$-paracomplete Σ-logics with subclassical negation $\sim$ (cf. Corollary 4.6):

Theorem 8.27. *Suppose C is $\veebar$-disjunctive and $(\veebar, \sim)$-paracomplete (viz., $\mathcal{A}$ is so; cf. Lemma 4.7). Then, C has no proper axiomatic/inferentially $(\veebar, \sim)$-paracomplete extension (i.e., C is maximally axiomatically/inferentially $(\veebar, \sim)$-paracomplete)/" iff either 2 does not form a subalgebra of $\mathfrak{A}$ or either K_3 or K_4 forms a subalgebra of $\mathfrak{A}^2$".*

And what is more, we have the following effective algebraic criterion of their structural completeness:

Theorem 8.28. *Suppose C is $\veebar$-disjunctive and $(\veebar, \sim)$-paracomplete (viz., $\mathcal{A}$ is so; cf. Lemma 4.7). Then, the following are equivalent:*

(i) *C is structurally complete;*
(ii) *C [has a theorem and] is maximally $(\veebar, \sim)$-paracomplete;*
(iii) *C has a theorem and, providing it is $\sim$-subclassical, either K_3 or K_4 forms a subalgebra of $\mathfrak{A}^2$ (i.e., C {viz., $\mathcal{A}$} is not implicative; cf. Lemmas 4.8 and 8.23(i)⇔(iv));*
(iv) *both $\{\frac{1}{2}\}$ does not form a subalgebra of $\mathfrak{A}$ and either 2 does not form a subalgebra of $\mathfrak{A}$ or either K_3 or K_4 forms a subalgebra of $\mathfrak{A}^2$.*

Proof. First, (i)⇒(iii) is by Remark 2.4 and Lemma 8.23(iv)⇒(v). Next, as $\mathcal{A}$ is then truth-singular, (iii)⇔(iv) is by Corollaries 2.13(i)⇔(iv) and 6.4. Further, in case C has a theorem, any extension of it does so, and so is $(\veebar, \sim)$-paracomplete iff it is inferentially so. Therefore, (iii)⇒(ii) is by Lemma 8.25. Finally, (ii)⇒(i) is by Lemma 8.22. □

Lemma 8.29. *Suppose C is $\veebar$-disjunctive and $(\veebar, \sim)$-paracomplete (viz., $\mathcal{A}$ is so; cf. Lemma 4.7). Then, C^{EM} is $\sim$-classical, whenever C is $\sim$-subclassical, in which case $C^{\mathrm{EM}} = C^{\mathrm{PC}}$, and inconsistent, otherwise.*

Proof. Then, by Remark 2.8(i)**d)**,(ii), C is not $\sim$-classical, while there is a non-$(\veebar, \sim)$-paracomplete submatrix of $\mathcal{A}$ iff 2 forms a subalgebra of $\mathfrak{A}$, in which case $\mathcal{A}{\restriction}2$ is the only non-$(\veebar, \sim)$-paracomplete submatrix of $\mathcal{A}$. In this way, Corollaries 2.15 and 6.4 complete the argument. □

Finally, by (2.14), Remarks 2.3, 2.5, 2.6, 2.8(i)**d)**,(ii), Lemmas 4.7, 8.23, 8.25, 8.29, Corollaries 2.13(i)⇔(iv), 6.4 and Theorem 7.12, we also get:

Theorem 8.30. *Suppose C is both $\veebar$-disjunctive, $(\veebar, \sim)$-paracomplete and [not] $\sim$-subclassical as well as has a/no theorem. Then, proper (arbitrary/"merely non-pseudo-axiomatic") extensions of C form the four-element diamond (resp., two-element chain) [resp., $(2(-1))$-element chain] depicted at Figure 1 (with merely solid circles) [(and) with solely big circles] iff either C is not $\sim$-subclassical or, otherwise, either K_3 or K_4 forms a subalgebra of $\mathfrak{A}^2$ {"that is,"/"in which case" C is not implicative}, $\mathrm{IC}_{\langle/+0\rangle}$ $|[=]C^{\mathrm{EM}}_{\langle/+0\rangle}$ being $\veebar$-disjunctive, relatively axiomatized by $(\langle/x_0 \vdash\rangle(x_1|(x_1 \veebar \sim x_1))$ and defined by $(\varnothing|([\varnothing\cap]\{\mathcal{A}{\restriction}([A\cup]2)\}))\langle/ \cup \{\mathcal{A}{\restriction}\{\frac{1}{2}\}\}\rangle$, respectively.*

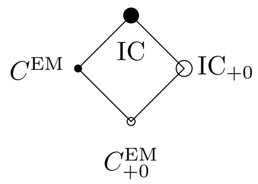

FIGURE 1. The lattice of proper extensions of C.

Perhaps, most representative subclassical instances of this discussion are those three-valued expansions of $\mathbb{K}_3$, which are ones by $\langle$"at least one"/none of$\rangle$ solely classical constants $\bot$ and $\top$ interpreted by 0 and 1, respectively, as instances $\langle$with/without theorems$\rangle$ with $K_{4[-1]}$ [not] forming a subalgebra of $\mathfrak{A}^2$, {the implication-free fragment of} G_3, as non-purely-inferential instances with $K_{3[+1]}$ [not] forming a subalgebra of $\mathfrak{A}^2$, and Ł_3 (as an implicative instance; cf. Example 7 of [19]). In this way, those of these instances, which are neither purely-inferential nor implicative, show that, as opposed to $\sim$-paraconsistent three-valued Σ-logics with subclassical negation $\sim$, the structural completeness of $\veebar$-disjunctive $(\veebar, \sim)$-paracomplete ones, though equally implying (even, being equivalent to) their maximal $(\veebar, \sim)$-paracompleteness, does not subsume absence of their $\sim$-classical extensions.

8.2.1. *Implicative paracomplete logics.* A Σ-matrix/-logic is said to be *[/maximally] $\sqsupset$-implicatively $\sim$-paracomplete*, provided the rule:

$$\{\sim^i x_0 \sqsupset \sim^{1-i} x_0 \mid i \in 2\} \vdash x_0 \tag{8.22}$$

is not true/satisfied in it [/and it has no proper $\sqsupset$-implicatively $\sim$-paracomplete extension], in which case it is "truth-non-empty and"/inferentially consistent. (Clearly, any $\sqsupset$-implicative $\sim$-negative/-classical Σ-matrix/-logic is not $\sqsupset$-implicatively $\sim$-paracomplete/, in view of Lemma 4.8.) By C^{INPC} we denote the least $\sqsupset$-implicatively non-$\sim$-paracomplete extension of C, that is, the extension of C relatively axiomatized by (8.22).

Throughout this subsubsection, it is supposed that C is both $\sqsupset$-implicative, $\veebar$-disjunctive and $(\veebar, \sim)$-paracomplete (viz., $\mathcal{A}$ is so; cf. Lemmas 4.7, 4.8), in which case $(\{\frac{1}{2}, \sim^{\mathfrak{A}}\frac{1}{2}\} \cap D^{\mathcal{A}}) = \varnothing$ (in particular, $\mathcal{A}$ is truth-singular), and so $(\frac{1}{2} \sqsupset^{\mathfrak{A}} \sim^{\mathfrak{A}}\frac{1}{2}) = 1 = (\sim^{\mathfrak{A}}\frac{1}{2} \sqsupset^{\mathfrak{A}} \frac{1}{2})$. In particular, $\mathcal{A}$ is $\sqsupset$-implicatively $\sim$-paracomplete (and so is C), for (8.22) is not true in it under $[x_0/\frac{1}{2}]$. Let $\top \triangleq (x_0 \supset x_0)$ and $\bot \triangleq \sim\top$ be secondary nullary connectives and $(\neg x_0) \triangleq (x_0 \supset \bot)$ a secondary unary connective of Σ. Then, as $\mathcal{A}$ is truth-singular, we have $\top^{\mathfrak{A}}(a) = 1$, for all $a \in A$, in which case $\bot^{\mathfrak{A}}(a) = 0$, and so $\mathcal{A}$ is $\neg$-negative. Hence, by Remark 2.8(i)**a)**, it is $\veebar^{\neg}$-conjunctive. And what is more, we have:

Theorem 8.31. *C is maximally $\sqsupset$-implicatively $\sim$-paracomplete.*

Proof. Let C' be an $\sqsupset$-implicatively $\sim$-paracomplete extension of C, in which case $x_0 \notin T \triangleq C'(\{\sim^i x_0 \sqsupset \sim^{1-i} x_0 \mid i \in 2\})$, while, by the structurality of C', $\langle \mathfrak{Fm}^{\omega}_{\Sigma}, T\rangle$ is a model of $C' \supseteq C$, and so is its finitely-generated $\sqsupset$-implicatively $\sim$-paracomplete submatrix $\mathcal{B} \triangleq \langle \mathfrak{Fm}^1_{\Sigma}, T \cap \mathrm{Fm}^1_{\Sigma}\rangle$, in view of (2.16). Hence, by Lemma 2.10, there are some finite set I, some $\overline{\mathcal{C}} \in \mathbf{S}_*(\mathcal{A})^I$ and some subdirect product $\mathcal{D} \in \mathbf{H}^{-1}(\mathbf{H}(\mathcal{B}))$ of it, in which case, by (2.16), $\mathcal{D}$ is an $\sqsupset$-implicatively $\sim$-paracomplete (and so both consistent and truth-non-empty) model of $C' \supseteq C$, and so, if $\mathcal{D}$ was not $(\veebar, \sim)$-paracomplete, then it would be a consistent truth-non-empty model of C^{EM}, in which case its logic C'' would be a[n inferentially] consistent extension of C^{EM}, and so, by Lemmas 4.8, 8.29 and Corollary 3.7,

C'' would be both $\sim$-classical and $\sqsupset$-implicative as well as $\sqsupset$-implicatively $\sim$-paracomplete, contrary to the fact that any $\sqsupset$-implicative $\sim$-classical Σ-logic is not $\sqsupset$-implicatively $\sim$-paracomplete. Therefore, $\mathcal{D}$ is $(\veebar, \sim)$-paracomplete. And what is more, since it is $\sqsupset$-implicatively $\sim$-paracomplete, there must be some $a \in D$ such that $\{a \sqsupset^{\mathfrak{D}} \sim^{\mathfrak{D}} a, \sim^{\mathfrak{D}} a \sqsupset^{\mathfrak{D}} a\} \subseteq D^{\mathcal{D}}$, in which case $D \ni a = (I \times \{\frac{1}{2}\})$, and so, by Claim 8.26, $\mathcal{A}$ is embeddable into $\mathcal{D}$. Thus, by (2.16), $C' = C$, as required. $\Box$

8.2.1.1. Extensions of logics with lattice conjunction and disjunction. Throughout this paragraph, it is also supposed that $\mathfrak{A}$ is a $(\bar{\wedge}, \veebar)$-lattice, in which case it is a chain (and so distributive) one with unit 1, for $\mathcal{A}$ is three-valued, truth-singular and $\veebar$-disjunctive, and so $\mathcal{A}$ is $\bar{\wedge}$-conjunctive.

