Interpolation in Description Logic: A Survey

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Plan

- Introduction to Ontologies/Description Logic
- Interpolation for Rewritings/Beth Definability
- Parallel Interpolation for Decomposition
- Uniform Interpolation

Ontologies

In Computer Science, ontologies $\mathcal{O} = (\boldsymbol{T}, \mathbf{Sig})$ consist of a

a finite axiomatization $\boldsymbol{T}$ *of a logical theory over a signature* Sig.

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- Ontologies are typically given in description logics (DLs) which underpin the W3C standard OWL.
- DLs: well-behaved fragments of first-order logic with convenient syntax.
- Data are not part of the ontology.

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- NCI: National Cancer Institute Thesaurus; 60,000 terms;
- GO: Gene ontology; more than 50,000 terms;
- GALEN: medical ontology; lot's of different versions.

Example

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\begin{aligned}
\text{Cystic\_Fibrosis} &\equiv \text{Fibrosis} \sqcap \exists \text{located\_In.Pancreas} \sqcap \exists \text{has\_Origin.Genetic\_Origin} \\
\text{Genetic\_Fibrosis} &\equiv \text{Fibrosis} \sqcap \exists \text{has\_Origin.Genetic\_Origin} \\
\text{Genetic\_Fibrosis} &\sqsupseteq \text{Fibrosis} \sqcap \exists \text{located\_In.Pancreas} \\
\text{Genetic\_Fibrosis} &\sqsubseteq \text{Genetic\_Disorder} \\
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Translation of first axiom into FO:

$$\forall x.(\textbf{Cystic\_Fibrosis}(x) \leftrightarrow C(x))$$

where

$$C(x) = \textbf{Fibrosis}(x) \sqcap \exists y.(\textbf{located\_In}(x, y) \wedge \textbf{Pancreas}(y)) \wedge \exists y.(\textbf{has\_Origin}(x, y) \wedge \textbf{Genetic\_Origin}(y))$$

Description Logics: $\mathcal{EL}$ and $\mathcal{ALC}$

$\mathcal{EL}$*-concepts* are constructed from concept names (unary predicates) $A_1, A_2, \ldots$ and binary relations $r_1, \ldots$

$$C := \top \mid A_i \mid C \sqcap C \mid \exists r_i.C.$$

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In a *model* $\mathcal{I} = (\Delta^{\mathcal{I}}, A_1^{\mathcal{I}}, \ldots, r_1^{\mathcal{I}}, \ldots)$ the interpretation $C^{\mathcal{I}} \subseteq \Delta$ of a concept $C^{\mathcal{I}}$ is defined inductively:

$$\begin{aligned}
(C_1 \sqcap C_2)^{\mathcal{I}} &= C_1^{\mathcal{I}} \cap C_2^{\mathcal{I}} \\
(\exists r.C)^{\mathcal{I}} &= \{w \in \Delta \mid \exists v\ (w,v) \in r^{\mathcal{I}} \wedge v \in C^{\mathcal{I}}\} \\
(\forall r.C)^{\mathcal{I}} &= \{w \in \Delta \mid \forall v\ (w,v) \in r^{\mathcal{I}} \Rightarrow v \in C^{\mathcal{I}}\}
\end{aligned}$$

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Deciding whether $\mathcal{O} \models C \sqsubseteq D$ is

- ExpTime-complete for $\mathcal{ALC}$;
- PTime-complete for $\mathcal{EL}$.

Explicit Definitions in Description Logic

Provide definitions of new terms using already defined terms.

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Possible aim: rewrite a given ontology into one that (mainly) consists of definitions of the form

$$A \equiv C$$

where A is a concept name. If no cyclic definitions occur, such ontologies are called acyclic TBoxes.

Concrete Application: Ontologies for Querying data

Assume a database schema is given by the signature

$$\Sigma = \{\textbf{diagnosis}, \textbf{heartdisease}\}$$

and a user wants to query $\textbf{heartpatient}(x)$ which is not in the schema.

