Some Considerations on Orthogonality, Strict Separation Theorems and Applications in Hilbert Spaces

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SOME CONSIDERATIONS ON ORTHOGONALITY, STRICT SEPARATION THEOREMS AND APPLICATIONS IN HILBERT SPACES

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Abstract

After presenting some structural notions on Hilbert spaces, which constitute a fundamental support for this work, we approach the goals of the chapter. First, a study about convex sets, projections and orthogonality, where we approach the optimization problem in Hilbert spaces with some generality. Then the approach to Riez representation theorem in this field, important in the rephrasing of the separation theorems. Then we give a look to the strict separation theorems as well as to the main results of convex programming: Kuhn-Tucker theorem and minimax theorem. Both these theorems are very important in the applications. Moreover, the strict separation theorems presented and the Riez representation theorem have a key importance in the demonstrations of Kuhn-Tucker and minimax theorems and respective corollaries.

Keywords: Hilbert spaces, convex sets, projections, orthogonality, Riez representation theorem, Kuhn-Tucker theorem, minimax theorem.

1. Introduction

Definition 1.1

A Hilbert space is a complex vector space with inner product that, as metric space, is complete.■

A Hilbert space is designated, usually, *H* or *I*. Remember that

Definition 1.2

An inner product in a complex vector space *H* is a sesquilinear Hermitian and strictly positive functional on *H*.■

Observation:

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-In real vector spaces, “sesquilinear Hermitian” must be replaced by “bilinear symmetric”,

-The inner product of two vectors *x* and *y* belonging to *H*, in this order, is denoted $[x, y]$,

-The norm of a vector x is given by $\|x\| = \sqrt{[x, x]}$,

-The distance between two elements *x* and *y* belonging to *H* is $d(x, y) = \|x - y\|$.

Proposition 1.1

The norm, in a space with inner product, satisfies the parallelogram rule:

$$\|x - y\|^2 + \|x + y\|^2 = 2(\|x\|^2 + \|y\|^2). \blacksquare$$

For more details on these concepts see, for instance, [1-5].

2. Convex Sets and Projections

Then we enounce and demonstrate a theorem that is a result of existence and uniqueness, fundamental in optimization, see [6].

Theorem 2.1

Every closed convex set in a Hilbert space has only one element with minimal norm**.**

Dem:

Call *C* the closed convex set and $d = inf\|x\|, x \in C$. Under the assumed conditions it is possible to find a sequence $\|x_n\|$ in *C*, called minimizing sequence, such that $d = \lim_n \|x_n\|$. By the parallelogram rule $\left\|\frac{x_n - x_m}{2}\right\|^2 = \frac{1}{2}(\|x_n\|^2 + \|x_m\|^2) - \left\|\frac{x_n}{2} + \frac{x_m}{2}\right\|^2$. Nevertheless, as the second parcel of the second member of this equality is the norm square of an element of *C*, $\left\|\frac{x_n - x_m}{2}\right\|^2 \leq \frac{1}{2}(\|x_n\|^2 + \|x_m\|^2) - d^2 \to 0$ and so x_n is a **Cauchy sequence**.

As *C* is closed and *H* is complete, the limit element *z* belongs to *C*. And, by the inequality $|\|x\| - \|y\|| \leq \|x - y\|$, it follws that $\|z\| = d$.

Suppose now that that z_1 and z_2 are two elements of *C* with norm *d*. So, again by the parallelogram rule, $\left\|\frac{1}{2}(z_1 - z_2)\right\|^2 = d^2 - \left\|\frac{z_1}{2} - \frac{z_2}{2}\right\|^2 \leq 0$ and then $z_1 = z_2$. ■

Be now a closed convex set *C*, in *H*, and an element *x*, anyone, of *H*. Noting that $x - C$ is a closed convex set, it results the following corollary of Theorem 2.1:

Corollary 2.1

Be *C* a closed convex set in *H*. For every element *x* in *H* there is only one element in *C* that is the closest of *x*; that is, there is only one element $z \in C$ such that $\|x - z\| = inf\|x - y\|, y \in C$. ■

