A Random Number Generating Algorithm of the Characteristic Function

Fatemeh Shahzamani Varnosfaderani

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Fatemeh Sahzamani Varnosfaderani
Department of Mathematical Sciences, Isfahan University of Technology
E-mail (fatemeh_shahzamany@yahoo.com)

ABSTRACT

In today's world of economics, physics, engineering, medicine, etc., one of the issues to consider is the prediction of data based on previous observations. This prediction is done based on random number generation.

Keywords
Characteristic Function, Fourier Transform, Monte Carlo Method, Simulation

1. DISCRETE FOURIER TRANSFORM

In the Fourier series, we usually have the function continuous and x-dependent. If the function is not known, the discrete Fourier series is introduced, wherein this series it is sufficient to have the values of the function at equal distances.

Theorem 1.1. Exponential functions $\varphi_n(k) = \omega^{kn} = e^{\frac{-2k\pi ni}{N}}$ are functions alternating with period N.
Proof: [1]

Suppose $\{f_k\}_{k=0}^{N-1}$ there is a sequence of data. For example, we select the number of N samples of continuous functions$f(x)$, assuming lengths at equal intervals T. that's mean,

$$f_k = f(KT), \qquad k = 0,1,2,\dots,N-1$$

In this case, the finite or discrete Fourier transform is defined as below for$-\infty < n < +\infty$.

$$F_n = \sum_{k=0}^{N-1} f_k \omega^{nk}$$

According to theorem 1.1, it has been said

$$F_{n+k} = \sum_{k=0}^{N-1} f_k \omega^{(n+N)k} = \sum_{k=0}^{N-1} f_k \omega^{nk} = F_n$$

Therefore the sequence $\{F_n\}_{n=-\infty}^{+\infty}$ is an alternating sequence of period$N$. Relationships

$$\sum_{k=0}^{N-1} f_k \omega^{nk} = F_n$$

And

$$f_k = \frac{1}{N}\sum_{n=0}^{N-1} F_n \omega^{-nk}.$$

We call the discrete Fourier transform pair.

2. FAST FOURIER TRANSFORM

Fast Fourier Conversion or FFT is an algorithm for fast and discrete Fourier transforms that are fast and efficient. In fact, computing $O(N^2)$ requires mathematical operations, while the fast Fourier transform calculates the same results with operations of the order O (NlogN). Suppose $\{f_k\}_{k=0}^{N-1}$ is a sequence of data obtained from a finite boundary function and N is large enough. To calculate the discrete Fourier transform of the relation, the number of N^2 times the multiplication and the number of $N(N-1)$is the sum of the operations. The fast Fourier transform FFT is an algorithm proposed by Cole and Tooky in year 1965, which greatly reduces the complexity of the computation. In this algorithm, the number of sample points must be a power of two, that is, $N = 2^m (m = 1,2, \dots)$. Here, for further understanding of this algorithm, we consider only the m=2 state, is N=4. Thus ω is the fourth root of the unit. That is, $\omega = e^{-2\pi i/4} = e^{-\pi i/2}$. To simplify the operation and formulate operations, the n root of the unit number is displayed withω_n. So $\omega_4 = e^{\frac{-i\pi}{2}}$ and so formula 1 is as follows:

$$\sum_{k=0}^{3} f_k \omega^{nk} \qquad n = 0,1,2,3$$

Converted sequence sentences are:

$$F_0 = f_0\omega_4{}^0 + f_1\omega_4{}^0 + f_2\omega_4{}^0 + f_3\omega_4{}^0.$$
$$F_1 = f_0\omega_4{}^0 + f_1\omega_4{}^1 + f_2\omega_4{}^2 + f_3\omega_4{}^3.$$
$$F_2 = f_0\omega_4{}^0 + f_1\omega_4{}^2 + f_2\omega_4{}^4 + f_3\omega_4{}^6.$$
$$F_3 = f_0\omega_4{}^0 + f_1\omega_4{}^3 + f_2\omega_4{}^6 + f_3\omega_4{}^9.$$