Lemma 8.32. *Let I be a finite set, $\overline{\mathcal{C}} \in \mathbf{S}_*(\mathcal{A})^I$, and $\mathcal{D}$ an $\sqsupset$-implcatively non-$\sim$-paracomplete consistent subdirect product of $\overline{\mathcal{C}}$. Then, 2 forms a subalgebra of $\mathfrak{A}$, while* $\hom(\mathcal{D}, \mathcal{A}{\restriction}2) \neq \varnothing$.

Proof. In that case, by (2.6) and Corollary 2.13(iv)⇒(i), $\mathcal{D}$ is truth-non-empty. Therefore, if, for each $i \in I$, $\frac{1}{2}$ was in $\pi_i[D] = C_i$, then, by Lemma 8.12 with $J = I$, $a \triangleq (I \times \{\frac{1}{2}\})$ would be in D, in which case (8.22) would not be true in $\mathcal{D}$ under $[x_0/a]$, for $I \neq \varnothing$, because $\mathcal{D}$ is consistent, and so $\mathcal{D}$ would be $\sqsupset$-implicatively $\sim$-paracomplete. Hence, there is some $i \in I$ such that $\frac{1}{2} \notin B \triangleq \pi_i[D] = C_i \neq \varnothing$, in which case $B \subseteq 2$ forms a subalgebra of $\mathfrak{A}$, and so $B = 2$, while $(\pi_i{\restriction}D) \in \hom(\mathcal{D}, \mathcal{A}{\restriction}B)$. $\Box$

Theorem 8.33. *The following are equivalent:*

(i) C^{INPC} *is consistent;*
(ii) C^{INPC} *is $\sim$-subclassical;*
(iii) C^{INPC} *is $(\veebar, \sim)$-paracomplete;*
(iv) C *is $\sim$-subclassical (i.e., 2 forms a subalgebra of $\mathfrak{A}$; cf. Corollary 6.4),*

in which case C^{INPC} is defined by $\mathcal{K}_6 \triangleq (\mathcal{A} \times (\mathcal{A}{\restriction}2))$, and so $C^{\mathrm{INPC}}(\varnothing) = C(\varnothing)$.

Proof. First, (i/iii) is a particular case of (ii[i]/iv), respectively/, for $\mathcal{A}$ is $(\veebar, \sim)$-paracomplete. Next, (iv)⇒(ii) is by the consistency of $\mathcal{K}_6$, (2.6) and Theorem 7.12. Further, (i)⇒(iv) is by (2.6) and Theorem 7.14.

Finally, assume (i,iv) hold. Then, by Theorem 2.14 with $\mathsf{M} \triangleq \{\mathcal{A}\}$ and $\mathsf{K} \triangleq \mathbf{P}^{\mathrm{SD}}_{\omega}(\mathbf{S}_*(\mathsf{M}))$, C^{INPC} is finitely-defined by the class S of all consistent members of $\mathsf{K} \cap \mathrm{Mod}(C')$. Consider any $\mathcal{D} \in \mathsf{S} \subseteq \mathrm{Mod}\,(8.22)$, in which case there are some finite set I and some $\overline{\mathcal{C}} \in \mathbf{S}_*(\mathcal{A})^I$ such that $\mathcal{D}$ is a subdirect product of it, and so, by Lemma 8.32, $\hom(\mathcal{D}, \mathcal{A}{\restriction}2) \neq \varnothing$. Take any $g \in \hom(\mathcal{D}, \mathcal{A}{\restriction}2)$. Consider any $a \in (D \setminus D^{\mathcal{D}})$. Then, there is some $i \in I$ such that $\pi_i(a) \notin D^{\mathcal{A}}$, while $f \triangleq (\pi_i{\restriction}D) \in \hom(\mathcal{D}, \mathcal{A})$, in which case $h \triangleq (f \times g) \in J \triangleq \hom(\mathcal{D}, \mathcal{K}_6)$, while $h(a) \notin D^{\mathcal{K}_6}$, and so $(\prod J) \in \hom_{\mathrm{S}}(\mathcal{D}, \mathcal{K}_6^J)$. Thus, by (2.16), C^{INPC} is finitely-defined by the finite $\mathcal{K}_6$, in which case it, being finitary, for (8.22) is so, while $\mathcal{A}$ is finite, is defined by $\mathcal{K}_6$, and so (2.17) and the fact that $(\pi_0{\restriction}K_6) \in \hom^{\mathrm{S}}(\mathcal{K}_6, \mathcal{A})$ complete the argument. $\Box$

Lemma 8.34. *Suppose C is $\sim$-subclassical. Then, $\mathcal{K}_6$ is generated by $K_1 \triangleq \{\langle \frac{1}{2}, 1\rangle\}$.*

Proof. Let $\mathcal{D}$ be the submatrix of $\mathcal{K}_6$ generated by K_1. Then, by (2.6) and the truth-singularity of $\mathcal{A}$, we have $D \ni a \triangleq (\langle \frac{1}{2}, 1\rangle \sqsupset^{\mathfrak{D}} \langle \frac{1}{2}, 1\rangle) = \langle 1, 1\rangle$, in which case $D \ni \sim^{\mathfrak{D}} a = \langle 0, 0\rangle$, and so $K_3 = (\Delta_2 \cup K_1) \subseteq D$. Hence, by Lemma 8.23(i)⇒[(ii)] and Claim 8.24, $D \ni b \triangleq \langle 1, 0\rangle$, in which case $D \ni \sim^{\mathfrak{D}} b = \langle 0, 1\rangle$, while there is some $\phi \in \mathrm{Fm}^1_\Sigma$ such that $\phi^{\mathfrak{A}}(\frac{1}{2}) = 1$, whereas $\phi^{\mathfrak{A}}(1) = 0$, in which case we have

$\varphi \triangleq (x_0 \bar{\wedge} \phi) \in \mathrm{Fm}^1_\Sigma$ such that $D \ni \varphi^{\mathfrak{D}}(\langle \frac{1}{2}, 1\rangle) = \langle \frac{1}{2}, 0\rangle$, for $\mathfrak{A}$ is a $(\bar{\wedge}, \underline{\vee})$-lattice with unit 1, and so $K_6 = (K_3 \cup M_2 \cup \{\langle \frac{1}{2}, 0\rangle\}) \subseteq D$, as required. □

As $\sim^{\mathfrak{A}} 1 = 0$, by Lemma 8.34, we immediately have:

Corollary 8.35. *Suppose C is $\sim$-subclassical, while $\sim^{\mathfrak{A}} \frac{1}{2} = \frac{1}{2}$. Then, $\mathcal{K}_6$ is generated by $\{\langle \frac{1}{2}, 0\rangle\}$.*

Lemma 8.36. *Let $\mathcal{K}'_5$ be the submatrix of $\mathcal{A}^2$ generated by $K_5 \triangleq (K_6 \setminus K_1)$. Suppose C is $\sim$-subclassical. Then, $\mathcal{K}'_5$ is a model of any $(\underline{\vee}, \sim)$-paracomplete extension of C^{INPC}. In particular, the structural completion of $C^{[\mathrm{INPC}]}$ is defined by $\mathcal{K}'_5$.*

Proof. Then, by Theorem 8.33, $C^{\mathrm{INPC}}(\varnothing) = C(\varnothing)$. while C^{INPC} is defined by $\mathcal{K}_6$. Let C' be any $(\underline{\vee}, \sim)$-paracomplete (in particular, having same theorems, for C is so) extension of C^{INPC}, in which case, by (2.6), (2.12) $\notin T \triangleq C'(\varnothing) \supseteq C(\varnothing) \ni$ (2.6), while, by the structurality of C', $\langle \mathfrak{Fm}^\omega_\Sigma, T\rangle$ is a model of C' (in particular, of C), and so is its truth-non-empty $(\underline{\vee}, \sim)$-paracomplete finitely-generated submatrix $\mathcal{B} \triangleq \langle \mathfrak{Fm}^1_\Sigma, T \cap \mathrm{Fm}^1_\Sigma\rangle$, in view of (2.16). Therefore, by Lemma 2.10, there are some finite set I, some $\overline{\mathcal{C}} \in \mathbf{S}(\mathcal{A})^I$ and some subdirect product $\mathcal{D}$ of it, being a strict homomorphic counter-image of a strict homomorphic image of $\mathcal{B}$, and so a $(\underline{\vee}, \sim)$-paracomplete (in particular, consistent, in which case $I \neq \varnothing$) truth-non-empty model of C' (in particular, of (8.22)), in view of (2.16), for $\mathcal{B}$ is so. Then, since $\mathcal{A}$, being truth-singular, for it is $\underline{\vee}$-disjunctive and $(\underline{\vee}, \sim)$-paracomplete, is not $\sim$-paraconsistent, by Claim 4.17, D contains both $a \triangleq (I \times \{1\})$ and $b \triangleq (I \times \{0\})$. Moreover, if D contained $c \triangleq (I \times \{\frac{1}{2}\})$, then (8.22) would not be true in $\mathcal{D}$ under $[x_0/c]$, for $I \neq \varnothing$. Consider the following complementary cases:

- $(\frac{1}{2} \underline{\vee}^{\mathfrak{A}} \sim^{\mathfrak{A}} \frac{1}{2}) = \frac{1}{2}$.
  Then, by Lemma 8.34, $\mathcal{K}_6$ is generated by $K_3 \supseteq K_1$. Hence, as $c \notin D$, by Lemma 8.26, $\mathcal{K}_6$ is embeddable into $\mathcal{D} \in \mathrm{Mod}(C')$, and so, by (2.16), a model of C', and so is its submatrix $\mathcal{K}'_5$.
- $(\frac{1}{2} \underline{\vee}^{\mathfrak{A}} \sim^{\mathfrak{A}} \frac{1}{2}) \neq \frac{1}{2}$,
  in which case $\sim^{\mathfrak{A}} \frac{1}{2} = 0$, and so $\flat^{\mathfrak{A}}_{\bar{\wedge}} = \frac{1}{2}$. Moreover, $\mathfrak{A}{\restriction}2$ is a $(\bar{\wedge}, \underline{\vee})$-lattice with zero 0, for $\mathfrak{A}$ is that with unit 1. And what is more, since $\mathcal{D}$ is $(\underline{\vee}, \sim)$-paracomplete, $J \triangleq \{i \in I \mid \frac{1}{2} \in \pi_i[D]\} \neq \varnothing$. Given any $x, y \in A$, set $(x \| y) \triangleq ((J \times \{x\}) \cup ((I \setminus J) \times \{y\})) \in A^I$. Then, $D \ni (a/b) = (1/0\|1/0)$. Moreover, by Lemma 2.2, $\mathfrak{D}$, being finite, is a $(\bar{\wedge}, \underline{\vee})$-lattice with zero $d \triangleq (\frac{1}{2}\|0) \in D$. Hence, $I \neq J$, for $c \notin D$. Then, $D \ni [\sim^{\mathfrak{D}}]\sim^{\mathfrak{D}} d = ([1-]0\|[1-]1)$. Thus, $\{(x\|y) \mid \langle x, y\rangle \in K_5\} \subseteq D$. In this way, since $J \neq \varnothing \neq (I \setminus J)$, $\{\langle\langle x, y\rangle, (x\|y)\rangle \mid \langle x, y\rangle \in K'_5\}$ is an embedding of $\mathcal{K}'_5$ into $\mathcal{D} \in \mathrm{Mod}(C')$, in which case, by (2.16), $\mathcal{K}'_5 \in \mathrm{Mod}(C')$.

Moreover, as $K_5 \subseteq K_6$, while $\pi_0[K_5] = A$, $\mathcal{K}'_5$ is a submatrix of $\mathcal{K}_6$, while $(\pi_0{\restriction}K'_5) \in \hom^{\mathrm{S}}(\mathcal{K}'_5, \mathcal{A})$, in which case, by (2.16) and (2.17), $\mathcal{K}'_5$ is a model of $C^{[\mathrm{INPC}]}$ such that $\mathrm{Cn}^\omega_{\mathcal{K}'_5}(\varnothing) = C^{[\mathrm{INPC}]}(\varnothing)$, and so the structural completion of $C^{[\mathrm{INPC}]}$ is defined by it. □

Lemma 8.37. *Suppose 2 forms a subalgebra of $\mathfrak{A}$ (i.e., C is $\sim$-subclassical; cf. Corollary 6.4). Then, (i)⇔(ii)⇔(iii)⇐(iv)⇔(v)⇔(vi), where:*

(i) *$((2.12)[x_0/(2.12)]) \in C(\varnothing)$;*
(ii) *neither $\sim^{\mathfrak{A}} \frac{1}{2} = \frac{1}{2}$ nor $0 \leq^{\mathfrak{A}}_{\bar{\wedge}} \frac{1}{2}$;*
(iii) *$(\frac{1}{2}(\underline{\vee})^{\mathfrak{A}} \sim^{\mathfrak{A}} \frac{1}{2}) \neq \frac{1}{2}$;*
(iv) *K_5 forms a subalgebra of $\mathfrak{A}^2$;*
(v) *$\mathfrak{K}_6$ is not generated by $K_2 \triangleq \{\langle \frac{1}{2}, 0\rangle, \langle 1, 1\rangle\}$;*
(vi) *neither $\mathfrak{K}_6$ is generated by K_2 nor $\mathfrak{A}$ has a discriminator.*

In particular, K_5 does not form a subalgebra of $\mathfrak{A}^2$, whenever $\sim^{\mathfrak{A}}\frac{1}{2} = \frac{1}{2}$.

Proof. First, (i)⇔(ii)⇔(iii) are immediate. Next, if $(\frac{1}{2}\veebar^{\mathfrak{A}}\sim^{\mathfrak{A}}\frac{1}{2}) = \frac{1}{2}$, then $(\langle\frac{1}{2},0\rangle\veebar^{\mathfrak{A}}\sim^{\mathfrak{A}}\langle\frac{1}{2},0\rangle) = \langle\frac{1}{2},1\rangle \notin K_5$, in which case $K_5 \ni \langle\frac{1}{2},0\rangle$ does not form a subalgebra of $\mathfrak{A}^2$, and so (iv)⇒(iii) holds.

Further, assume K_5 does not form a subalgebra of A^2, in which case $K_5' = K_6$. Let $\mathfrak{B}$ be the subalgebra of $\mathfrak{A}^2$ generated by $K_2 \subseteq K_6$. Then, in case $\sim^{\mathfrak{A}}\frac{1}{2} = \frac{1}{2}$, by Corollary 8.35, $\mathfrak{K}_6$ is generated by $K_2 \ni \langle\frac{1}{2},0\rangle$. Otherwise, $\sim^{\mathfrak{A}}\frac{1}{2} = 0$, in which case $B \ni \sim^{\mathfrak{B}}\langle\frac{1}{2},0\rangle = \langle 0,1\rangle$, and so $K_5 = (K_2 \cup \Delta_2 \cup M_2) \subseteq B$, in which case $K_6 = K_5' \subseteq B \subseteq K_6$, and so $B = K_6$. Thus, (v)⇒(iv) holds. Furthermore, (v) is a particular case of (vi). Finally, if $\mathfrak{K}_6$ is generated by $K_2 \subseteq K_5$, then $K_6 \subseteq K_5' \subseteq K_6$, in which case $K_5' = K_6$, and so K_5 does not form a subalgebra of $\mathfrak{A}^2$, for, otherwise, $K_5' \supseteq K_5$ would be equal to $K_5 \neq K_6$. Likewise, if $\mathfrak{A}$ has a discriminator δ, then so does $\mathfrak{A}{\restriction}2$, in which case δ is a congruence-permutation term for both $\mathfrak{A}$ and $\mathfrak{A}{\restriction}2$, being simple, and so so K_5 does not form a subalgebra of $\mathfrak{A}^2$, for, otherwise, $\mathfrak{D} \triangleq (\mathfrak{A}^2{\restriction}K_5)$ would be a subdirect product of $\langle\mathfrak{A},\mathfrak{A}{\restriction}2\rangle$, in which case, by Lemma 2.1, it would be isomorphic to either $\mathfrak{K}_6$ or $\mathfrak{A}$ or $\mathfrak{A}{\restriction}2$, and so $5 = |D|$ would be equal to either $6 = |K_6|$ or $3 = |A|$ or $2 = |2|$. Thus, (iv)⇒(vi) holds, as required. □

Next, by $C^{\mathrm{INPC+DN}}$ we denote the extension of C^{INPC} relatively axiomatized by the *Double Negation* rule:

$$\sim\sim x_0 \vdash x_0, \tag{8.23}$$

the inverse one being satisfied in C.

Lemma 8.38. *Suppose C is $\sim$-subclassical $\langle$i.e., 2 forms a subalgebra of $\mathfrak{A}$; cf. Corollary 6.4$\rangle$ (while $\sim^{\mathfrak{A}}\frac{1}{2} \neq \frac{1}{2}$ {in particular, K_5 forms a subalgebra of $\mathfrak{A}^2$; cf. Lemma 8.37}). Then,* (8.23) *is true in $\mathcal{A}{\restriction}2$ ({as well as in $\mathcal{K}_5 \triangleq (\mathcal{A}^2{\restriction}K_5)$} but not true in $\mathcal{K}_6$).*

Proof. First, (8.23) is true in $\mathcal{A}{\restriction}2$, for $\sim^{\mathfrak{A}}\sim^{\mathfrak{A}}i = i$, for all $i \in 2$, and so in $(\mathcal{A}{\restriction}2)^2(\{= (\mathcal{K}_5{\restriction}2^2)\})$. (Finally, using the truth-singularity of $\mathcal{A}$, it is routine checking that it is [not] true in $\mathcal{A}^2$ under $[x_0/\langle\frac{1}{2},[1-]0\rangle]$). □

Theorem 8.39. *Suppose C is $\sim$-subclassical. Then, the following are equivalent:*

(i) *C^{INPC} has a proper $(\veebar,\sim)$-paracomplete extension;*
(ii) *C^{INPC} is not structurally complete;*
(iii) *$C^{\mathrm{INPC}} \neq C^{\mathrm{INPC+DN}} \neq C^{\mathrm{PC}}$;*
(iv) *$C^{\mathrm{INPC+DN}} \neq C^{\mathrm{INPC}}$ is $(\veebar,\sim)$-paracomplete;*
(v) *K_5 forms a subalgebra of $\mathfrak{A}^2$,*

in which case $\frac{1}{2} \leq^{\mathfrak{A}}_{\wedge} 0 = \sim^{\mathfrak{A}}\frac{1}{2}$, while the logic of $\mathcal{K}_5$ has no proper $(\veebar,\sim)$-paracomplete extension, whereas it is the structural completion of $C^{[\mathrm{INPC}]}$.