For the moment, there is a result of existence and uniqueness for the optimization problem. However, unhappily, the demonstration is not constructive. It is not said how to determine that unique element. However, it is possible a better characterization, through a variational inequality, as we point in the following result, see [7, 8]:

Theorem 2.2

Be ***C*** a closed convex set in *H*. For every *x* belonging to *H*, *z* is the only element in *C* closest - in norm - of *x* if and only if

$$Re[x-z,z-y] \geq 0, \underset{y \in C}{\forall} \qquad (2.1)$$

Dem:

Every characterization of this type comes through a variational argument. So, suppose that *z* is the only element closest in *C*, granted by Corollary 2.1. So, for any $\theta, 0 \leq \theta \leq 1$, $(1-\theta)z + \theta y \in C$ since $y \in C$, as *C* is convex. So,

$$g(\theta) = \left\|x - \left((1-\theta)z + \theta y\right)\right\|^{2} \qquad (2.2)$$

is a function twice continuously differentiable of θ. In fact is a quadratic function of θ. More:

$$g'(\theta) = 2\, Re[x - \theta y - (1-\theta)z, z-y] \qquad (2.3)$$

$$g''(\theta) = 2Re[z-y, z-y] \qquad (2.4).$$

Then, so that *z* is the minimizing element, it is evident that it has to be $g'(0) \geq 0 \Leftrightarrow Re[x-z, z-y] \geq 0$.

Suppose now that (2.1) is fulfilled for a given element *z* of *C*. Therefore, building again $g(\theta)$, as in (2.2), (2.1) allows concluding that $g'(0)$ is non-negative and, owing to (2.4), $g''(\theta)$ is non-negative. So $g(0) \leq g(1)$ for any$y \in C$ that is

$$\|x-z\|^2 \leq \|x-y\|^2, \underset{y \in C}{\forall}.$$

Therefore, it proofs that *z* is the minimizing element in *C*. As already seen, such element is unique.■

Observation:

-It is interesting to interpret geometrically (2.1). So consider the set of elements *h* belonging to *H* such that

$$Re[x-z, h] = c = Re[x-z, z].$$

Indeed, a hyper-plane that contains *z*. This hyper-plane, which normal is $x-z$, is a convex set *C* support plane in the sense that

$$Re[x-z, z] = c, z \in C \qquad (2.5),$$

$$Re[x-z, y] \leq c, \underset{y \in C}{\forall} \qquad (2.6).$$

As $Re[x-z, z-y] \geq 0 \Leftrightarrow Re[x-z, z] - Re[x-z, y] \geq 0 \Leftrightarrow Re[x-z, y] Re[x-z, z]$, the point *z* is the support point.

Now it is pertinent to present the following definitions, see [9]:

Definition 2.1

Given any closed convex set *C* in *H* , the application of *H* in *H* , making to correspond to each *x* the closest element of *x* in *C* , is called *projection over C* and is designated $P_C(.)$. $P_C(x)$ is said the projection of *x* over C. ■

Observation:

-$P_C(.)$ is not necessarily linear and lets *C* invariant.

Definition 2.2
A cone is a set with the following property: $tx, t \geq 0$, belongs to it since x belongs. ■

Observation:
-A cone is not necessarily convex[1].
-Note that *C* is a convex cone if, whenever x_1 and x_2 belong to *C* also $t_1x_1 + t_2x_2$ belong to *C* for any $t_1, t_2 \geq 0$. ■

Then it follows a corollary of Theorem 2.2:

Corollary 2.2
Suppose that *C* is a closed convex cone. Be *z* the projection of *x* over *C*. Then

$$Re[x - z, z] = 0 \ and \ Re[x - z, y] \leq 0, \ \forall_{y \in C} \qquad (2.7).$$

In addition, if an element z of C satisfies these relations, it is the projection of *x* over *C*.■

This section ends with the following corollary see again [9]:

Corollary 2.3
Be *M* a closed vector subspace. So, for each $x \in H$ there is one only element of *M*, that is the closest of x, being the projection of *x* over *M* such that

$$[x - P_M(x), m] = 0 \, , \ \forall_{m \in M} \qquad (2.8).$$

In this case $P_M(.)$ is linear and called projection operator corresponding to *M*.■

3. Orthogonality and Orthonormal Basis

Following [10-12]:

Definition 3.1
Vector x is orthogonal to vector y if $[x, y] = 0$. ■

Definition 3.2
The set *S* orthogonal complement in a Hilbert space is the set of the whole elements orthogonal to any element of *S*. Designate it $S^{\perp}$. ■

Proposition 3.1

i) If $S \neq \emptyset$ $S^{\perp}$ is a closed vector subspace.

ii) If *M* is a closed vector subspace

a) $(M^{\perp})^{\perp} = M$,

b) After (2.8)

[1] It is enough to think in two straight lines passing through the origin.

$x = P_M(x) + (x - P_M(x)), P_M(x) \in M, (x - P_M(x)) \in M^{\perp}$ (3.1). ■

Observation:

-In (3.1) it is patent an orthogonal decomposition of x. That is, x is decomposed in the sum of two elements orthogonal to each other. One belongs to the subspace M and the other to its orthogonal complement. Such a decomposition is unique in the sense that if $x = z_1 + z_2$ where $z_1 \in M$ and $z_2 \in M^{\perp}$ it must be $z_1 = P_M(x)$ and $z_2 = x - P_M(x)$, since $0 = (P_M(x) - z_1) + (x - (P_M(x) - z_2)$ and the elements between parenthesis are orthogonal.

Definition 3.3

Call an orthonormal set everyone in which any two of its elements are orthogonal to each other, and each element has norm 1.■

Definition 3.4

Be S a non-empty set of H. $L(S)$ designates the closure of the set of every S elements finite linear combinations.■

Definition 3.5

An orthonormal set S is a basis of $L(S)$.■

Observation:

-If S has a finite number of elements $x_i, i = 1, \dots, n$ the subspace $L(S)$ is precisely the set of the whole elements of the form $\sum_{k=1}^{n} a_k x_k$. And, in this case, the projection operator corresponding to L(S) is given by $P_{\mathrm{L}(S)}(x) = \sum_{k=1}^{n} a_k x_k$ fulfilling the coefficients a_k the equation $\left[x - \sum_{j=1}^{n} a_j x_j, x_i\right] = 0, i = 1, \dots, n$ or:

$$\sum_{j=1}^{n} a_j \left[x_j, x_i\right] = [x, x_i], i = 1, \dots, n \qquad (3.2).$$

-If the set $x_i, i = 1, \dots, n$ is orthonormal, the projection has the simple form

$$P_{\mathrm{L}(S)}(x) = \sum_{i=1}^{n} [x, x_i]\, x_i \qquad (3.3)$$

and also

$$\|x\|^2 \geq \left\|P_{\mathrm{L}(S)}(x)\right\|^2 = \sum_{i=1}^{n} |[x, x_i]|^2 \text{ (Bessel's Inequality)}.$$

-Call, now, S a sequence $\{x_i\}$ of elements $x_i, i = 1, \dots, n$. S can be made orthonormal mean this that is possible to determine an orthonormal basis for $L(S)$: $L(S)$=$L(0)$ being 0 orthonormal. Such a basis may be obtained through the known Gram-Schmidt method, since not the whole $\{x_i\}$ are 0.

With the whole generality:

Theorem 3.1

Every non-trivial Hilbert space, that is: not constituted exclusively by 0, has an orthonormal basis.

Dem:

It is possible to find orthonormal sets in the space, unless it is trivial. Introduce a partial ordination in the class of the orthonormal sets, through the inclusion relation:

-Given two orthonormal sets A and B, $A < B$ if and only if $A \subset B$.

Be $\{A_\alpha\}$ a subclass totally ordered: a chain – maximal, that is: not strictly contained in another chain. The Haudsdorf maximal chain theorem grants the existence of a maximal chain.