Since the sum of the roots of N is a complex number $(1, \omega, \omega^2, \dots, \omega^{N-1})$ equal to zero, we have $\omega_4^1 = \omega_4^9$, $\omega_4^2 = \omega_4^6$ and $\omega_4^0 = \omega_4^4$ so we can rewrite 4 as follows.

$$\begin{bmatrix} F_0 \\ F_1 \\ F_2 \\ F_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & \omega_4{}^1 & \omega_4{}^2 & \omega_4{}^3 \\ 1 & \omega_4{}^2 & \omega_4{}^0 & \omega_4{}^2 \\ 1 & \omega_4{}^3 & \omega_4{}^2 & \omega_4{}^1 \end{bmatrix} \begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \end{bmatrix}$$

We now divide the coefficients matrix into 5 as follows.

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & \omega_4{}^1 & \omega_4{}^2 & \omega_4{}^3 \\ 1 & \omega_4{}^2 & \omega_4{}^0 & \omega_4{}^2 \\ 1 & \omega_4{}^3 & \omega_4{}^2 & \omega_4{}^1 \end{bmatrix} = \begin{bmatrix} 1 & \omega_4{}^0 & 0 & 0 \\ 1 & \omega_4{}^2 & 0 & 0 \\ 0 & \omega_4{}^1 & 1 & \omega_4{}^1 \\ 0 & 0 & 1 & \omega_4{}^3 \end{bmatrix} \begin{bmatrix} 1 & 0 & \omega_4{}^0 & 0 \\ 0 & 1 & 0 & \omega_4{}^0 \\ 1 & 0 & \omega_4{}^2 & 0 \\ 0 & 1 & 0 & \omega_4{}^2 \end{bmatrix}$$

The left-hand matrix 6 is the coefficient matrix5, whose second and third rows are displaced. So 5 would be as follows.

$$\begin{bmatrix} F_0 \\ F_1 \\ F_2 \\ F_3 \end{bmatrix} = \begin{bmatrix} 1 & \omega_4{}^0 & 0 & 0 \\ 1 & \omega_4{}^2 & 0 & 0 \\ 0 & \omega_4{}^1 & 1 & \omega_4{}^1 \\ 0 & 0 & 1 & \omega_4{}^3 \end{bmatrix} \begin{bmatrix} 1 & 0 & \omega_4{}^0 & 0 \\ 0 & 1 & 0 & \omega_4{}^0 \\ 1 & 0 & \omega_4{}^2 & 0 \\ 0 & 1 & 0 & \omega_4{}^2 \end{bmatrix} \begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \end{bmatrix}.$$

Now consider the $g = (g_0\ g_1\ g_2\ g_3)^T$ column vector as follows:

$$\begin{bmatrix} g_0 \\ g_1 \\ g_2 \\ g_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & \omega_4{}^0 & 0 \\ 0 & 1 & 0 & \omega_4{}^0 \\ 1 & 0 & \omega_4{}^2 & 0 \\ 0 & 1 & 0 & \omega_4{}^2 \end{bmatrix} \begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \end{bmatrix}.$$

It results from multiplying two right-hand matrices:

$$g_0 = f_0 + \omega_4{}^0 f_2.$$
$$g_1 = f_1 + \omega_4{}^0 f_3.$$
$$g_2 = f_0 + \omega_4{}^2 f_2.$$
$$g_3 = f_1 + \omega_4{}^2 f_3.$$

To calculate g_0 and g_1 , two complex multiplications and two additional are required, and since $\omega_4{}^2 = -\omega_4{}^0$, the computation of g_2 and g_3 involves two further operations, so the g vector is obtained by two multiplication operations and four addition operations. By putting the g vector in relation 7 we have:
Then

$$F_0 = g_0 + \omega_4{}^0 g_1.$$
$$F_1 = g_0 + \omega_4{}^2 g_1.$$
$$F_2 = g_2 + \omega_4{}^1 g_3.$$
$$F_3 = g_2 + \omega_4{}^3 g_3.$$

Because ${\omega_4}^2 = -{\omega_4}^0$ calculates F_0 and F_1 depends on one multiplication operation and two additional operations. It also results from the equation ${\omega_4}^3 = -{\omega_4}^1$ that for F_2 and F_3 requires one multiplication operation and two addition operations.