Proof. First, since $(K_6 \setminus K_5) = K_1$ is a singleton, K_5 forms a subalgebra of $\mathfrak{A}^2$ iff $K_5' = [\neq]K_{5[+1]}$. In this way, (2.6), (2.16), Remark 2.8(i)**d)**,(ii), Corollaries 6.4, Lemmas 8.29, 8.36, 8.37, 8.38, Theorem 8.33 and the linearity of the poset $\langle A, \leq^{\mathfrak{A}}_{\wedge}\rangle$ complete the argument. □

Given any $\varphi \in \mathrm{Fm}^1_\Sigma$, by $C^{\mathrm{INPC}+\varphi}$ we denote the extension of C^{INPC} relatively axiomatized by:

$$\varphi \vdash x_0. \tag{8.24}$$

In this way, $C^{\mathrm{INPC+DN}} = C^{\mathrm{INPC}+(\sim\sim x_0)}$. A *characteristic formula for* a $K \subseteq (K_6 \setminus \Delta_2)$ is any $\varphi \in \mathrm{Fm}^1_\Sigma$ such that, for all $a \in K_6$, it holds that $(a \in K) \Rightarrow (\varphi^{\mathfrak{K}_6}(a) = \langle 1,1\rangle) \Rightarrow (a \neq \langle 0,0\rangle)$, in which case, unless $K = \varnothing$, (8.24) is not true in $\mathcal{K}_6$ under $[x_0/a]$, where $a \in K \not\ni \langle 1,1\rangle$.

Lemma 8.40. *Let φ be any characteristic formula for K_1 (in particular, $\varphi = {\sim}{\sim}x_0$, unless ${\sim}^{\mathfrak{A}}\frac{1}{2} = \frac{1}{2}$). Then, C^{INPC} has no proper extension not satisfying* (8.24) *(in particular,* (8.23)*, unless ${\sim}^{\mathfrak{A}}\frac{1}{2} = \frac{1}{2}$). In particular, $C^{\mathrm{INPC}+\varphi} = C^{\mathrm{INPC}+\mathrm{DN}}$, unless ${\sim}^{\mathfrak{A}}\frac{1}{2} = \frac{1}{2}$.*

Proof. The case, when C^{INPC} is inconsistent, is evident. Now, assume it is consistent. Then, by Theorem 8.33, C is ${\sim}$-subclassical (i.e., 2 forms a subalgebra of $\mathfrak{A}$; cf. Corollary 6.4), while C^{INPC} is defined by $\mathcal{K}_6$. Consider any extension C' of C^{INPC} not satisfying (8.24). Then, by Theorem 2.14, there are some set I, some $\overline{\mathcal{C}} \in \mathbf{S}(\mathcal{A})^I$ and some subdirect product $\mathcal{D} \in \mathrm{Mod}(C') \subseteq \mathrm{Mod}\,(8.22)$ of it, not satisfying (8.24), for this is finitary, in which case there is some $a \in (D \setminus D^{\mathcal{D}})$ such that $\varphi^{\mathfrak{D}}(a) \in D^{\mathcal{D}} = \{I \times \{1\}\}$, and so $(I \times \{1\}) \neq a \in \{\frac{1}{2}, 1\}^I$, for $\varphi^{\mathfrak{A}}(0) \neq 1$. Hence, $\varnothing \neq J \triangleq \{i \in I \mid \pi_i(a) = \frac{1}{2}\} \neq I$, for, otherwise, (8.22) would not be true in $\mathcal{D}$ under $[x_0/a]$. Given any $\bar{a} \in A^2$, set $(a_0\|a_1) \triangleq ((J \times \{a_0\}) \cup ((I \setminus J) \times \{a_1\})) \in A^I$. Then, $D \ni a \triangleq (\frac{1}{2}\|1)$. In this way, as $J \neq \varnothing \neq (I \setminus J)$, by Lemma 8.34, $\{\langle \bar{a}, (a_0\|a_1)\rangle \mid \bar{a} \in K_6\}$ is an embedding of $\mathcal{K}_6$ into $\mathcal{D} \in \mathrm{Mod}(C')$, in which case, by (2.16), $\mathcal{K}_6 \in \mathrm{Mod}(C')$, and so $C' = C^{\mathrm{INPC}}$. Finally, the fact that (8.24) is not true in $\mathcal{K}_6$ under $[x_0/\langle \frac{1}{2}, 1\rangle]$ completes the argument. □

Finally, combining (2.6), Theorems 7.12, 7.14, 8.31, 8.33, 8.39, Lemmas 8.29, 8.37, 8.40 and Corollary 6.4, we get:

Theorem 8.41. *Suppose C is [not] non-${\sim}$-subclassical [i.e., 2 forms a subalgebra of $\mathfrak{A}$, while K_5 is (not) non-$\mathfrak{A}^2$-closed (in which case ${\sim}^{\mathfrak{A}}\frac{1}{2} \neq \frac{1}{2}$, whereas $C^{\mathrm{INPC}+\mathrm{DN}}$ is {not} defined by $\mathcal{K}_5$)]. Then, the following hold:*

(i) *[({some of})] extensions of C form the $(2[+2(+1\{+1\})])$-element chain $C \subsetneq C^{\mathrm{INPC}} = [\mathrm{Cn}^{\omega}_{\mathcal{K}_6} \subsetneq (C^{\mathrm{INPC}+\mathrm{DN}} = \{\subsetneq\}\, \mathrm{Cn}^{\omega}_{\mathcal{K}_5} \subsetneq)]C^{\mathrm{EM}} = [C^{\mathrm{PC}} = \mathrm{Cn}^{\omega}_{\mathcal{A}\restriction 2} \subsetneq\,]\,\mathrm{IC}$ [({others being simultaneously extensions of $C^{\mathrm{INPC}+\mathrm{DN}}$ and sublogics of $\mathrm{Cn}^{\omega}_{\mathcal{K}_5}$})];*

(ii) *C is [(not) pre]maximally $(\veebar, {\sim})$-paracomplete;*

(iii) *$C^{[INPC]}[(\cup\, \mathrm{Cn}^{\omega}_{\mathcal{K}_5})]$ is the structural completion of C.*

The []-optional ()-non-optional particular case of Theorem 8.41, covering the both ${\sim}$-subclassical and implicative (cf. Example 7 of [19]) Ł$_3$ [7], equally ensues from Theorem 3.3 of [16], Corollaries 4.6, 4.12 and 4.13 with $\Lambda = \{\barwedge, \veebar\}$ of [20], the linearity of the poset $\langle A, \leq^{\mathfrak{A}}_{\barwedge}\rangle$, the fact that $\mathfrak{A}$ is generated by $\{\frac{1}{2}, 1\}$, for $0 = {\sim}^{\mathfrak{A}}1$, while it is a $(\barwedge, \veebar)$-lattice with unit 1, whereas $(a = (a \sqsupset^{\mathfrak{A}} a)) \Leftrightarrow (a \in D^{\mathcal{A}})$, for all $a \in A$, Lemma 8.37(v)⇒(iv) and the following observation:

Remark 8.42. [Suppose $0 \leq^{\mathfrak{A}}_{\barwedge} \frac{1}{2}$, while 2 forms a subalgebra of $\mathfrak{A}$ (i.e., C is ${\sim}$-subclassical; cf. Corollary 6.4). Then, $\mathfrak{A}$ is is a $(\barwedge, \veebar)$-lattice with unit 1 and zero 0. Moreover,] $\Upsilon \triangleq \{x_0, {\sim}x_0\}$ is a unitary equality determinant for $\mathcal{A}$, because ${\sim}^{\mathfrak{A}}\frac{1}{2} \notin D^{\mathcal{A}} = \{1\} \not\ni 0$, while ${\sim}^{\mathfrak{A}}i = (1 - i)$, for all $i \in 2$, in which case, by the $\sqsupset$-implicativity of $\mathfrak{A}$, $\{\phi \sqsupset \psi \mid (\phi \vdash \psi) \in \varepsilon_{\Upsilon}\}$ is an axiomatic equality determinant for $\mathcal{A}$, and so is $(x_0 \leftrightarrow x_1) \triangleq (\barwedge\langle\barwedge\langle{\sim}^i x_j \sqsupset {\sim}^i x_{1-j}\rangle_{j\in 2}\rangle_{i\in 2})$, in view of the $\barwedge$-conjunctivity of $\mathcal{A}$. Therefore, since $\mathcal{A}$ is $\sqsupset$-implicative and truth-singular, $(x_0 \approx (x_0 \sqsupset x_0))[x_0/((x_0 \leftrightarrow x_1) \sqsupset (x_2 \leftrightarrow x_3))]$ is an implicative system for $\mathfrak{A}$. [And what is more, $(\mathrm{img}\,\neg^{\mathfrak{A}}) \subseteq 2$, for 2 forms a subalgebra of $\mathfrak{A}$, in which case $({\sim}^{\mathfrak{A}} \circ \neg^{\mathfrak{A}}) = \chi^{\mathcal{A}}$, and so $(({\sim}\neg(x_0 \leftrightarrow x_1) \barwedge x_2) \veebar (\neg(x_0 \leftrightarrow x_1) \barwedge x_0))$ is a discriminator for $\mathfrak{A}$.] □