Be $A = \underset{\alpha}{\cup} A_\alpha$. A is orthonormal. Then we show that L (A), the subspace generated by A is in fact the whole Hilbert space.

Proceed by absurd. Suppose that $z \in H$ is not in L (A). Call P the projection operator corresponding to $L(A)$. So $e = \frac{z-Pz}{\|z-Pz\|}$ is orthogonal to A and the family obtained postponing to the chain $\{A_\alpha\}$ the set $A \cup \{e\}$ violates the chain maximally. ■

Observation:

-There may be, evidently, many sets as the set A referred in this demonstration, but it is demonstrated that all of them have the same cardinal.

-An orthonormal basis may not be finite and the space is of infinite dimension. Moreover, it is not necessarily countable. However, it results from Bessel's inequality that, for every $x \in H$, only a countable number of $[x, e], e \in O$, may be different from zero.

4. Riez Representation Theorem

An important theorem, about the representation of a continuous linear functional by elements of the space is the Riesz representation theorem, see again [11] and [13]:

Theorem 4.1 (Riesz representation**)**

Every continuous linear functional $f(\cdot)$ may be represented in the form $f(x) = [x, \tilde{q}]$ where

$$\tilde{q} = \frac{\overline{f(q)}^2}{[q,q]} q.$$

Dem:

Begin noting that for every continuous linear functional $f(.)$, the *Nucleus* of $f(.)$ [3] is a closed vector subspace. If the functional under consideration is not the null functional, there is an element y such that $f(y) \neq 0$. Be z the projection of y over *Nuc*(*f*)and make $q = y - z$. So, q is orthogonal to *Nuc*(*f*) and $f(q) = f(y)$ and, in consequence, $f(q) \neq 0$. Then, for every $x \in H, x - \frac{f(x)}{f(q)} q$ belongs evidently to *Nuc*(*f*). So, $x - \frac{f(x)}{f(q)} q$ is orthogonal to q and, in consequence, $[x, q] - \frac{f(x)}{f(q)} [q, q] = 0 \Leftrightarrow [x, q] = \frac{f(x)}{f(q)} [q, q]$ $that\ is: f(x) = \left[x, \frac{\overline{f(q)}}{[q,q]} q \right]$. ■

Observation:

-From the theorem, it results $\|f\|_{H'} = \|\tilde{q}\|_H$, where the H dual space is H'[4].

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[2] $\overline{f(q)}$ is the conjugate complex number of *f*(*q*)**.**

[3] The *Nucleus* of $f(\cdot)$is designated *Nuc*(f) and *Nuc*(f)={x: $f(x) = 0$}.

[4] Consider a continuous linear functional f in a normed space E. It is called f norm, and designated $\|f\|$:

$$\|f\| = \sup_{\|x\| \le 1} |f(x)|.$$

5. Convex Sets Strict Separation

Convex sets separation is very important in convex programming, which is a very potent mathematical instrument for operations research, management and economics see, for example, [14-16]. The target of this work is to present Theorem 5.1 that gives sufficient conditions for the strict separation of convex sets. First the following definitions:

Definition 5.1

Two closed convex subsets A and B, in a Hilbert space H, are at finite distance from each other if $\inf_{x\in A, y\in B}\|x-y\| = d > 0$.■

Definition 5.2

Two closed convex subsets A and B, in a Hilbert space H, are strictly separated if, for some $v \in H$, $\inf_{x\in A}[v,x] > \sup_{y\in B}[v,y]$.■

Then it follows, see again [12],

Theorem 5.1(strict separation)

Two closed convex subsets A and B, in a Hilbert space H, at finite distance from each other can be strictly separated.

Dem: In fact, as zero is then a $A-B$ complement interior point, taking its projection over the $A-B$ closure and calling it v, $[-v, v-q] \geq 0, \forall q \in A-B$, by Theorem 2.2. So $[v,q] \geq [v,v]$ and $[v,x]-[v,y] \geq [v,v], x \in A, y \in B$ leading to $\inf_{x\in A}[v,x] \geq \sup_{y\in B}[v,y]$.■

It is also possible to show that

Theorem 5.2 Being H a finite dimension Hilbert space, if A and B are non-empty disjoint convex sets, they can always be separated.■

6. Convex Programming

Now we outline a class of convex programming problems, at which it we intend to minimize convex functionals subject to convex restrictions. Begin presenting a basic result that characterizes the minimum point of a convex functional subject to convex inequalities, see [17]. Note that it is not mandatory to impose any continuity conditions.