So the total number of operations required to produce the desired conversion is given below

$$(F_0\, F_1\, F_2\, F_3)^T$$

There are four complex multiplication operations and eight complex multiplication operations. In contrast, the direct calculation of the discrete Fourier transform requires $N^2 = 16$ of complex multiplication operation and $N(N-1) = 12$ of complex multiplication.

Generally when $N = 2^m$. The FFT algorhythm consists of multiplying m by the $N * N$ matrix of form 6. Each step contains the $\frac{N}{2}$ multiplication action, and so the total number of multiplication operations is $\frac{Nm}{2}$, and since $m = log_2 N$ is, then the number of multiplication operations is approximately equal to the $Nlog_2 N$ operation. After each step, N requires addition action. So the total number of additional operations required is $N_m = Nlog_2 N$ [3].

The FFT algorithm is a well-known algorithm coded for its importance in the R language.

3. Characteristic function

Suppose $f(x)$ is the probability density function of the random variable X, then the Fourier transform of the $f(x)$ function is called the X characteristic function, that is,

$$\emptyset(w) = E(e^{iwX}) = \int_{-\infty}^{+\infty} e^{iwX} f(x)dx$$

In fact, the characteristic function and density function of the Fourier pairs are so that the density function can be obtained from its characteristic function by inverting the Fourier transform as follows.

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-\omega x}\emptyset(\omega)\, d\omega$$

The characteristic function is always available. By integrating the density function, we can also obtain the distribution function. So the distribution function is obtained as follows.

$$F(x) = \int_{-\infty}^{x} f(\tau)d\tau = \frac{1}{2\pi}\int_{-\infty}^{x}\int_{-\infty}^{+\infty} e^{-\omega x\tau}\emptyset(\omega)\, d\omega d\tau$$

To avoid computing the dual numerical integrals, it is preferred to obtain the distribution function directly from the characteristic function. For this purpose, the $e^{-\alpha x}F(x)$ function conversion function is used, such that $e^{-\alpha x}$ is $\alpha > 0$ modulation factor.

$$\int_{-\infty}^{+\infty} e^{iwx}e^{-i\alpha x} F(x)dx = \int_{-\infty}^{+\infty} e^{-(\alpha - i\omega)x} F(x)dx$$

The following equation is obtained by fractional integration and since the first part tends to zero.

$$\int_{-\infty}^{+\infty} e^{-(\alpha - i\omega)x} F(x)dx = \frac{1}{\alpha - i\omega}\int_{-\infty}^{+\infty} e^{-(\alpha - i\omega)x} f(x)dx$$
$$= \frac{1}{\alpha - i\omega}\int_{-\infty}^{+\infty} e^{i(i\alpha + \omega)x} f(x)dx = \frac{1}{\alpha - i\omega}\emptyset(i\alpha + \omega)$$

So

$$\int_{-\infty}^{+\infty} e^{iwx}e^{-i\alpha x} F(x)dx = \frac{1}{\alpha - i\omega}\emptyset(i\alpha + \omega)$$

Now using the Fourier transform inverse in the above relation, the distribution function can be obtained as follows from the characteristic function.