In this connection, recall that it is this alternative argumentation (more specifically, its "discriminator" particular case based upon Corollary 4.12 of [20]) that has been invoked therein to find the lattice of extensions of Ł$_3$ upon the basis of Example B.2 therein. On the other hand, the "discriminator" subcase does not at all exhaust the []-optional ()-non-optional case of Theorem 8.41, in view of the following double counterexample equally showing the possibility of the []-optional ()-optional case of this theorem:

Example 8.43. Let $\Sigma \triangleq (\Sigma_{+,\sim} \cup \{\top\})$, while $\mathcal{A}$ truth-singular with $\sim^{\mathfrak{A}}\frac{1}{2} \triangleq (0[+\frac{1}{2}])$, $\top^{\mathfrak{A}} \triangleq 1$, $\bar{\wedge} \triangleq \wedge$, $\veebar \triangleq \vee$ and $\frac{1}{2} \leq^{\mathfrak{A}}_{\wedge} 0 \leq^{\mathfrak{A}}_{\wedge} 1$ [whereas $\mathfrak{B}$ the Σ-algebra with $(\mathfrak{B}{\restriction}(\Sigma \setminus \{\sim\})) \triangleq (\mathfrak{D}_{2,01}{\restriction}(\Sigma \setminus \{\sim\}))$ and $\sim^{\mathfrak{B}} \triangleq \Delta_2$]. Then, 2 forms a subalgebra of $\mathfrak{A}$, in which case 2^2 forms a subalgebra of $\mathfrak{A}^2$, and so $K_5 = (2^2 \cup \{\langle \frac{1}{2}, 0\rangle\})$, [though] forming a subalgebra of $(\mathfrak{A}{\restriction}(\Sigma \setminus \{\sim\}))^2$, does [not] form a subalgebra of $\mathfrak{A}^2$, for $\langle 0[+\frac{1}{2}], 1\rangle = \sim^{\mathfrak{A}^2}\langle \frac{1}{2}, 0\rangle$ does [not] belong to K_5, while, by Theorem 6.3, C is $\sim$-subclassical, whereas, by Lemma 8.23[(ii)⇒(i)], $\mathcal{A}$ is implicative, for $\top \in C(\varnothing)$, while $\langle 1, 0\rangle = \sim^{\mathfrak{A}^2}(\langle \frac{1}{2}, 1\rangle \veebar^{\mathfrak{A}^2} \sim^{\mathfrak{A}^2}\top^{\mathfrak{A}^2}) \in K'_3 \supseteq K_3 \supseteq K_1$. [And what is more, $\chi^2_A \in \hom(\mathfrak{A}, \mathfrak{B})$ is surjective. Therefore, if $\mathfrak{A}$ had a discriminator, then this would be a congruence-permutation term for $\mathfrak{B}$, being simple, for it is two-element, in which case, by Lemma 2.1, the subdirect square $\mathfrak{D} \triangleq (\mathfrak{B}^2{\restriction}(2^2 \setminus \{\langle 0, 1\rangle\}))$ of $\mathfrak{B}$ would be isomorphic to either $\mathfrak{B}$ or $\mathfrak{B}^2$, and so $3 = |D|$ would be even.] Thus, anyway, $\mathfrak{A}$ has no discriminator, in view of Lemma 8.37(iv)⇒(vi). □

This — in addition to Subsection 5.5 of [20] — highlights the "non-discriminator" advance of the mentioned study.

In this way, Remarks 2.4, 2.6, Corollary 4.6, Lemmas 4.7, 8.23 and Theorems 8.28 and 8.41 exhaust the issue of structural completions of $(\veebar, \sim)$-paracomplete Σ-logics with subclassical negation $\sim$ as well as lattice conjunction and disjunction $\veebar$.

On the other, the condition of existence of *lattice* conjunction and disjunction is essential for the above advanced results to hold, as it is demonstrated in Subsubsection 8.3.2.

8.3. **Extensions of classically-valued disjunctive conjunctive non-classical logics.** Here, it is supposed that $\mathcal{A}$ is both classically-valued (and so classically-hereditary, in which case C is $\sim$-subclassical; cf. Theorem 6.3) and $\diamond$-conjunctive|-disjunctive (viz., C is so| cf. Lemma 4.7), where $\diamond$ is a (possibly, secondary) binary connective of Σ, in which case $(a \diamond^{\mathfrak{A}} a) = \chi^{\mathcal{A}}(a)$, for all $a \in A$, and so $\mathcal{A}$ is $\neg$-negative, where $(\neg x_0) \triangleq \sim(x_0 \diamond x_0)$, as well as hereditarily-simple (i.e., C is not $\sim$-classical; cf. Corollary 4.13), in which case, by Theorem 3.4(i)⇒(iii), $\mathcal{A}$ has a unary equality determinant ε. Then, by Remark 2.8(i)**a)**, $\mathcal{A}$ is both $\bar{\wedge}$-conjunctive and $\veebar$-disjunctive, where $\bar{\wedge} \triangleq (\diamond|\diamond^{\neg})$ and $\veebar \triangleq (\diamond^{\neg}|\diamond)$, in which case we have secondary nullary connectives $(\bot/\top) \triangleq (x_0(\bar{\wedge}/\veebar)\neg x_0)$ of Σ such that $(\bot/\top)^{\mathfrak{A}}(a) = (0/1)$, for all $a \in A$, while, by Remark 2.8(i)**c)**, $\mathcal{A}$ is $\sqsupset$-implicative, where $\sqsupset \triangleq \sqsupset^{\neg}_{\veebar}$, and so $\{\phi \sqsupset \psi \mid (\phi \vdash \psi) \in \varepsilon\}$ is an axiomatic equality determinant for it.

8.3.1. *Paraconsistent logics.* Here, it is also supposed that $\mathcal{A}$ (viz., C) is $\sim$-paraconsistent (cf. Remark 2.8(i)**d)**), in which case it is false-singular, while $\sim^{\mathfrak{A}}\frac{1}{2} = 1$.

Let $n \in (\omega \setminus 1)$, C_n the finitary (for C, being three-valued, is so) extension of C relatively axiomatized by the finitary rule $R_n \triangleq ((\{\sim x_i \mid i \in n\} \cup \{\veebar \bar{x}_n\}) \vdash x_n)$ and C'_ω the finitary (for both C, being three-valued, is so) extension of C relatively axiomatized by the finitary Σ-calculus $\{R_m \mid m \in (\omega \setminus 1)\}$.

Lemma 8.44. *For any $n \in (\omega \setminus (1(+1)))$, there is a consistent subdirect n-power $\mathcal{A}_n \in \mathrm{Mod}(C)$ of $\mathcal{A}$ such that ($D^{\mathcal{A}_n} = \{n \times \{1\}\}$ and) R_n is [not] true in $\mathcal{A}_{n+1[-1]}$, in which case $\mathcal{A}_{n+1} \in (\mathrm{Mod}(C_n) \setminus \mathrm{Mod}(C_{n+1}))$, and so $C_{n+1} \nsubseteq C_n$.*

Proof. Since $\mathcal{A}$ is classically-valued, the set $A_n \triangleq (2^n \cup \{\{\langle i, \frac{1}{2}\rangle\} \cup ((n \setminus \{i\}) \times \{0\}) \mid i \in n\}) \ni (n \times \{0\})$ forms a subalgebra of $\mathfrak{A}^n$, so we have the consistent {for $n \neq 0$} subdirect n-power $\mathcal{A}_n \triangleq (\mathcal{A}^n{\restriction}A_n) \in \mathrm{Mod}(C)$ {cf. (2.16)} of $\mathcal{A}$ (with

$D^{\mathcal{A}_n} = \{n \times \{1\}\}$, as $n \neq 1$). Then, as $\mathcal{A}$ is $\veebar$-disjunctive, R_n is not true in $\mathcal{A}_n$ under $[x_i/(\{\langle i, \frac{1}{2}\rangle\} \cup ((n \setminus \{i\}) \times \{0\})); x_n/(n \times \{0\})]_{i\in n}$ but is true in $\mathcal{A}_{n+1}$, for $\sim^{\mathfrak{A}}1 = 0$, while, for every $\bar{b} \in (\{\frac{1}{2}, 0\}^{n+1} \cap A_{n+1})^+$, $(\veebar^{\mathfrak{A}^{n+1}}\bar{b}) \in \{\frac{1}{2}, 1\}^{n+1}$ only if, for each $i \in (n+1)$, there is some $j \in (\operatorname{dom}\bar{b})$ such that $\pi_i(b_j) = \frac{1}{2}$ (that is, $b_j = (\{\langle i, \frac{1}{2}\rangle\} \cup (((n+1) \setminus \{i\}) \times \{0\})))$ iff $(A_{n+1} \setminus 2^{n+1}) \subseteq (\operatorname{img}\bar{b})$, and so, for no $\bar{b} \in (\{\frac{1}{2}, 0\}^{n+1} \cap A_{n+1})^n$, $(\veebar^{\mathfrak{A}^{n+1}}\bar{b}) \in \{\frac{1}{2}, 1\}^{n+1}$, because, otherwise, we would have $(n+1) = |A'_{n+1} \setminus 2^{n+1}| \leqslant |\operatorname{img}\bar{b}| \leqslant n$. □

Theorem 8.45. *$\langle C_n\rangle_{i\in n}$ is a strictly increasing countable chain of finitary axiomatically-equivalent (and so consistent) non-$\sim$-paraconsistent (and so proper) extensions of C, and so is C_ω that is not [relatively] finitely-axiomatizable.*