That is, the supreme of the values assumed by $|f(x)|$ in the E unitary ball. The class of the continuous linear functionals, with the norm above defined, is a normed vector space, called the E dual space, designated E'. Of course, a Hilbert space is a normed space.

Theorem 6.1 (Kuhn-Tucker**)**

Be *f(x)*, $f_i(x), i = 1, \dots, n$, convex functionals defined in a convex subset *C* of a Hilbert space. Consider the problem $\min_{x \in C} f(x)$, $sub.: f_i(x) \leq 0, i = 1, \dots, n$, be x_0 a point where the minimum, supposed finite, is reached. Suppose also that for each vector *u* in E_n, Euclidean space with dimension *n*, non-null and such that $u_k \geq 0$, there is a point *x* in *C* such that $\sum_1 u_k f_k(x) < 0$,designating u_k the components of *u*. So,

i) There is a vector *v*, with non-negative components $\{v_k\}$, such that

$$\min_{x \in C} \left\{ f(x) + \sum_1^n v_k f_k(x) \right\} = f(x_0) + \sum_1^n v_k f_k(x_0) = f(x_0) \qquad (6.1)\,,$$

ii) For every vector u in E_n with non-negative components, that is: belonging to the positive cone of E_n ,

$$f(x) + \sum_1^n v_k f_k(x) \geq f(x_0) + \sum_1^n v_k f_k(x_0) \geq f(x_0) + \sum_1^n u_k f_k(x_0) \quad (6.2).\ \blacksquare$$

Corollary 6.1 (Lagrange duality**)**

In the conditions of Theorem 6.1 $f(x_0) = \sup_{u \geq 0} \inf_{x \in C} f(x) + \sum_1^n u_k f_k(x)$. ■

Observation:

-This corollary is useful supplying a process to determine the problem optimal solution,

-If the whole v_k in expression (6.2) are positive, x_0 is a point in the border of the convex set defined by the restrictions,

-If the whole $v_k$are zero, the inequalities do not influence the problem, that is: the minimum is equal to the one of the restrictions free problem.

Considering non-finite inequalities see [18]:

Theorem 6.2 (Kuhn-Tucker in infinite dimension**)**

Be *C* a convex subset of a Hilbert space *H* and *f* (*x*) a real convex functional defined in *C*. Be *I* a Hilbert space with a closed convex cone $\wp$, with non-empty interior, and *F*(*x*) a convex transformation from *H* to *I* (convex in relation to the order introduced by cone $\wp$: if $x, y \in \wp, x \geq y \; if \; x - y \in \wp$). Be x_0 a *f* (*x*) minimizing in *C* subjected to the inequality $F(x) \leq 0$.Consider $\wp^* = \left\{ x: [x, p] \geq 0, \forall_{x \in \wp} \right\}$ (dual cone). Admit that given any $u \in \wp^*$ it is possible to determine *x* in *C* such that $[u, F(x)] < 0$. So, there is an element *v* in the dual cone $\wp^*$, such that for *x* in *C* $f(x) + [v, F(x)] \geq f(x_0) + [v, F(x_0)] \geq f(x_0) + [u, F(x_0)]$, being *u* any element of $\wp^*$. ■

Corollary 6.2 (Lagrange duality in infinite dimension**)**

$f(x_0) = \sup_{v \in \wp^*} \inf_{x \in C} (f(x) + [v, F(x)])$ in the conditions of Theorem 6.2. ■

7. Minimax Theorem

Although belonging to the field of convex programming we make the option of giving a privileged treatment to the minimax theorem, see [19, 20].