Religiously, we obtain the distribution function by integrating a single numeric from the characteristic function.

$$F(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{1}{\alpha - i\omega} e^{-i\omega x}\, \emptyset(i\alpha + \omega)d\omega.$$

4. Monte Carlo method for calculating integrals

Since it is difficult to calculate integrals using the integral method for some functions, by using the Monte Carlo method, we obtain a good approximation for the integral value. The Monte Carloaz method follows the following algorithm:

Suppose $X_1, X_2, \ldots, X_n$ are random variables independent of the probability density function of$f(x)$. We put:

$$T_i = g(X_i)$$

For n big enough and according to the strong law of large numbers, we have:

$$M_n = \frac{\sum_0^n T_i}{n} \to E[T]$$

Then:

$$E[T] = \int g(x)f(x)dx.$$

Algorithm

Step 1: Get n big enough.
Step 2: Generate a random number n from the probability density function of (x) .
Step 3: Insert:

$$T_i = g(X_i)$$

Step 4: The $M_n = \frac{\sum_0^n T_i}{n}$ Monte Carlo approximation is desirable. Suppose

$$A = \int z(x)dx.$$

Be it. Calculate A by Monte Carlo method as follows:

Multiply and divide the standard normal density function ($\frac{1}{\sqrt{2\pi}}e^{\frac{-1}{2}x^2}$) on the right-hand side of the relation17. Then A is equal to:

$$y(x) = \frac{z(x)}{f(x)} \times f(x)dx.$$

According to the Monte Carlo algorithm we have:
Step 1: Make the n large enough.
Step 2: n We generate a random number from the standard normal probability density function.
Step 3: Put:

$$T_i = \frac{z(X_i)}{f(X_i)}.$$

Step 4: $M_n = \frac{\sum_0^n T_i}{n}$ is an approximation of A.

4. *A* RANDOM NUMBER GENERATING ALGORITHM OF THE CHARACTERISTIC FUNCTION

Set for each x value:

$$f(x) = \frac{1}{2\pi} \int \emptyset(t) e^{-itx} dt.$$

$f(x)$ Is calculated as in Example 3.
Step One: Put:

$$a \to \frac{1}{2\pi} \int |\emptyset(t)| \quad , \quad b \to \frac{1}{2\pi} \int |\emptyset(t)''|$$

a And b are calculated as in Example 3.
Step Two: Generate two independent random numbers U and V from the $[-1,1]$ uniform distribution.
Step Three: If $U < 0$, set:

$$X \to \sqrt{\frac{b}{a} V} \quad , \quad T \to |U|a$$

Otherwise:

$$X \to \sqrt{\frac{b}{a} \frac{1}{V}} \quad , \quad T \to \frac{|U|b}{X^2}$$

If $T \leq f(x)$, then X is a random number. Otherwise, we will return to the second step. Note that in this algorithm the real data must be checked and its size for complex numbers taken into account.
This algorithm generates a random number. Repeat the algorithm to the required number of random numbers[2].

4. CONCLUSION

The above algorithm is coded in the R language. A few applications are presented below. The normal distribution with the parameter (μ, σ^2) has the characteristic function$\emptyset(t) = e^{it\mu - \frac{1}{2}\sigma^2 t^2}$. The histogram of random numbers is plotted using the above algorithm and based on 1000 random numbers from the $N(0,1)$ distribution in Fig1.