Proof. We use Theorem 2.14 with $\mathsf{K} \triangleq \operatorname{Mod}(C)$ Then, as C is weakly $\veebar$-disjunctive, and so is any $\mathcal{B} \in \mathsf{K}$, whenever R_n is not true in $\mathcal{B}$ under any $v : V_{n+1} \to B$, for every $m \in (\omega \setminus n)$, R_m is not true in $\mathcal{B}$ under $v \cup [x_j/v(x_0); x_m/v(x_n)]_{j\in(m\setminus n)}$. Hence, $\langle C_n\rangle_{i\in n}$ is an increasing denumerable chain of finitary non-$\sim$-paraconsistent extensions of C, for $R_1 = (2.10)$. Moreover, by Claim 8.44, the increasing chain $\langle C_n\rangle_{n\in(\omega\setminus 1)}$ is injective, and so countable, in which case C_ω is a proper extension of C_n, for any $n \in (\omega \setminus 1)$, and so, by the Compactness Theorem for classes of algebraic systems closed under ultra-products (cf. [9]) — in particular, finitary logic model classes, being finitary equality-free universal Horn model classes axiomatized by finitary calculi axiomatizing finitary logics, C_ω is not [relatively] finitely axiomatizable. Finally, by Lemma 8.44, for each $n \in (\omega \setminus 1)$, $\mathcal{A}_{n+1} \in \operatorname{Mod}(C_n)$, while $(\pi_0{\restriction}A_{n+1}) \in \hom^{\mathrm{S}}(\mathcal{A}_{n+1}, \mathcal{A})$, in which case, by (2.17), $C_n \equiv_1 C$, and so $C(\varnothing) \in (\operatorname{img} C_n)$. Hence, $C(\varnothing) \in (\wp(\mathrm{Fm}^{\omega}_{\Sigma}) \cap \bigcap_{n\in(\omega\setminus 1)}(\operatorname{img} C_n)) = (\operatorname{img} C_\omega)$, for C_ω is the join of $\{C_n \mid n \in (\omega \setminus 1)\}$. Thus, $C_\omega \equiv_1 C$, as required. □

As it has been demonstrated in Subsubsection 8.1.1, the condition of $\mathcal{A}$'s being classically-valued cannot be omitted in the formulation of Theorem 8.45. It is remarkable that C_ω, being a consistent extension of C, is a sublogic of C^{PC}, in view of Theorem 7.7(i)⇒(v) and Corollary 7.9, and so, by Theorem 8.45, the infinite chain involved appears intermediate between C^{NP} and C^{PC}, in contrast to Theorem 8.21, unless L_5 forms a subalgebra of $\mathfrak{A}^2$. And what is more, in contrast to Corollary 8.13, we have:

Lemma 8.46. $\mathcal{B} \triangleq \mathcal{A}_2 \in \operatorname{Mod}(C^{\mathrm{MP}}) \subseteq \operatorname{Mod}(C^{\mathrm{NP}})$ *(cf. Lemma 8.44) is a consistent subdirect square of $\mathcal{A}$ such that* $\hom(\mathcal{B}, \mathcal{A}{\restriction}2) = \varnothing$.

Proof. Then, $\mathcal{B} \triangleq \mathcal{A}_2 \in \operatorname{Mod}(C)$ is a consistent subdirect square of $\mathcal{A}$. Moreover, as $2 \notin 2$, $D^{\mathcal{B}} = \{\langle 1, 1\rangle\}$, while, for every $b \in B$, it holds that $(\sim^{\mathfrak{B}}\langle 1, 1\rangle \veebar^{\mathfrak{B}} b) = (\langle 0, 0\rangle \veebar^{\mathfrak{B}} b) \in D^{\mathcal{B}}$ implies $b \in D^{\mathcal{B}}$, in view of the $\veebar$-disjunctivity of $\mathcal{A}$ and the fact that $0 \notin D^{\mathcal{A}}$. Hence, (2.8) is true in $\mathcal{B}$. Finally, let us prove, by contradiction, that $\hom(\mathcal{B}, \mathcal{A}{\restriction}2) = \varnothing$. For suppose $\hom(\mathcal{B}, \mathcal{A}{\restriction}2) \neq \varnothing$. Take any $h \in \hom(\mathcal{B}, \mathcal{A}{\restriction}2)$, in which case $h(\langle 1, 1\rangle) = 1$, for $\langle 1, 1\rangle \in D^{\mathcal{B}}$, while $D^{\mathcal{A}{\restriction}2} = \{1\}$. Therefore, if, for any $a \in \{\langle \frac{1}{2}, 0\rangle, \langle 0, \frac{1}{2}\rangle\} \subseteq B$, it did hold that $h(a) = 1$, we would have $0 = \sim^{\mathfrak{A}}1 = h(\sim^{\mathfrak{B}}a) = h(\langle 1, 1\rangle) = 1$. Hence, $h(\langle \frac{1}{2}, 0\rangle) = 0 = h(\langle 0, \frac{1}{2}\rangle)$. Then, we get $0 = (0 \veebar^{\mathfrak{A}} 0) = h(\langle \frac{1}{2}, 0\rangle \veebar^{\mathfrak{B}} \langle 0, \frac{1}{2}\rangle) = h(\langle 1, 1\rangle) = 1$. This contradiction completes the argument. □

As a consequence, in contrast to Theorem 8.14/8.15, we get:

Corollary 8.47. $C^{\mathrm{NP/MP}}$ *is not defined by* $\mathcal{D} \triangleq ((\mathcal{A} \times (\mathcal{A}{\restriction}2))/(\mathcal{A}{\restriction}2))$.

Proof. By contradiction. For suppose $C^{\mathrm{NP/MP}}$ is defined by $\mathcal{D}$. Then, by Lemma 8.46, $\mathcal{B} \triangleq \mathcal{A}_2 \in \operatorname{Mod}(C^{\mathrm{NP/MP}})$ is a consistent subdirect square of $\mathcal{A}$ such that

line	formula	triple
0	x_0	$\langle 0,1,\frac{1}{2}\rangle$
1	$\sim\sim x_0$	$\langle 0,1,0\rangle$
2	$\neg x_0$	$\langle 1,0,0\rangle$
3	$\sim x_0$	$\langle 1,0,1\rangle$
4	$\neg\neg x_0$	$\langle 0,1,1\rangle$
5	$\neg\neg x_0 \mathbin{\bar\wedge} \sim x_0$	$\langle 0,0,1\rangle$
6	$\sim\sim x_0 \mathbin{\underline\vee} \neg x_0$	$\langle 1,1,0\rangle$
7	$\top$	$\langle 1,1,1\rangle$
8	$\bot$	$\langle 0,0,0\rangle$

TABLE 1. An isomorphism from $\mathcal{F}^1_{\mathcal{A}}$ onto $\mathcal{B}$.

$\hom(\mathcal{B},\mathcal{A}\upharpoonright 2)=\varnothing$, in which case it is finite, for A is so, and so is a finitely-generated consistent model of $C^{\mathrm{NP/MP}}$. Therefore, by Lemmas 2.10, 3.2, 3.3, 3.5 and Remark 2.7, there are some set I, some $\overline{\mathcal{C}}\in\mathbf{S}(\mathcal{D})^I$, some subdirect product $\mathcal{E}$ of it and some injective $g\in\hom^{\mathrm{S}}_{\mathrm{S}}(\mathcal{E},\mathcal{B})$, in which case $\mathcal{E}$ is consistent, for $\mathcal{B}$ is so, and so $I\neq\varnothing$. Then, $((\pi_1/\Delta_2)\circ\pi_i\circ g^{-1})\in\hom(\mathcal{B},\mathcal{A}\upharpoonright 2)=\varnothing$, where $i\in I\neq\varnothing$. This contradiction completes the argument. □

Finally, P^1 collectively with Theorem 8.45 show that, despite of Theorem 8.21, three-valued (even both conjunctive, implicative [and so disjunctive] and subclassical) paraconsistent logics with subclassical negation need not have finitely many (even merely finitary) extensions.

8.3.1.1. The structural completion of P^1. Let $\Sigma\triangleq\{\supset,\sim\}$ with binary $\supset$ and $\mathcal{A}$ both false-singular, $\supset$-implicative and classically-hereditary, in which case it is both $\uplus_{\supset}$-disjunctive and $\sim$-paraconsistent, while $C=P^1$.