In a two players game with null sum be $\Phi(x,y)$ a real function of two variables $x, y \in H$ and A and B convex sets in H. One of the players chooses strategies (points) in A in order to maximize $\Phi(x,y)$ (or minimize $-\Phi(x,y)$): it is the maximizing player. The other player chooses strategies (points) in B in order to minimize $\Phi(x,y)$ (or maximize $-\Phi(x,y)$); it is the minimizing player. The function $\Phi(x,y)$ is the payoff function. The function $\Phi(x_0,y_0)$ represents, simultaneously, the gain of the maximizing player and the loss of the minimizing player in a move at which they chose, respectively the strategies x_0 and y_0. So, the gain of one of the players is equal to the other's loss. That is why the game is a null sum game. A game in these conditions value is c if

$$\sup_{x\in A}\inf_{y\in B}\Phi(x,y) = c = \inf_{y\in B}\sup_{x\in A}\Phi(x,y) \qquad (7.1).$$

If, for any (x_0,y_0), $\Phi(x_0,y_0) = c$, (x_0,y_0) is a pair of optimal strategies. There will be a saddle point if also

$$\Phi(x,y_0) \leq \Phi(x_0,y_0) \leq \Phi(x_0,y), \text{x}\in A, y \in B \qquad (7.2).$$

So, see again [6]:

Theorem 7.1(minimax)

Consider A and B closed convex sets in H, being A bounded. Be $\Phi(x,y)$a real functional defined for *x* in *A* and *y* in *B* fulfilling:

-$\Phi(x,(1-\theta)y_1+\theta y_2) \leq (1-\theta)\Phi(x,y_1)+\theta\Phi(x,y_2)$ for *x* in *A* and y_1, y_2 in *B*, $0 \leq \theta \leq 1$ (that is: $\Phi(x,y)$ is convex in *y* for each *x*),

- $\Phi\big((1-\theta)x_1+\theta x_2,y\big) \geq (1-\theta)\Phi\big(x_{1,},y\big)+\theta\Phi(x_2,y)$ for *y* in *B* and x_1, x_2 in *A*, $0 \leq \theta \leq 1$ (that is: $\Phi(x,y)$ is concave in *x* for each *y*),

- $\Phi(x,y)$ is continuous in x for each y,

so (7.1) holds, that is: the game has a value. ■

Dem:

Beginning by the most trivial part of the demonstration:

$$\inf_{\boldsymbol{y}\in B}\Phi(\boldsymbol{x},\boldsymbol{y}) \leq \Phi(\boldsymbol{x},\boldsymbol{y}) \leq \sup_{\boldsymbol{x}\in A}\Phi(\boldsymbol{x},\boldsymbol{y})$$

and so

$$\sup_{\boldsymbol{x}\in A}\inf_{\boldsymbol{y}\in B}\Phi(\boldsymbol{x},\boldsymbol{y}) \leq \inf_{\boldsymbol{y}\in B}\sup_{\boldsymbol{x}\in A}\Phi(\boldsymbol{x},\boldsymbol{y}).$$

Then, as $\Phi(\boldsymbol{x},\boldsymbol{y})$ is concave and continuous in $\boldsymbol{x} \in A$, A convex, closed and bounded, it follows that $\sup_{\boldsymbol{x}\in A}\Phi(\boldsymbol{x},\boldsymbol{y}) < \infty$.

Be $C = \inf_{\boldsymbol{y}\in B}\sup_{\boldsymbol{x}\in A}\Phi(\boldsymbol{x},\boldsymbol{y})$. Suppose now that there is $\boldsymbol{x}_0 \in A$ such that $\Phi(\boldsymbol{x}_0,\boldsymbol{y}) \geq C$, for any $\boldsymbol{y}$ in *B*. In this case, $\inf_{\boldsymbol{y}\in B}\Phi(\boldsymbol{x}_0,\boldsymbol{y}) \geq C$ or $\sup_{\boldsymbol{x}\in A}\inf_{\boldsymbol{y}\in B}\Phi(\boldsymbol{x},\boldsymbol{y}) \geq C$ as it is appropriate. Then the existence of such a $\boldsymbol{x}_0$ will be proved.