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Problem: Given an ontology $\mathcal{O}$, a schema Σ, and a query q, can q be equivalently rewritten into a Σ-query?

Explicit Definitions

Let C be a concept, $\mathcal{O}$ an ontology, and Σ a signature. C is explicitly definable using Σ in $\mathcal{O}$ iff there exists a concept D over Σ such that

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$$\begin{array}{rcl} \textsf{Parent} & \equiv & \exists \textsf{hasChild}.\top \\ \textsf{Parent} & \equiv & \textsf{Father} \sqcup \textsf{Mother} \\ \textsf{Father} & \sqsubseteq & \textsf{Man} \\ \textsf{Mother} & \sqsubseteq & \textsf{Woman} \\ \textsf{Man} & \sqsubseteq & \neg\textsf{Woman} \end{array}$$

Then **Mother** and **Father** are explicitly definable from $\Sigma = \{\textsf{hasChild}, \textsf{Woman}\}$ in $\mathcal{O}$ by

$$\textsf{Mother} \equiv \textsf{Woman} \sqcap \exists \textsf{hasChild}.\top, \quad \textsf{Father} \equiv \textsf{Man} \sqcap \exists \textsf{hasChild}.\top$$

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C is implicitly definable from Σ in $\mathcal{O}$ iff for any two models $\mathcal{I}$ and $\mathcal{J}$ with the same domain and the same interpretation of Σ-symbols,

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C is implicitly definable using Σ in $\mathcal{O}$ iff

$$\mathcal{O} \cup \mathcal{O}' \models C \equiv C'$$

where $'$ is the result of replacing non-Σ-symbols by fresh symbols.

Interpolants as explicit definitions

Assume $\mathcal{O} \cup \mathcal{O}' \models C \sqsubseteq C'$. Then there exists an interpolant I with

- $\textbf{sig}(I) \subseteq \textbf{sig}(C, \mathcal{O}) \cap \textbf{sig}(C', \mathcal{O}')$.
- $\mathcal{O} \cup \mathcal{O}' \models C \sqsubseteq I$.
- $\mathcal{O} \cup \mathcal{O}' \models I \sqsubseteq C'$.

Tableau-based algorithms for computing I for various DLs (including $\mathcal{ALC}$) developed in recent JAIR paper.

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$\exists S.\top$ is explicitly defined using $\{R_1, R_2\}$ by

$$\exists S.\top \equiv \exists (R_1 \cap R_2).\top.$$

This is, however, not in the OWL standard.

Decompositions of Ontologies

Assume $\mathcal{O}$ is an ontology.

A partition $\Sigma_1, \ldots, \Sigma_n$ of $\textbf{sig}(\mathcal{O})$ is a decomposition of $\mathcal{O}$ if there are $\mathcal{O}_1, \ldots, \mathcal{O}_n$ such that

- $\textbf{sig}(\mathcal{O}_i) \subseteq \Sigma_i$;
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Assume $\mathcal{O}$ is an ontology and $\Delta \subseteq \mathbf{sig}(\mathcal{O})$ a signature.

A partition $\Sigma_1, \ldots, \Sigma_n$ of $\mathbf{sig}(\mathcal{O}) \setminus \Delta$ is a Δ-decomposition of $\mathcal{O}$ if there are $\mathcal{O}_1, \ldots, \mathcal{O}_n$ such that

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Problems:

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- Do decompositions in a given DL coincide with decompositions in SO?
- Compute (unique finest) decomposition.