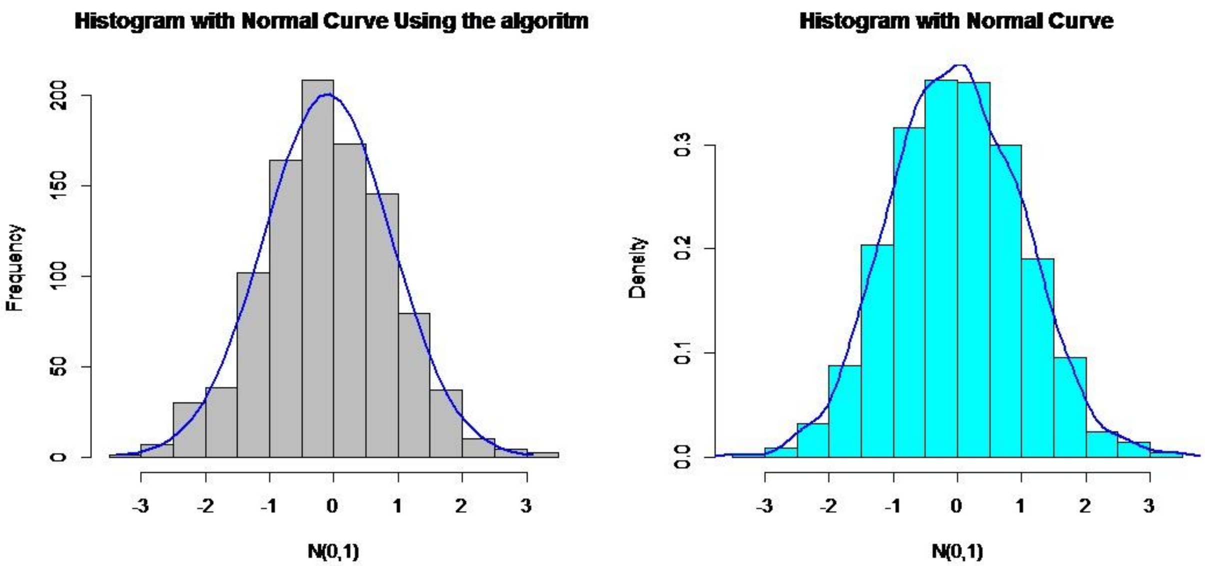

Figure 1 Comparison of Standard Normal Distribution Histogram with Histogram Algorithm

The exponential distribution with parameter λ has the characteristic function$(1 - it\lambda^{-1})^{-1}$. The histogram of random numbers is plotted using the above algorithm and based on 1000 random numbers of the $\exp(1)$distribution in Fig2.

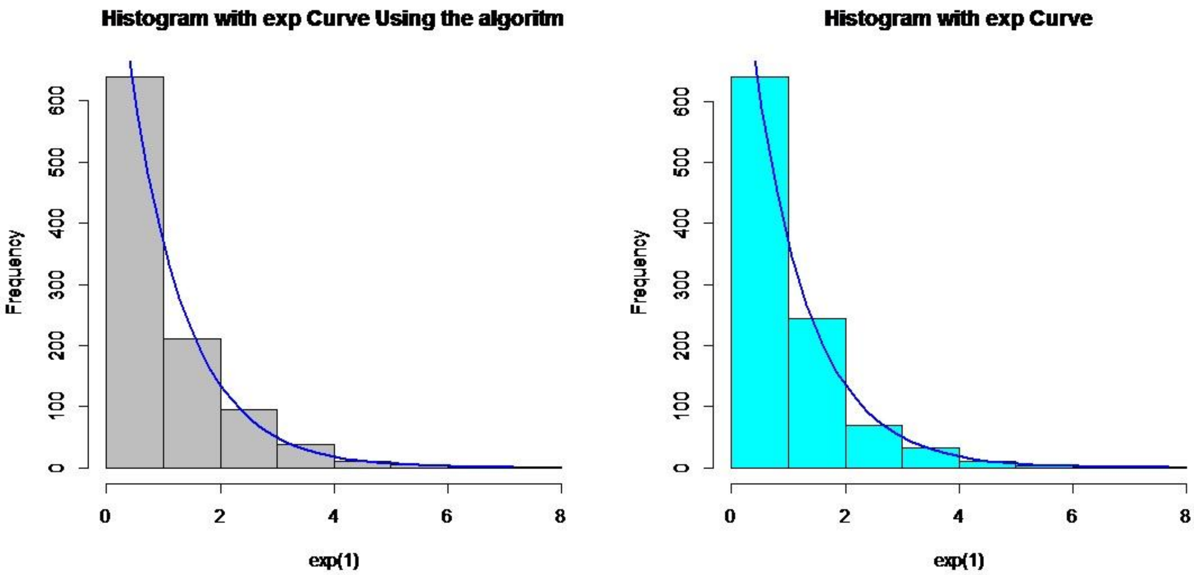

Figure 2 Comparison of Exponential Distribution Histograms with Histogram Algorithms

6. ATTACHMENT

Applications are written inR language from version i3863.3.1.

7. REFERENCES

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