For any $\boldsymbol{y}$ in B, be $A_{\boldsymbol{y}} = \{\boldsymbol{x} \in A: \Phi(\boldsymbol{x},\boldsymbol{y}) \geq C\}. A_{\boldsymbol{y}}$ is closed, limited and convex. Suppose that, for a finite set $(\boldsymbol{y}_1, \boldsymbol{y}_2, \dots, \boldsymbol{y}_n), \bigcap_{i=1}^{n} A_{\boldsymbol{y}_i} = \emptyset$. Consider the transformation from A to E_n defined by

$$f(\boldsymbol{x}) = (\Phi(\boldsymbol{x},\boldsymbol{y}_1) - C, \Phi(\boldsymbol{x},\boldsymbol{y}_2) - C, \dots, \Phi(\boldsymbol{x},\boldsymbol{y}_n) - C).$$

Call G the $f(A)$ convex hull closure. Be P the E_n closed positive cone. Now we show $P \cap G = \emptyset$: indeed, being $\Phi(\boldsymbol{x},\boldsymbol{y})$ concave in $\boldsymbol{x}$, for any $\boldsymbol{x}_k$ in A, $k = 1, 2, \dots, n, 0 \leq \theta_k \leq 1, \sum_{k=1}^{n} \theta_k = 1$,

$$\sum_{k=1}^{n} \theta_k (\Phi(\boldsymbol{x}_k,\boldsymbol{y}) - C) \leq \Phi(\sum_{k=1}^{n} \theta_k \boldsymbol{x}_k,\boldsymbol{y}) - C.$$

Therefore, the convex extension of $f(A)$ does not intersect P.

Consider now a sequence $\boldsymbol{x}_n$ of elements of A, such that $f(\boldsymbol{x}_n)$ converges for $\boldsymbol{v}, \boldsymbol{v} \in E_n$. As A is closed, limited and convex, it is possible to define a subsequence, designated $\boldsymbol{x}_m$ such that $\boldsymbol{x}_m$ converges weakly for an element of A (call it $\boldsymbol{x}_0$). In addition, for any $\boldsymbol{y}_i$ as $\Phi(\boldsymbol{x},\boldsymbol{y}_i)$ is concave in $\boldsymbol{x}$,

$$\overline{lim}\Phi(\boldsymbol{x}_m,\boldsymbol{y}_i) \leq \Phi(\boldsymbol{x}_0,\boldsymbol{y}_i), \text{or } f(\boldsymbol{x}_0) \geq \overline{lim} f(\boldsymbol{x}_m = \boldsymbol{v}).$$

So $P \cap G = \emptyset$. Then, G and P may be strictly separated, and it is possible to find a vector in E_n with coordinates a_k, such that

$$\sup_{x \in A} \sum_{i=1}^{n} a_i (\Phi(\boldsymbol{x},\boldsymbol{y}_i) - C) < \sum_{i=1}^{n} a_i \boldsymbol{e}_i,$$

with the whole a_i greater or equal than zero.

Obviously, the a_i cannot be simultaneously null. So dividing for $\sum_{i=1}^{n} a_i$ and taking in account the convexity of $\Phi(\boldsymbol{x},\boldsymbol{y})$ in $\boldsymbol{y}$

$$\sup_{x \in A} \Phi(\boldsymbol{x},\overline{\boldsymbol{y}}) - C < 0, \text{where } \overline{\boldsymbol{y}} = \frac{\sum_{k=1}^{n} a_k y_k}{\sum_{k=1}^{n} a_k}.$$