Parallel Interpolation (without Δ)

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is a parallel interpolant of $\mathcal{O}_1, \mathcal{O}_2$ and α if

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- $\mathcal{O}_i \models \mathcal{O}'_i$;
- $\mathbf{sig}(\mathcal{O}'_i) \subseteq \mathbf{sig}(\mathcal{O}_i) \cap \mathbf{sig}(\alpha)$;

Parallel interpolation: parallel interpolant exists if $\mathcal{O}_1 \cup \mathcal{O}_1 \models \alpha$, $\mathbf{sig}(\mathcal{O}_1) \cap \mathbf{sig}(\mathcal{O}_2) = \emptyset$, and $\mathcal{O}_1, \mathcal{O}_2$ have the same consequences over empty signature.

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Parallel interpolation implies:

- There is a unique finest Δ-decomposition.
- Decompositions in DL coincide with decompositions in SO.
- Interpolants are axiomatizations of components.

Uniform interpolation

Standard interpolation: if $\mathcal{O} \models \alpha$, then there exists $\mathcal{O}'$ with

- $\textbf{sig}(\mathcal{O}') \subseteq \textbf{sig}(\mathcal{O}) \cap \textbf{sig}(\alpha)$;
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Definition: A uniform Σ-interpolant $\mathcal{O}'$ of $\mathcal{O}$ has the following properties:

- $\mathcal{O} \models \mathcal{O}'$;
- $\mathbf{sig}(\mathcal{O}') \subseteq \Sigma$;
- if $\mathcal{O} \models \alpha$ and $\mathbf{sig}(\alpha) \cap \mathbf{sig}(\mathcal{O}) \subseteq \Sigma$, then $\mathcal{O}' \models \alpha$.

In FO (and DLs) uniform interpolants do not always exist

Let

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and $\Sigma = \{A, r\}$.

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$\mathcal{ALC}\mu$ (modal μ-calculus) is an extension of $\mathcal{ALC}$ with uniform interpolation.

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- Re-use: from an ontology of size 300 000 one typically needs only a small fraction of its terms for an application. Work with the corresponding uniform Σ-interpolant.
- Ontology summary: a uniform interpolant summarises what an ontology says about Σ.
- Predicate-Hiding: if one does not want to publish what the ontologies says about non-Σ-symbols.

Uniform interpolants for acyclic $\mathcal{EL}$-TBoxes

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Proof that exponentially many axioms are required: Let

$$\mathcal{O} = \{A \equiv B_1 \sqcap \cdots \sqcap B_n\} \cup \{A_{ij} \sqsubseteq B_i \mid 1 \leq i, j \leq n\}.$$

and

$$\Sigma = \{A\} \cup \{A_{ij} \mid 1 \leq i, j \leq n\}.$$

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and

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Then

$$\mathcal{O}' = \{A_{1j_1} \sqcap \cdots \sqcap A_{nj_n} \sqsubseteq A \mid 1 \leq j_1, \ldots, j_n \leq n\}$$

is a smallest uniform Σ-interpolant.

Exponential size axioms in uniform interpolants

Let

$$\mathcal{O} = \{A_i \sqsubseteq \exists r.A_{i+1} \sqcap \exists s.A_{i+1} \mid i \leq n\}$$

and $\Sigma = \{A_0, r, s\}$.

Exponential size axioms in uniform interpolants

Let

$$\mathcal{O} = \{A_i \sqsubseteq \exists r.A_{i+1} \sqcap \exists s.A_{i+1} \mid i \leq n\}$$

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Then

$$\mathcal{O}' = \{A_0 \sqsubseteq \text{ binary tree of depth } n\}$$

is smallest uniform Σ-interpolant.