Theorem 8.48. *Let $\theta\triangleq\theta^1_{\mathfrak{A}}$ and $\mathcal{D}\triangleq\langle\mathfrak{Fm}^1_\Sigma, C(\varnothing)\cap\mathrm{Fm}^1_\Sigma\rangle$. Then, the structural completion of P^1 is defined by $\mathcal{F}^1_{\mathcal{A}}\triangleq(\mathcal{D}/\theta)$ isomorphic to $\mathcal{B}\triangleq(\mathcal{A}^3\upharpoonright(2^3\cup\{\langle 0,1,\frac{1}{2}\rangle\}))$, an isomorphism from the former onto the latter being given by table 1 (under identification of any $\varphi\in\mathrm{Fm}^1_\Sigma$ with $[\varphi]_\theta$).*

Proof. Then, $\mathfrak{A}$ is generated by the singleton $\{\frac{1}{2}\}$. Hence, by Theorem 3.8, the structural completion of $C=P^1$ is defined by $\mathcal{F}^1_{\mathcal{A}}\triangleq(\mathcal{D}/\theta)$. Given any $a\in A$, let $h_a\in\hom(\mathfrak{Fm}^1_\Sigma,\mathfrak{A})$ extend $[x_0/a]$ and F_9 the set of all Σ-formulas appearing in the second column of Table 1. Then, as $F_9\subseteq\mathrm{Fm}^1_\Sigma$ includes $\{x_0\}$ generating $\mathfrak{Fm}^1_\Sigma$, the latter is equally generated by F_9. Moreover, $h\triangleq((h_0\times h_1)\times h_{\frac{1}{2}})\in\hom(\mathfrak{Fm}^1_\Sigma,\mathfrak{A}^3)$, while $h\upharpoonright F_9$ is given by Table 1 (in particular, $h[F_9]=B$), in which case $h\in\hom(\mathfrak{Fm}^1_\Sigma,\mathfrak{B})$ is surjective, for B forms a subalgebra of $\mathfrak{A}^3$, because $\mathcal{A}$ is classically-valued, whereas

$$\hom(\mathfrak{Fm}^1_\Sigma,\mathfrak{A})=\{h_a\mid a\in A\},\tag{8.25}$$

in which case $\theta=(\bigcap_{a\in A}(\ker h_a))=(\ker h)$, and so, by the Homomorphism Theorem, $e\triangleq(h\circ\nu_\theta^{-1})$ is an isomorphism from $\mathfrak{F}^1_{\mathfrak{A}}=\mathfrak{F}^1_{\mathcal{A}}$ onto $\mathfrak{B}$. And what is more, as C is consistent, $x_0\notin C(\varnothing)$, in which case, for every $\varphi\in D^{\mathcal{D}}$, $h(\varphi)=\langle 1,1,1\rangle$, because $\mathcal{A}$ is classically-valued, and so $h\in\hom^{\mathrm{S}}_{\mathrm{S}}(\mathcal{D},\mathcal{B})$, for $D^{\mathcal{B}}=\{\langle 1,1,1\rangle\}$, while $1\in D^{\mathcal{A}}$ (in particular, $h^{-1}[D^{\mathcal{B}}]\subseteq C(\varnothing)$, in view of (8.25)). Thus, e is an isomorphism from $\mathcal{F}^1_{\mathcal{A}}$ onto $\mathcal{B}$, in which case $h=(e\circ\nu_\theta)$, and so $F^1_{\mathcal{A}}=(F_9/\theta)$, for $h[F_9]=B$, while $F^1_{\mathcal{A}}=e^{-1}[B]$, as required. □

8.3.2. *Paracomplete logics.* Here, it is also supposed that $\mathcal{A}$ (viz., C) is $(\veebar,\sim)$-paracomplete (cf. Remark 2.8(i)**d)**), in which case it is truth-singular, while $\sim^{\mathfrak{A}}\frac{1}{2}=0$, and so $\{\frac{1}{2}\}$ does not form a subalgebra of $\mathfrak{A}$.

Let $n\in(\omega\setminus 1)$, C'_n the finitary (for C, being three-valued, is so) extension of C relatively axiomatized by the finitary rule $R'_n\triangleq((\{\sim\sim x_i\mid i\in n\}\cup\{\veebar(\neg\circ\bar{x}_n)\})\vdash x_n)$ and C'_ω the finitary (for both C, being three-valued, is so) extension of C relatively axiomatized by the finitary Σ-calculus $\{R'_m\mid m\in(\omega\setminus 1)\}$.

Lemma 8.49. *For any $n\in(\omega\setminus(1(+1)))$, there is a consistent subdirect n-power $\mathcal{A}'_n\in\mathrm{Mod}(C)$ of $\mathcal{A}$ such that R'_n is [not] true in $\mathcal{A}'_{n+1[-1]}$, in which case $\mathcal{A}'_{n+1}\in(\mathrm{Mod}(C'_n)\setminus\mathrm{Mod}(C'_{n+1}))$, and so $C'_{n+1}\nsubseteq C'_n$.*

Proof. Since $\mathcal{A}$ is classically-valued, the set $A'_n\triangleq(2^n\cup\{\{\langle i,\frac{1}{2}\rangle\}\cup((n\setminus\{i\})\times\{1\})\mid i\in n\})\ni(n\times\{0\})$ forms a subalgebra of $\mathfrak{A}^n$, so we have the consistent {for $n\neq 0$} subdirect n-power $\mathcal{A}'_n\triangleq(\mathcal{A}^n{\restriction}A'_n)\in\mathrm{Mod}(C)$ {cf. (2.16)} of $\mathcal{A}$. Then, as $\mathcal{A}$ is $\veebar$-disjunctive, R'_n is not true in $\mathcal{A}'_n$ under $[x_i/(\{\langle i,\frac{1}{2}\rangle\}\cup((n\setminus\{i\})\times\{1\}));x_n/(n\times\{0\})]_{i\in n}$ but is true in $\mathcal{A}'_{n+1}$, for $\sim^{\mathfrak{A}}\sim^{\mathfrak{A}}0=0$, while $\neg^{\mathfrak{A}}1=0$, as 2 forms a subalgebra of $\mathfrak{A}$, in which case, for every $\bar{b}\in(\{\frac{1}{2},1\}^{n+1}\cap A'_{n+1})^+$, $(\veebar^{\mathfrak{A}^{n+1}}(\neg^{\mathfrak{A}^{n+1}}\circ\bar{b}))=((n+1)\times\{1\})$ only if, for each $i\in(n+1)$, there is some $j\in(\mathrm{dom}\,\bar{b})$ such that $\pi_i(b_j)=\frac{1}{2}$ (that is, $b_j=(\{\langle i,\frac{1}{2}\rangle\}\cup(((n+1)\setminus\{i\})\times\{1\})))$ iff $(A'_{n+1}\setminus 2^{n+1})\subseteq(\mathrm{img}\,\bar{b})$, and so, for no $\bar{b}\in(\{\frac{1}{2},1\}^{n+1}\cap A'_{n+1})^n$, $(\veebar^{\mathfrak{A}^{n+1}}(\neg^{\mathfrak{A}^{n+1}}\circ\bar{b}))=((n+1)\times\{1\})$, because, otherwise, we would have $(n+1)=|A'_{n+1}\setminus 2^{n+1}|\leqslant|\,\mathrm{img}\,\bar{b}|\leqslant n$. $\square$

Theorem 8.50. *$\langle C'_n\rangle_{i\in n}$ is a strictly increasing countable chain of finitary axiomatically-equivalent (and so $(\veebar,\sim)$-paracomplete {in particular, consistent}) proper (and so $\sqsupset$-implicatively non-$\sim$-paracomplete) extensions of C, and so is C'_ω that is not [relatively] finitely-axiomatizable.*

Proof. We use Theorem 2.14 with $\mathsf{K}\triangleq\mathrm{Mod}(C)$ tacitly. Then, as C is weakly $\veebar$-disjunctive, and so is any $\mathcal{B}\in\mathsf{K}$, for any $n\in(\omega\setminus 1)$, whenever R'_n is not true in $\mathcal{B}$ under any $v:V_{n+1}\to B$, for every $m\in(\omega\setminus n)$, R_m is not true in $\mathcal{B}$ under $v\cup[x_j/v(x_0);x_m/v(x_n)]_{j\in(m\setminus n)}$. Hence, $\langle C'_n\rangle_{i\in n}$ is an increasing denumerable chain of finitary proper extensions of C, for R'_1 is not true in $\mathcal{A}$ under $[x_i/\frac{1}{2}]_{i\in 2}$. Moreover, by Lemma 8.49, the increasing chain $\langle C'_n\rangle_{n\in(\omega\setminus 1)}$ is injective, and so countable, in which case C'_ω is a proper extension of C'_n, for any $n\in(\omega\setminus 1)$, and so, by the Compactness Theorem for classes of algebraic systems (in particular, Σ-matrices) closed under ultra-products (cf. [9]) — in particular, finitary logic model classes, being finitary equality-free universal Horn model classes axiomatized by finitary calculi axiomatizing finitary logics, C'_ω is not [relatively] finitely axiomatizable. And what is more, by Lemma 8.49, for each $n\in(\omega\setminus 1)$, $\mathcal{A}'_{n+1}\in\mathrm{Mod}(C'_n)$, while $(\pi_0{\restriction}A'_{n+1})\in\hom^{\mathrm{S}}(\mathcal{A}'_{n+1},\mathcal{A})$, in which case, by (2.17), $C'_n\equiv_1 C$, and so $C(\varnothing)\in(\mathrm{img}\,C'_n)$. Hence, $C(\varnothing)\in(\wp(\mathrm{Fm}^\omega_\Sigma)\cap\bigcap_{n\in(\omega\setminus 1)}(\mathrm{img}\,C'_n))=(\mathrm{img}\,C'_\omega)$, for C'_ω is the join of $\{C'_n\mid n\in(\omega\setminus 1)\}$. Thus, $C'_\omega\equiv_1 C$. Finally, Theorem 8.31 completes the argument. $\square$

As it has been demonstrated in Subsubsection 8.2.1.1, the condition of $\mathcal{A}$'s being classically-valued cannot be omitted in the formulation of Theorem 8.50. It is remarkable that C_ω, being a consistent extension of C, is a sublogic of C^{PC}, in view of Theorem 7.7(i)⇒(v) and Corollary 7.9, and so, by Theorem 8.50, the infinite chain involved appears intermediate between C^{INPC} and C^{PC}, in contrast to Theorem 8.41, unless K_5 forms a subalgebra of $\mathfrak{A}^2$. And what is more, in contrast to Lemma 8.32, we have:

Lemma 8.51. $\mathcal{B}\triangleq\mathcal{A}'_2\in\mathrm{Mod}(C'_1)\subseteq\mathrm{Mod}(C^{\mathrm{INPC}})$ *(cf. Lemma 8.49 and Theorem 8.50) is a consistent subdirect square of $\mathcal{A}$ such that* $\hom(\mathcal{B},\mathcal{A}{\restriction}2)=\varnothing$.

line	formula	triple
0	x_0	$\langle 0,1,\frac{1}{2}\rangle$
1	$\sim\sim x_0$	$\langle 0,1,1\rangle$
2	$\neg x_0$	$\langle 1,0,1\rangle$
3	$\sim x_0$	$\langle 1,0,0\rangle$
4	$\neg\neg x_0$	$\langle 0,1,0\rangle$
5	$\sim\sim x_0 \mathbin{\overline{\wedge}} \neg x_0$	$\langle 0,0,1\rangle$
6	$\neg\neg x_0 \mathbin{\underline{\vee}} \sim x_0$	$\langle 1,1,0\rangle$
7	$\top$	$\langle 1,1,1\rangle$
8	$\bot$	$\langle 0,0,0\rangle$

TABLE 2. An isomorphism from $\mathcal{F}^1_{\mathcal{A}}$ onto $\mathcal{B}'$.