In addition, evidently, or $\overline{\boldsymbol{y}} \in B$or $\inf_{y \in B} \sup_{x \in A} \Phi(\boldsymbol{x},\boldsymbol{y}) < C$.This contradicts the definition of C. So,

$$\bigcap_{i=1}^{n} A_{\boldsymbol{y}_i} \neq \emptyset.$$

Indeed,

$$\bigcap_{y \in B} A_{\boldsymbol{y}} \neq \emptyset$$

as it will be seen in the sequence using that result and proceeding by absurd. Note that $A_{\boldsymbol{y}}$ is a closed and convex set and so it is also weakly closed. And being bounded it is compact in the weak topology[5], as *A*. Calling $G_{\boldsymbol{y}}$ the complement of $A_{\boldsymbol{y}}$ it results that $G_{\boldsymbol{y}}$ is open in the weak topology. So, if $\bigcap_{\boldsymbol{y}\in B} A_{\boldsymbol{y}}$ is empty, $\bigcap_{\boldsymbol{y}\in B} G_{\boldsymbol{y}} \supset H \supset A$. But, being *A* compact, a finite number of $G_{\boldsymbol{y}_i}$ is enough to cover *A*:

$$\bigcup_{i=1}^{n} G_{\boldsymbol{y}_i} \supset A;$$

that is: $\bigcap_{i=1}^{n} A_i$ is in the complement of A and so it must be $\bigcap_{i=1}^{n} A_{\boldsymbol{y}_i} = \emptyset$, leading to a contradiction.

Suppose then that $\boldsymbol{x}_0 \in \bigcap_{\boldsymbol{y}\in B} A_{\boldsymbol{y}}$. So, in fact $\boldsymbol{x}_0$ satisfies $\Phi(\boldsymbol{x}_0, \boldsymbol{y}) \geq C$, as requested.■

Then it follows a Corollary of Theorem 7.1, obtained strengthening its hypothesis.

Corollary 7.1

Suppose that the functional $\Phi(\boldsymbol{x}, \boldsymbol{y})$ defined in Theorem 2.1 is continuous in both variables, separately, and that *B* is limited. Therefore, there is an optimal pair of strategies, with the property of being a saddle point.

Dem:

It was already seen that exists $\boldsymbol{x}_0$ such that

$$\Phi(\boldsymbol{x}_0, \boldsymbol{y}) \geq C \qquad (7.3)$$

for each $\boldsymbol{y}$. As $\Phi(\boldsymbol{x}_0, \boldsymbol{y})$ is continuous in $\boldsymbol{y}$ and *B* is limited

$$\inf_{\boldsymbol{y}\in B} \Phi(\boldsymbol{x}_0, \boldsymbol{y}) = \Phi(\boldsymbol{x}_0, \boldsymbol{y}_0) \geq C \qquad (7.4)$$

for any $\boldsymbol{y}_0$ in *B*[6]. But $\inf_{\boldsymbol{y}\in B} \Phi(\boldsymbol{x}_0, \boldsymbol{y}) \leq \sup_{\boldsymbol{x}\in A} \inf_{\boldsymbol{y}\in B} \Phi(\boldsymbol{x}, \boldsymbol{y}) = C$ and, so

$$\Phi(\boldsymbol{x}_0, \boldsymbol{y}_0) = C. \qquad (7.5)$$

The saddle point property follows immediately from (7.3), (7.4) and (7.5). ■

To see more details about this approach of minimax theorem, see [21-26].

One last reference to Nash theorem, [27], which generalizes the minimax theorem:

Theorem 7.2 (Nash)

The mixed extension of every finite game has, at least, one strategic equilibrium. ■

Its demonstration demands, among other results, an important contribution of Kakutani theorem, see [28].

[5] See, for instance, [3].

[6] A continuous convex functional in a Hilbert space has minimum in any limited closed convex set.

8. Conclusions

The Hilbert spaces are one of the mathematical fields more considered in the optimization problems fundamentals. Therefore, their structure and respective consequences deserve study and reflection. This was what we tried to do here in as simple ways as we could. It is always important to emphasize the fruitfulness of the results in part presented in convex programming, for instance in Kuhn-Tucker theorem and in the minimax theorem. It is never too much to point out the importance of strict separation theorems in achieving these results. Also to refer the importance of the Riesz representation theorem in the rephrasing of the separation theorems, key tools in functional optimization, here in strict separation theorems. Ant its direct contribution in getting the Lagrange duality results. Finally, to highlight Theorem 2.1, by its comprehensiveness, fundamental in optimization.

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