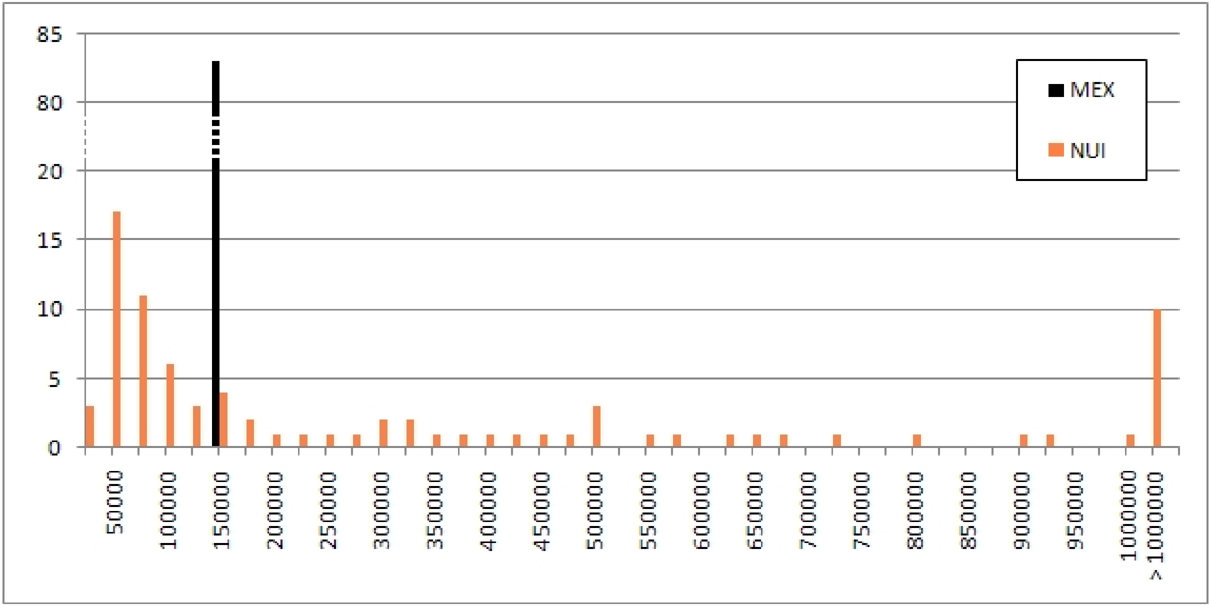
Computing uniform interpolants for SNOMED CT and NCI

100 randomly generated signatures.

$\lvert\Sigma\rvert$	SNOMED CT	$\lvert\Sigma\rvert$	NCI
2 000	**100.0%**	**5 000**	**97.0%**
3 000	**92.2%**	**10 000**	**81.1%**
4 000	**67.0%**	**15 000**	**72.0%**
5 000	**60.0%**	**20 000**	**59.2%**

Comparing the size of Σ-modules and Σ-interpolants for SNOMED CT

- Signatures containing $3\,000$ concept names and 20 role names



Σ-module

Let $\mathcal{O}$ be an ontology and Σ a signature.

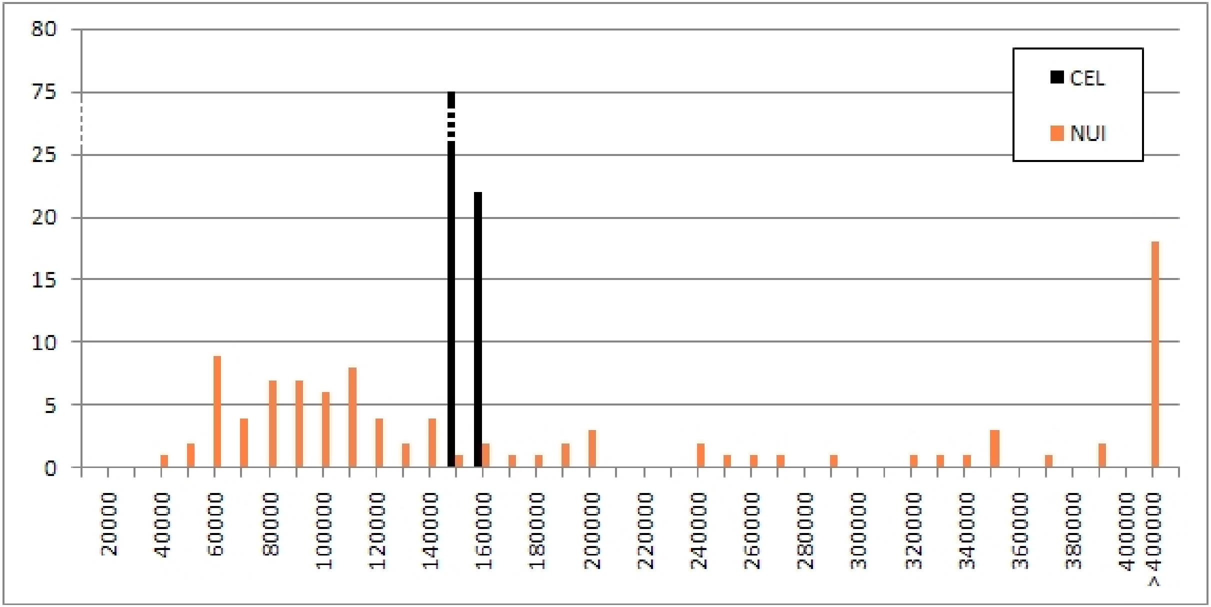
A Σ-module $\mathcal{M} \subseteq \mathcal{O}$ has the following property:

$$\mathcal{M} \models \alpha \quad \Leftrightarrow \quad \mathcal{O} \models \alpha$$

for all α over Σ.

Comparing the size of Σ-modules and Σ-interpolants for NCI

- Σ contains $7\,000$ concept names and 20 role names



Complexity and Size of Uniform Interpolants

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Work on computing uniform interpolants at this workshop, IJCAR 2014, and KR 2014.

Where do the α come from?

Let

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is a uniform Σ-interpolant of $\mathcal{O}$ for $\mathcal{ALC}$ concept inclusions.

This is not a uniform Σ interpolant for FO (or certain DLs).

$\mathcal{EL}$ uniform interpolants are not always $\mathcal{ALC}$ uniform interpolants

$$\mathcal{O} = \{A \sqsubseteq \exists r.B, A_0 \sqsubseteq \exists r.(A_1 \sqcap B), E \equiv A_1 \sqcap B \sqcap \exists r.(A_2 \sqcap B)\}$$

is an acyclic $\mathcal{EL}$-TBox. So uniform interpolants for $\mathcal{EL}$ consequences always exist.

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is an acyclic $\mathcal{EL}$-TBox. So uniform interpolants for $\mathcal{EL}$ consequences always exist.

However, for $\Sigma = \{A, r, A_0, A_1, E\}$, there is no uniform Σ-interpolant for $\mathcal{ALC}$ consequences.

Other α

Important for ontology-based data access, where one uses queries $\boldsymbol{q}$ (e.g., conjunctive queries) to query data sets $\mathcal{D}$ taking into account ontology $\mathcal{O}$:

$$\mathcal{O} \cup \mathcal{D} \models \boldsymbol{q}$$

Other α

Important for ontology-based data access, where one uses queries q (e.g., conjunctive queries) to query data sets $\mathcal{D}$ taking into account ontology $\mathcal{O}$:

$$\mathcal{O} \cup \mathcal{D} \models q$$

Investigate existence and computation of $\mathcal{O}'$ such that

- $\mathcal{O} \models \mathcal{O}'$;
- $\mathbf{sig}(\mathcal{O}') \subseteq \Sigma$;
- if $\mathcal{O} \cup \mathcal{D} \models q$ and $\mathbf{sig}(D, q) \cap \mathbf{sig}(\mathcal{O}) \subseteq \Sigma$, then $\mathcal{O}' \cup \mathcal{D} \models q$.

For $\mathcal{EL}$ very similar to concept inclusions; for $\mathcal{ALC}$ no results yet.

Conclusion

- Many potential applications of interpolation in Description Logic.
- Many theoretical results: existence of interpolants, size of interpolants, complexity of computing interpolants.
- Implemented algorithms and evaluation needed.

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