Proof. By contradiction. For suppose $\hom(\mathcal{B},\mathcal{A}\restriction 2) \neq \varnothing$. Take any $h \in \hom(\mathcal{B}, \mathcal{A}\restriction 2)$, in which case $h(\langle 1,1\rangle) = 1$, for $\langle 1,1\rangle \in D^{\mathcal{B}}$, while $D^{\mathcal{A}\restriction 2} = \{1\}$, and so $0 = \sim^{\mathfrak{A}} 1 = h(\sim^{\mathfrak{B}}\langle 1,1\rangle) = h(\langle 0,0\rangle)$. Therefore, if, for any $a \in \{\langle \frac{1}{2},1\rangle, \langle 1,\frac{1}{2}\rangle\} \subseteq B$, it did hold that $h(a) = 0$, we would have $1 = \sim^{\mathfrak{A}} 0 = h(\sim^{\mathfrak{B}} a) = h(\langle 0,0\rangle) = 0$. Hence, $h(\langle \frac{1}{2},1\rangle) = 1 = h(\langle 1,\frac{1}{2}\rangle)$. Then, we get $1 = (1 \mathbin{\overline{\wedge}^{\mathfrak{A}}} 1) = h(\langle \frac{1}{2},1\rangle \mathbin{\overline{\wedge}^{\mathfrak{B}}} \langle 1,\frac{1}{2}\rangle) = h(\langle 0,0\rangle) = 0$. This contradiction completes the argument. □

As a consequence, in contrast to Theorem 8.33, we get:

Corollary 8.52. *C^{INPC} is not defined by $\mathcal{D} \triangleq (\mathcal{A} \times (\mathcal{A}\restriction 2))$.*

Proof. By contradiction. For suppose C^{INPC} is defined by $\mathcal{D}$. Then, by Lemma 8.51, $\mathcal{B} \triangleq \mathcal{A}_2 \in \mathrm{Mod}(C^{\mathrm{INPC}})$ is a consistent subdirect square of $\mathcal{A}$ such that $\hom(\mathcal{B},\mathcal{A}\restriction 2) = \varnothing$, in which case it is finite, for A is so, and so is a finitely-generated consistent model of C^{INPC}. Therefore, by Lemmas 2.10, 3.2, 3.3, 3.5 and Remark 2.7, there are some set I, some $\overline{\mathcal{C}} \in \mathbf{S}(\mathcal{D})^I$, some subdirect product $\mathcal{E}$ of it and some injective $g \in \hom^{\mathrm{S}}_{\mathrm{S}}(\mathcal{E},\mathcal{B})$, in which case $\mathcal{E}$ is consistent, for $\mathcal{B}$ is so, and so $I \neq \varnothing$. Then, $((\pi_1/\Delta_2) \circ \pi_i \circ g^{-1}) \in \hom(\mathcal{B},\mathcal{A}\restriction 2) = \varnothing$, where $i \in I \neq \varnothing$. This contradiction completes the argument. □

Finally, the instance, dual to P^1 and discussed in the next paragraph, collectively with Theorem 8.50 show that, despite of Theorem 8.41, three-valued implicative (even both conjunctive and subclassical) paracomplete logics with subclassical negation need not have finitely many (even merely finitary) extensions.

8.3.2.1. The structural completion of the paracomplete counterpart of P^1. Let $\Sigma \triangleq \{\supset, \sim\}$ with binary $\supset$ and $\mathcal{A}$ both truth-singular, $\supset$-implicative and classically-hereditary, in which case it is both $\uplus_{\supset}$-disjunctive and $(\uplus_{\supset},\sim)$-paracomplete.

Theorem 8.53. *Let $\theta \triangleq \theta^1_{\mathfrak{A}}$ and $\mathcal{D}' \triangleq \langle \mathfrak{Fm}^1_\Sigma, C(\varnothing) \cap \mathrm{Fm}^1_\Sigma\rangle$. Then, the structural completion of C is defined by $\mathcal{F}^1_{\mathcal{A}} \triangleq (\mathcal{D}'/\theta)$ isomorphic to $\mathcal{B}' \triangleq (\mathcal{A}^3\restriction(2^3 \cup \{\langle 0,1,\frac{1}{2}\rangle\}))$, an isomorphism from the former onto the latter being given by table 2 (under identification of any $\varphi \in \mathrm{Fm}^1_\Sigma$ with $[\varphi]_\theta$).*

Proof. Then, $\mathfrak{A}$ is generated by the singleton $\{\frac{1}{2}\}$. Hence, by Theorem 3.8, the structural completion of C is defined by $\mathcal{F}^1_{\mathcal{A}} \triangleq (\mathcal{D}'/\theta)$. Given any $a \in A$, let $h_a \in \hom(\mathfrak{Fm}^1_\Sigma,\mathfrak{A})$ extend $[x_0/a]$ and F'_9 the set of all Σ-formulas appearing in the second column of Table 2. Then, as $F'_9 \subseteq \mathrm{Fm}^1_\Sigma$ includes $\{x_0\}$ generating $\mathfrak{Fm}^1_\Sigma$, the latter is equally generated by F'_9. Moreover, $h \triangleq ((h_0 \times h_1) \times h_{\frac{1}{2}}) \in \hom(\mathfrak{Fm}^1_\Sigma,\mathfrak{A}^3)$, while $h\restriction F'_9$ is given by Table 2 (in particular, $h[F'_9] = B'$), in which case $h \in \hom(\mathfrak{Fm}^1_\Sigma, \mathfrak{B}')$ is surjective, for B' forms a subalgebra of $\mathfrak{A}^3$, because $\mathcal{A}$ is classically-valued, whereas

$$\hom(\mathfrak{Fm}^1_\Sigma, \mathfrak{A}) = \{h_a \mid a \in A\}, \tag{8.26}$$

in which case $\theta = (\bigcap_{a \in A}(\ker h_a)) = (\ker h)$, and so, by the Homomorphism Theorem, $e \triangleq (h \circ \nu_\theta^{-1})$ is an isomorphism from $\mathfrak{F}^1_\mathfrak{A} = \mathfrak{F}^1_\mathcal{A}$ onto $\mathfrak{B}'$. And what is more, for every $\varphi \in D^{\mathcal{D}'}$, $h(\varphi) = \langle 1, 1, 1\rangle$, because $\mathcal{A}$ is truth-singular, in which case $h \in \hom^{\mathrm{S}}_{\mathrm{S}}(\mathcal{D}, \mathcal{B}')$, for $D^{\mathcal{B}'} = \{\langle 1, 1, 1\rangle\}$ (in particular, $h^{-1}[D^{\mathcal{B}'}] \subseteq C(\varnothing)$, in view of (8.25)). Thus, e is an isomorphism from $\mathcal{F}^1_\mathcal{A}$ onto $\mathcal{B}'$, in which case $h = (e \circ \nu_\theta)$, and so $F^1_\mathcal{A} = (F'_9/\theta)$, for $h[F'_9] = B'$, while $F^1_\mathcal{A} = e^{-1}[B']$, as required. □

9. Conclusions

Aside from quite useful general results and their equally illustrative generic applications to infinite classes of particular logics, the paper demonstrates the value of the conception of equality determinant going back to [18].

Among other things, profound connections between the structural completeness of paraconsistent/"disjunctive paracomplete" three-valued logics with subclassical negation and their [pre]maximal paraconsistency/paracompleteness discovered here deserve a particular emphasis within the context of Many-Valued Logic. Likewise, the deep characterizations (in particular, yielding effective algebraic criteria) of implicativity of $\veebar$-disjunctive "$\sim$-paraconsistent and conjunctive"/$(\veebar, \sim)$-paracomplete three-valued $\sim$-subclassical Σ-logics given by Lemma 8.2/8.23, respectively, are equally valuable within the context involved.

Perhaps, most acute problems remained still open within the framework of those $\sim$-paraconsistent/ "implicative $(\veebar, \sim)$-paracomplete" three-valued $\sim$-subclassical Σ-logics with lattice conjunction and disjunction $\veebar$, the direct square of the underlying algebra of whose characteristic matrices have the subalgebras with carrier $(L/K)_5$, are the following quite non-trivial universal problems:

(1) What is a relative axiomatization of the logic of $(\mathcal{L}/\mathcal{K})_5$?
(2) What is the lattice of those extensions of $C^{\mathrm{DMP}/(\mathrm{INPC}+\mathrm{DN})}$, which have the model $(\mathcal{L}/\mathcal{K})_5$?
(3) What is a class of matrices defining $C^{\mathrm{DMP}/(\mathrm{INPC}+\mathrm{DN})}$?

We conjecture that $C^{\mathrm{DMP}/(\mathrm{INPC}+\mathrm{DN})}$ is defined by $(\mathcal{L}/\mathcal{K})_5$. On the other hand, though being technically quite non-trivial, these problems are not especially acute logically, because they deal with rather extraordinary algebraic stipulations not typical of any already known instances.

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