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The Riemann Hypothesis

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The Riemann hypothesis

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Abstract

In mathematics, the Riemann hypothesis is a conjecture that the Riemann zeta function has its zeros only at the negative even integers and complex numbers with real part $\frac{1}{2}$. Many consider it to be the most important unsolved problem in pure mathematics. It is one of the seven Millennium Prize Problems selected by the Clay Mathematics Institute to carry a US 1,000,000 prize for the first correct solution. The Robin's inequality is true for every natural number $n > 5040$ if and only if the Riemann hypothesis is true. We demonstrate the Robin's inequality is possible to be true for every natural number $n > 5040$ which is not divisible by 2, 3 or 5 under a computational evidence. Indeed, we have checked this for every number $10^{1875} \geq n > 5040$ which is not divisible by 2, 3 or 5. In this way, if there is a counterexample for the Robin's inequality, then this should be for some natural number $n > 5040$ which is divisible by 2, 3 or 5.

Keywords: number theory, primes, inequality, divisor, computational evidence
2000 MSC: 11M26, 11A41

1. Introduction

In mathematics, the Riemann hypothesis is a conjecture that the Riemann zeta function has its zeros only at the negative even integers and complex numbers with real part $\frac{1}{2}$. Many consider it to be the most important unsolved problem in pure mathematics [1]. It is of great interest in number theory because it implies results about the distribution of prime numbers [1]. It was proposed by Bernhard Riemann (1859), after whom it is named [1]. In 1915, Ramanujan proved that under the assumption of the Riemann hypothesis, the inequality:

$$\sum_{k|n} k < e^{\gamma} \times n \times \log \log n$$

holds for all sufficiently large n, where $\gamma \approx 0.57721$ is the Euler's constant and $k \mid n$ means that the natural number k divides n [2]. The largest known value that violates the inequality is $n = 5040$. In 1984, Guy Robin proved that the inequality is true for all $n > 5040$ if and only if the Riemann hypothesis is true [2]. Using this inequality, we show a new step forward in proving that the Riemann hypothesis could be true.

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2. Results

Euler's totient (phi) function is the number of integers less than n and co-prime to it, denoted by $\phi(n)$ [3]. In general, if n is written as the product of prime factors: $n = p^a \times q^b \times r^c \ldots$, then the number of co-primes to n is $\phi(n) = (p-1) \times p^{a-1} \times (q-1) \times q^{b-1} \times (r-1) \times r^{c-1} \ldots$ [3].

Definition 2.1. *We define another function φ such that if n is written as the product of prime factors: $n = p^a \times q^b \times r^c \ldots$, then the value of $\varphi(n)$ is $\varphi(n) = \frac{p}{(p-1)} \times \frac{q}{(q-1)} \times \frac{r}{(r-1)} \ldots$.*

Theorem 2.2. *For every natural number n, we obtain that $n = \varphi(n) \times \phi(n)$.*

Proof. This is true as a consequence of the definitions of these functions. □

Theorem 2.3. *For every natural number $n \geq 2$, the inequality*

$$\sum_{k|n} k \leq \varphi(n) \times n$$

is true.

Proof. We know that

$$\sum_{k|n} \phi(k) = n$$

is true [3]. If we multiply both sides of this equation by $\varphi(n)$, then we obtain that

$$\sum_{k|n} \varphi(n) \times \phi(k) = \varphi(n) \times n.$$

In addition, we know that

$$\sum_{k|n} k = \sum_{k|n} \varphi(k) \times \phi(k)$$

as result of Theorem 2.2. However, we know that

$$\sum_{k|n} \varphi(k) \times \phi(k) \leq \sum_{k|n} \varphi(n) \times \phi(k)$$

since we have that $\varphi(k) \times \phi(k) \leq \varphi(n) \times \phi(k)$ for every divisor k of $n \geq 2$. Using the transitivity, we finally have that

$$\sum_{k|n} k \leq \varphi(n) \times n.$$

□

Definition 2.4. *A number will be a simple primorial if it is prime or it is the product of prime numbers. For the j^{th} prime number $p_j \geq 7$ (j is equal to 1 for the prime 7), the incomplete primorial $p_j\#$ is defined as the product of the first j primes without the numbers 2, 3 and 5.*

Theorem 2.5. *A computational verification shows that for every simple primorial number $n \geq 7$, the inequality*

$$\varphi(n) < e^{\gamma} \times \log\log n$$

is possible to be true. We have studied this behavior for every simple primorial number $10^{1875} \geq n \geq 7$.

Proof. We have checked that the value of $s(n)$

$$s(n) = e^{\gamma} \times \log\log n - \varphi(n)$$

is always greater than 0 for the first simple primorial numbers $n \geq 7$. It is trivial that for every prime number $n \geq 7$, the number 7 has the smallest possible value of $s(n)$. Based on this trivial argument, we can see that for every product of j prime numbers which are greater than or equal to 7, then the incomplete primorial $p_j\#$ has the smallest possible value of $s(n)$. Consequently, we only need to study the value of $s(n)$ evaluated in the incomplete primorial $p_j\#$ for every $j \geq 1$. We computationally analyze this behavior and we note that this tends to be greater than 0 as long as the incomplete primorial $p_j\#$ increases its value. Certainly, we study this behavior for every incomplete primorial $p_j\#$ lesser than 10^{1875}. In this way, we have checked this for every simple primorial number $n \geq 7$ lesser than 10^{1875} at the same time. Therefore, we obtain that the inequality

$$\varphi(n) < e^{\gamma} \times \log\log n$$

should be possible true for every simple primorial number $n \geq 7$. □

Theorem 2.6. *The Robin's inequality is possible to be true for every natural number* $n > 5040$ *which is not divisible by* 2*,* 3 *or* 5 *under a computational evidence.*

Proof. This is a direct consequence of Theorems 2.3 and 2.5. From the Theorem 2.3, we have that if we prove

$$\varphi(n) \times n < e^{\gamma} \times n \times \log\log n$$

for all $n > 5040$, then we could prove the Robin's inequality since we have that

$$\sum_{k|n} k \leq \varphi(n) \times n.$$

If we divide by n, then we would have that we only need to prove

$$\varphi(n) < e^{\gamma} \times \log\log n.$$

By a computational evidence, we know that this should be true for every simple primorial number $n \geq 7$ due to Theorem 2.5. Note that, $\varphi(n)$ is the same as $\varphi(m)$ when n and m have the same prime factors. Indeed, if we prove the inequality for every n that is a simple primorial, then we are proving the same for every other number m with the same prime factors, because of $\log\log n < \log\log m$. As a consequence, we have checked this for every number $10^{1875} \geq n > 5040$ which is not divisible by 2, 3 or 5 according to Theorem 2.5. □

3. Conclusions

The practical uses of the Riemann hypothesis include many propositions known true under the Riemann hypothesis, and some that can be shown equivalent to the Riemann hypothesis [1]. Certainly, the Riemann hypothesis is close related to various mathematical topics such as the distribution of prime numbers, the growth of arithmetic functions, the Lindelöf hypothesis, the large prime gap conjecture, etc [1]. In this way, a possible proof of the Riemann hypothesis could spur considerable advances in many mathematical areas, such as the number theory and pure mathematics [1]. This paper shows if there is a counterexample for the Robin's inequality, then this should be for some natural number $n > 5040$ which is divisible by 2, 3 or 5. We know if the Robin's inequality is false for some natural number $n > 5040$, then the Riemann hypothesis could be false. In this way, we provide a proof that could help in that direction.

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